**Engineering Mathematics**
2018; 2(2): 63-67
http://www.sciencepublishinggroup.com/j/engmath
doi: 10.11648/j.engmath.20180202.11
ISSN: 2640-0855 (Print); ISSN: 2640-088X (Online)

# Qualitative Behavior for Fourth-Order Nonlinear Differential Equations

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**To cite this article:**
Omar Bazighifan, Elmetwally Elabbasy, Osama Moaaz. Qualitative Behavior for Fourth-Order Nonlinear Differential Equations. *Engineering Mathematics*. Vol. 2, No. 2, 2018, pp. 63-67. doi: 10.11648/j.engmath.20180202.12

**Received**: October 1, 2018; **Accepted**: October 16, 2018; **Published**: November 5, 2018

**Abstract:** In this paper, new an oscillation criterion for a class of fourth-order nonlinear delay differential equations are established by using the double generalized Riccatti substitutions. Recently, there has been increasing interest related to the theory of delay differential equations (DDEs), this has been attributed to the important of understanding of application in delay differential equations. Recent applications that include delay differential equations continue to appear with increasing frequency in the modeling of diverse phenomena in physics, biology, ecology, and physiology. So, other authors have been attracted to finding the solutions of the differential equations or deducing important characteristics of them has received the attention of many authors. The solution to this equation is important in order to understand these issues and phenomena, or at least to know the characteristics of the solutions to these equations. But, differential equations such as those used to solve real life problems may not necessarily be directly solvable, i.e., do not have closed form solutions. The main objective of this work is to provide an opportunity to study the new trends and analytical insights of the delay differential equations, existence and uniqueness of the solutions, boundedness and persistence, oscillatory behavior of the solutions, stability and bifurcation analysis, parameter estimations and sensitivity analysis, and numerical investigations of solutions. One objective of our paper is to further simplify and complement some well-known results which were published recently in the literature. An illustrative example is included.

**Keywords:** Oscillation, Fourth-Order, Delay Differential Equations

## 1. Introduction

This paper deals with the oscillation of a fourth-order delay differential equation

$$[b(t)(y'''(\ell))^{\gamma}]' + q(\ell) f(y(\tau(\ell))) = 0, \quad \ell \geq \ell_0. \tag{1}$$

The following conditions are assumed to hold:

($H_1$) $\gamma$ is a quotient of odd positive integers;

($H_2$) $b, q, \tau \in C^1[\ell_0, \infty)$, $b'(\ell) \geq 0$, $b(\ell) > 0$, $f \in C(\Re, \Re)$, $-f(-xy) = f(xy) = f(x) f(y)$, for $xy > 0$, q $> 0$, $\tau \leq \ell$, $\lim_{\ell \to \infty} \tau(\ell) = \infty$.

($H_3$) there exist constants $r > 0$ such that $f(u)/u^{\gamma} \geq r$, for $u \neq 0$.

The function $y(\ell)$ is a solution of (1), we mean a non-trivial real function n $y(\ell) \in C([\ell_y, \infty))$, $\ell_y \geq \ell_0$ satisfying (1) on $[t_x, \infty)$ and moreover, having the properties: $y(\ell)$, $y'(\ell)$, $y''(\ell)$, $b(\ell)(y'''(\ell))$ are continuously differentiable for all $\ell \in [\ell_y, \infty)$. We consider only those solutions $y(\ell)$ of (1) which satisfy $\sup\{|y(\ell)| : \ell \geq L\} > 0$, for any $L \geq \ell_y$. A solution of (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

The oscillations of higher and fourth order differential equations have been studied by several authors and several techniques have been proposed for obtaining oscillatory criteria for higher and fourth order differential equations. For treatments on this subject, we refer the reader to the texts [2, 6, 17, 13-15] and the articles [1, 3-12, 16-25]. In what follows, we review some results that have provided the background and the motivation, for the present work.

Elabbasy, et al.[10] consider the oscillation of a fourth-order delay differential equation

$[b(\ell)(y''(\ell))^{\gamma}]' + \sum_{i=1}^{m} q_i(\ell) f(y(\tau_i(\ell))) = 0$ , $\ell \geq \ell_0$. Grace, et al. [12] studied the oscillation behavior of the fourth-order nonlinear differential equation

$$[b(\ell)(y'(\ell))^{\alpha}]''' + q(\ell) f(y(\tau(\ell))) = 0 , \quad \ell \geq \ell_0.$$

Agarwal, et al.[3] examined the oscillation of the fourth-order nonlinear delay differential equation

$$[b(\ell)(y''(\ell))^{\beta}]'' + q(\ell) y^{\beta}(\ell) = 0 , \quad \ell \geq \ell_0.$$

Zhang, et al.[24] consider the oscillatory properties ofthe higher-order differential equation

$$[b(\ell)(y^{(n-1)}(\ell))^{\gamma}]' + q(\ell) y^{\gamma}(\tau(\ell)) = 0 , \quad \ell \geq \ell_0.$$

under the conditions

$$\int_{\ell_0}^{\infty} \frac{1}{b^{1/\gamma}(\ell)} d\ell < \infty,$$

and

$$\int_{\ell_0}^{\infty} \frac{1}{b^{1/\gamma}(\ell)} d\ell = \infty. \tag{2}$$

Our aim in the present paper, we have studied the oscillatory behavior of the fourth-order delay differential equation (1) under condition (2). As it has been illustrated through example, the results obtained improve a large number of the existing ones. Our technique lies in using comparison theorem and generalized Riccatti substitutions. An illustrative example is included.

## 2. Main Results

In this section, we shall establish some oscillation criteria for equation (1). The proof of our main results are essentially based on the following lemmas.

Lemma 2.1: Let $z \in \left(C^n\left[\ell_0,\infty\right),\Re^+\right)$ and assume that $z^{(n)}$ is of fixed sign and not identically zero on a subrayof $\left[\ell_0,\infty\right)$. If moreover, $z(\ell) > 0,\ z^{(n-1)}(\ell) z^{(n)}(\ell) \leq 0.$ and $\lim_{\ell\to\infty} z(\ell) \neq 0,$ then, for every $\lambda \in (0,1)$, there exists $\ell_\lambda \geq \ell_0$ such that

$$z(\ell) \geq \frac{\lambda}{(n-1)} \ell^{n-1} \left| z^{(n-1)}(\ell) \right|, \text{ for } \ell \in [\ell_\lambda, \infty).$$

Lemma 2.2: Let $\beta \geq 1$ be a ratio of two numbers. Then

$$A^{\frac{\beta+1}{\beta}} - (A-B)^{\frac{\beta+1}{\beta}} \leq \frac{B^{\frac{1}{\beta}}}{\beta}\left[(1+\beta)A - B\right], \quad AB \geq 0,$$

and

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^{\beta}},\ V > 0.$$

Lemma 2.3: If the function $z$ satisfies $z^{(i)} > 0,\ i = 0,1,...,n$ and $z^{(n+1)} < 0,$ then $\frac{z(\ell)}{\ell^n / n!} \geq \frac{z'(\ell)}{\ell^{n-1}/(n-1)!}.$

We are now ready to state and prove the main results. For convenience, we denote

$$\pi(s) := \int_{\ell_0}^{\infty} \frac{1}{b(s)} ds \quad \text{and} \quad \vartheta'_+(\ell) := \max\{0, \vartheta'(\ell)\}.$$

Theorem 2.1: Let $(H_1), (H_2), (H_3)$ and (1.2) holds. Further, assume that for some constant λ ∈ (0, 1), the differential equation

$$x'(\ell) + q(\ell) f\left[\frac{\lambda_0}{6 b^{1/\gamma}(\tau(\ell))} \tau^3(\ell)\right] f(x^{1/\gamma}(\tau(\ell))) = 0 , \tag{3}$$

is oscillatory. If there exist positive functions $\vartheta \in C\left(\left[\ell_0,\infty\right)\right)$ such that

$$\int_{\ell_0}^{\infty} \left(\psi^*(s) - \frac{1}{4}\vartheta(s)\left(\varphi^*(s)\right)^2\right) ds = \infty, \tag{4}$$

where

$$\psi^*(\ell) := \vartheta(\ell)\left[\int_{\ell}^{\infty}\left[\frac{r}{b(v)}\int_{v}^{\infty} q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}} ds\right]^{1/\gamma} dv - \frac{1 + b^{\frac{1}{\gamma}}(\ell)}{\pi^2(\ell)}\right],$$

$$\varphi^*(\ell) := \frac{\vartheta'_+(\ell)}{\vartheta(\ell)} + \frac{2}{\pi(\ell)},$$

for some $\mu \in (0,1)$. Then every solution of (1) is oscillatory.

Proof: Assume that (1) has a nonoscillatory solution $y$ Without loss of generality, we can assume that $y(\ell) > 0$. It follows from (1) that there exist two possible cases for $\ell \geq \ell_1$, where $\ell_1 \geq \ell_0$ is sufficiently large:

Case 1:

$$y'(\ell)>0, y''(\ell)>0, y'''(\ell)>0, y^{(4)}(\ell)\le 0, \left(b\left(y'''\right)^{\gamma}\right)'(\ell)\le 0.$$

Case 2:

$$y'(\ell)>0, y''(\ell)<0, y'''(\ell)>0, y^{(4)}(\ell)\le 0, \left(b\left(y'''\right)^{\gamma}\right)'(\ell)\le 0.$$

For $\ell>\ell_1$, $\ell_1$ is large enough.

Assume that we have Case (1) holds. From Lemma 2.1, we have

$$y(\tau(\ell)) \ge \frac{\lambda\tau^3(\ell)}{6\, b^{1/\gamma}(\ell)}\left(b^{1/\gamma}(\ell)y'''(\tau(\ell))\right), \tag{5}$$

for every λ∈(0, 1). Using (5) in Eq. (1), we see that

$$x(\ell)=b(\ell)(y'''(\ell))^{\gamma},$$

is a positive solution of the differential inequality

$$x'(\ell)+k\,q(\ell)f\left[\frac{\lambda_0}{6b^{1/\gamma}(\tau(\ell))}\tau^3(\ell)\right]f(x^{1/\gamma}(\tau(\ell)))\le 0\,.$$

By Theorem 1 in Philos [21], we conclude that the corresponding equation (1) also has a positive solution. This contradiction.

Assume that we have Case (2)holds. From Lemma 2.3, we get that $y(\ell)\ge \ell y'(\ell)$, by integrating this inequality from $\tau(\ell)$ to $\ell$, we get

$$y(\tau(\ell))\ge\frac{\tau(\ell)}{\ell}y(\ell).$$

Hence, we have

$$f\left(y\left(\tau(\ell)\right)\right)\ge r\frac{\tau^{\gamma}(s)}{s^{\gamma}}y^{\gamma}(\ell).$$

Integrating (1) from $\ell$ to $u$ and using $y'(\ell)>0$, we obtain

$$\begin{aligned} b(u)\left(y'''(u)\right)^{\gamma}-b(\ell)\left(y'''(\ell)\right)^{\gamma} &= -\int_{\ell}^{u}q(s)f\left(y\left(\tau(s)\right)\right)ds \\ &\le -ry^{\gamma}(\ell)\int_{\ell}^{u}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds. \end{aligned}$$

Letting $u\to\infty$, we see that

$$b(\ell)\left(y'''(\ell)\right)^{\gamma}\ge ry^{\gamma}(\ell)\int_{\ell}^{\infty}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds,$$

and so,

$$y'''(\ell)\ge y(\ell)\left[\frac{r}{b(\ell)}\int_{\ell}^{\infty}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds\right]^{1/\gamma}.$$

Integrating again from $\ell$ to $\infty$, we get

$$y''(\ell)\le -y(\ell)\int_{\ell}^{\infty}\left[\frac{r}{b(v)}\int_{v}^{\infty}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds\right]^{1/\gamma}dv.$$

Now, we define

$$w(\ell)=\vartheta(\ell)\left[\frac{y'(\ell)}{y(\ell)}+\frac{1}{\pi(\ell)}\right] \tag{6}$$

Then $w(\ell)>0$ for $\ell\ge\ell_1$. By differentiating and using (6), we find

$$w'(\ell)=\frac{\vartheta'(\ell)}{\vartheta(\ell)}w(\ell)+\vartheta(\ell)\frac{y''(\ell)}{y(\ell)}-\vartheta(\ell)\frac{\left(y'(\ell)\right)^2}{y^2(\ell)}-\frac{\vartheta(\ell)}{b^{\frac{1}{\gamma}}(\ell)\pi^2(\ell)}$$

Thus, we obtain

$$w'(\ell)=\frac{\vartheta'(\ell)}{\vartheta(\ell)}w(\ell)+\vartheta(\ell)\frac{y''(\ell)}{y(\ell)}-\vartheta(\ell)\left[\frac{w(\ell)}{\rho(\ell)}-\frac{1}{\pi(\ell)}\right]^2-\frac{\vartheta(\ell)}{b^{\frac{1}{\gamma}}(\ell)\pi^2(\ell)} \tag{7}$$

Using Lemma 2.2 with $A=\frac{w(\ell)}{\rho(\ell)}, B=\frac{1}{\pi(\ell)}$ and $\gamma=1$, we get

$$\left[\frac{w(\ell)}{\vartheta(\ell)}-\frac{1}{\pi(\ell)}\right]^2\ge\left(\frac{w(\ell)}{\vartheta(\ell)}\right)^2-\frac{1}{\pi(\ell)}\left(\frac{2w(\ell)}{\vartheta(\ell)}-\frac{1}{\pi(\ell)}\right) \tag{8}$$

From (1), (7) and (8), we obtain

$$\begin{aligned} w'(\ell)\le{}&\frac{\vartheta'(\ell)}{\vartheta(\ell)}w(\ell)-\vartheta(\ell)\int_{\ell}^{\infty}\left[\frac{r}{b(v)}\int_{v}^{\infty}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds\right]^{1/\gamma}dv \\ &-\vartheta(\ell)\left[\left(\frac{w(\ell)}{\vartheta(\ell)}\right)^2-\frac{1}{\pi(\ell)}\left(\frac{2w(\ell)}{\vartheta(\ell)}-\frac{1}{\pi(\ell)}\right)\right]-\frac{\vartheta(\ell)}{b^{\frac{1}{\gamma}}(\ell)\pi^2(\ell)} \end{aligned}$$

This implies that

$$\begin{aligned} w'(\ell)\le{}&\left(\frac{\vartheta'_{+}(\ell)}{\vartheta(\ell)}+\frac{2}{\pi(\ell)}\right)w(\ell)-\frac{1}{\vartheta(\ell)}w^2(\ell) \\ &-\vartheta(\ell)\left[\int_{\ell}^{\infty}\left[\frac{r}{b(v)}\int_{v}^{\infty}q(s)\frac{\tau^{\gamma}(s)}{s^{\gamma}}ds\right]^{1/\gamma}dv-\frac{1+b^{\frac{1}{\gamma}}(\ell)}{\pi^2(\ell)}\right] \end{aligned} \tag{9}$$

Thus, by (9) yield

$$w'(\ell)\le-\psi^*(\ell)+\varphi^*(\ell)\omega(\ell)-\frac{1}{\vartheta(\ell)}w^2(\ell).$$

Applying the lemma 2.2 with $U=\varphi^*(\ell), V=\frac{1}{\vartheta(\ell)}$, $\gamma=1$ and $y=w$, we get

$$\omega'(\ell) \le -\psi^*(\ell) + \frac{1}{4}\vartheta(\ell)\left(\varphi^*(\ell)\right)^2.$$

Integrating from $\ell_1$ to $\ell$, we get

$$\int_{\ell_1}^{\ell}\left(\psi^*(s) - \frac{1}{4}\vartheta(s)\left(\varphi^*(s)\right)^2\right)ds \le \omega(\ell_1),$$

this contradicts (4).

Theorem 1 is proved.

Corollary 2.1: Let $(H_1)$, $(H_2)$, $(H_3)$ and (2) holds. If

$$\liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} q(s) f\left(\frac{\lambda_0}{6b^{1/\gamma}(\tau(s))}\tau^3(s)\right)ds \ge \frac{1}{e}, \tag{10}$$

and

$$\limsup_{\ell\to\infty} \int_{\ell_0}^{\ell}\left[rq(s)\left(\frac{\lambda}{2}\tau^2(s)\right)^{\gamma}\pi^{\gamma}(s) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}\pi(s)b^{1/\gamma}(s)}\right]ds = \infty \tag{11}$$

hold for some constant $\lambda \in (0,1)$. Then every solution of (1) is oscillatory.

## 3. Examples

In this section, we give the following example to illustrate our main results.

Example 3.1: Consider a differential equation

$$[\ell^3(y'''(\ell))^3]' + a\,\ell\, y^3(\ell) = 0\ ,\ \ell \ge 1. \tag{12}$$

Where a > 0 a constant. Let

$$\gamma = 3,\ \ \mathrm{b}(\ell) = \ell^3\ ,\ q(\ell) = a\,\ell,\ \ \tau(\ell) = \ell,$$

we see ($H_1$), (H2) and (H3) hold. Then, we find

$$\pi(s) := \int_{\ell_\circ}^{\infty}\frac{1}{b(s)}ds = \infty,$$

we get

$$\limsup_{\ell\to\infty} \int_{\ell_0}^{\ell}\left[rq(s)\left(\frac{\lambda}{2}\tau^2(s)\right)^{\gamma}\pi^{\gamma}(s) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}\pi(s)b^{1/\gamma}(s)}\right]ds = \infty$$

It is easy to see that all conditions of Corollary 2.1 hold. Hence every solution of (12) is oscillatory.

## 4. Conclusion

Therehas beenan openproblem regarding the study of sufficientconditions ensuring oscillationof all solutions of fourth-order differential equation withdelay. The present paper aimsto fill thisgap. In thefirst part, results for oscillation of (1) can be obtained by comparison withordinary differentialequations of thesameor a lowerorder. We offera comparison theorem thatrelates properties of solutions of (1) withthoseof first-order differential equations. There after, we suggest a new oscillation criterion for the fourth-order delay differential equation (1) using the Riccatti transformation technique.

## Acknowledgements

The authors express their sincere gratitude to the editors and the anonymous referee for careful reading of the original manuscript and useful comments.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All authors read and approved the final manuscript.

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