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A New Approach on Imprecise Stochastic Orders of Fuzzy Random Variables

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Abstract: In this paper the extension of stochastic dominance to an imprecise frame work are discussed in fuzzy nature. Also stochastic dominance between sets of fuzzy Probabilities can be studied by means of a P-box representation. The extension of pair of sets of distribution function by means of fuzzy random variables has been carried out.

Keywords: Fuzzy Distribution Function, Stochastic Dominance, Imprecise Stochastic Dominance, Fuzzy Random Variable, Probability Boxes

1. Introduction

One of the most popular stochastic ordering is stochastic dominance which is used in many fields. This notion was employed in 1930 and it has increasing popularity in many areas namely economics, Social welfare, agriculture operational research etc.

In some situations where there is uncertainty above the probability distributions underlying the random variables have vague, which results in the impossibility of producing the probability distribution. In this paper our aim is to extend the notion of stochastic dominance to the comparison of sets of cumulative distribution functions.

Stochastic dominance are used in economics and finance [2, 13] and can be given the following interpretation: $F \geq_{SD} G$ means that the choice of F over G is rational, in the sense that we prefer the alternative that provides greater probability of having a great profit. The notion has also been used in the other frame work such as reliability theory, statistical physics, epidemiology, etc.

In section 2 we discuss the some of the preliminary definitions and in section3 we extend the pairs of sets of distribution functions by means of P-boxes and some of the prepositions of imprecise stochastic dominance in the field of imprecise frame work have been discussed on the basis of fuzzy random variables.

2. Preliminary Concepts

2.1. Elementary Definitions

Definition:2.1.1

Let X be a universal set. Then a fuzzy set $\tilde{A} = \{(x, \mu_A(x))/x \in X\}$ of X is defined by its membership function $\mu_{\tilde{A}}: X \to [0,1]$.

Definition: 2.1.2

For each $0 \leq \alpha \leq 1$, the α-cut of set of $\tilde{A}$ is denoted by its $\tilde{A}_\alpha = \{x \in X; \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition:2.1.3

A fuzzy number is a fuzzy set of R such that the following conditions are satisfied:

i. $\tilde{A}$ is normal if there exists $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.
ii. $\tilde{A}$ is called convex if $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda x_2)) \geq$ min $(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$.
iii. $\tilde{A}$ is called upper semi continuous with compact support; that is for every $\in > 0$, there exists $\delta > 0; |x - y| < \delta \Rightarrow \mu_{\tilde{A}}(x) < \mu_{\tilde{A}}(y) + \in$.
iv. The α-cut of fuzzy number is closed interval denoted by $A_\alpha = [A_\alpha^L, A_\alpha^U]$, where
v. $A_\alpha^L =$ inf $\{x \in R; \mu_{\tilde{A}}(x) \geq \alpha\}$ and $A_\alpha^U =$ sup $\{x \in R: \mu_{\tilde{A}}(x) \geq \alpha\}$.
vi. If $\tilde{A}$ is closed and bounded fuzzy number with A_α^L, A_α^U and its membership function is strictly increasing on $[A_\alpha^L, A_\alpha^U]$ and strictly decreasing on $[A_1^L, A_\alpha^U]$ then $\tilde{A}$ is

called canonical fuzzy number.

Definition:2.1.4

A fuzzy random variable is a fuzzy set of a membership function and a basic set of underlying variables. A fuzzy random X is a map $X: \Omega \longrightarrow$ F(R) satisfying the following conditions:

i. For each $\alpha \in (0,1]$ both X_α^L and X_α^U defined as $X_\alpha^L(\omega)(x) = \inf\{x \in R; X(\omega)(x) \geq \alpha\}$ and $X_\alpha^U(\omega)(x) = \sup\ \{x \in R; X(\omega)(x) \geq \alpha\}$ are finite real valued random variables defined on such (Ω, A, P) that the mathematical expectations EX_α^L and EX_α^U exist.

ii. For each $\omega \in \Omega$ and $\alpha \in (0,1]$, $X_\alpha^L(\omega)(x) \geq \alpha$ and $X_\alpha^U(\omega)(x) \geq \alpha$.

Definition:2.1.5

$X(\omega)$ is a fuzzy random variables if and only if $X_\alpha(\omega) = [X_\alpha^L(\omega), X_\alpha^U(\omega)]$, where $X_\alpha^L(\omega)$ and $X_\alpha^U(\omega)$ are both random variables for each $\alpha \in (0,1]$ and $X(\omega) = \bigcup_{\alpha \in (0,1]} \alpha X_\alpha(\omega)$.

2.2. Stochastic Dominance

The notion of stochastic dominance between random variables is based on comparison of their corresponding distribution functions. In this paper we are going to work with random variables taking values on [0,1]. The distribution function is thus defined in the following way:

Definition: 2.2.1 [12]

A cumulative distribution function (cdf) is a function F: $[0,1] \rightarrow [0,1]$ satisfying following properties:

i. $x \leq y \Longrightarrow F(x) \leq F(y) \forall x, y$ [Monotonicity]

ii. $F(1) = 1$ [Normalisation]

iii. $F(x) = \lim_{\epsilon \downarrow 0} F(x + \varepsilon)\ \forall x < 1$ [Right continuity]

When F satisfies the properties of monotonicity and normalization, it is associated to finite additive probability distribution and we shall call it a finitely additive distribution function.

Definition: 2.2.2[12]

Given two cumulative distribution function F and G, we say that F is stochastically dominates G, and denote if $F \geq_{SD} G$, if $F(t) \leq G(t)$ for every $t \in [0,1]$. This definition produces a partial order in the space of cumulative distribution function, from which we can derive the notions of strict stochastic dominance, indifference and incomparability:

i. We say that F strictly stochastically dominates G, and denote it by $F >_{SD} G$ if $F \geq_{SD} G$ but $G \ngeq_{SD} F$. This holds if and only if $F \leq G$ and there is some $t \in [0,1]$. Such that $F(t) < G(t)$

ii. F and G are stochastically indifferent if $F \geq_{SD} G$ and $G \geq_{SD} F$ or equivalently, if $F = G$.

iii. F and G are stochasticallyincomparable if $F \ngeq_{SD} G$ and $G \ngeq_{SD} F$.

Definition: 2.2.3[12]

For the two random variables U and V such that $P(U \leq V) = 1$. We define both belief function and plausibility function as follows:

bel $(A) = P\ ([U,V] \subseteq A)$ and Pl$(A) = P\ ([U,V] \cap A \neq \phi)$.

Definition: 2.2.4[12]

The associated set of probability measures P is given by $\wp$ ={ $\wp$ Probability: bel (A)$\leq$ P(A)$\leq$ Pl (A) for every A$\in \mathcal{B}(\mathbb{R})$}.

Note: 2.2.5[12]

We consider two random closed interval [U,V] and $[U', V`']$ one possible way of comparing them is to compare their associated sets of probabilities:

$\wp$ ={$\wp$ Probability: bel (A)$\leq$ P(A)$\leq$ Pl(A) for every A$\in \mathcal{B}(\mathbb{R})$}.

$\wp'$={$\wp'$Probability: bel$'$(A)$\leq$ P(A)$\leq$ Pl$'(A)$ for every A$\in \mathcal{B}(\mathbb{R})$}.

Proposition: 2.2.6 [12Proposition 3]

Let [U,V] and $[U', V`']$ be two random closed intervals and let $\wp$ and $\wp'$ their associated sets of probability measures thefollowing equivalences hold:

- $\wp \gg \wp' \Leftrightarrow U \geq_{SD} V'$
- $\wp \succcurlyeq \wp' \Leftrightarrow U \geq_{SD} U'$
- $\wp \succcurlyeq \wp' \Leftrightarrow V \geq_{SD} V'$
- $\wp \gtrsim \wp' \Leftrightarrow V \geq_{SD} U'$

Note: 2.2.7

We consider the set of distribution functions induced by $\wp$, we obtain $\{F: F_V \leq F \leq F_U\}$ that is the P-box determined by F_V and F_U. Similarly, the set $\wp'$ induces the P-box (F'_U, F'_V).

2.3. Extension to Pairs of Sets of Distribution Functions

Definition: 2.3.1 [12]

Given a set of probability measures $\wp$ on [0, 1], we shall denote by $\mathcal{F}$= $\{F_\wp$: $P \in \wp\}$ its associated set of cumulative distribution functions.

Definition: 2.3.2 [12]

Let $\wp_1$,$\wp_2$ be two sets of probability measures on [0, 1], and let $\mathcal{F}_1, \mathcal{F}_2$ be their associated sets of distribution functions. We say that $\wp_1$:

- (SD_1) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_2} \mathcal{F}_2$ if and only if for every $F_1 \epsilon \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ it holds that $F_1 \leq F_2$.
- (SD_2) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_2} \mathcal{F}_2$ if andonly if there is some $F_1 \epsilon \mathcal{F}_1$ such that $F_1 \leq F_2$ for every $F_2 \in \mathcal{F}_2$.
- (SD_3) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_3} \mathcal{F}_2$ if and only if for every $F_2 \in \mathcal{F}_2$ there is some $F_1 \epsilon \mathcal{F}_1$ such that $F_1 \leq F_2$.
- (SD_4) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_4} \mathcal{F}_2$ if and only if there are $F_1 \epsilon \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ such that $F_1 \leq F_2$.
- (SD_5) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_5} \mathcal{F}_2$ if and only if there is $F_2 \in \mathcal{F}_2$ such that $F_1 \leq F_2$ for every $F_1 \epsilon \mathcal{F}_1$.
- (SD_6) stochastically dominates $\wp_2$ and denote it by $\mathcal{F}_1 \succeq_{SD_6} \mathcal{F}_2$ if and only if for every $F_1 \epsilon \mathcal{F}_1$ there is $F_2 \in \mathcal{F}_2$ such that $F_1 \leq F_2$.

Preposition 2.3.3 [12]

Let $\mathcal{F}_1$ and $\mathcal{F}_1$ be two sets of cumulative distribution functions on [0,1]

i. The implications between the conditions of stochastic dominance are given in the following figure.

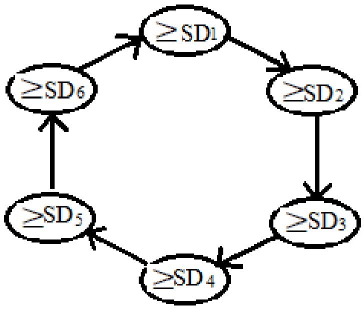

Figure 1. *Conditions of Stochastic dominance.*

ii. Concerning strict stochastic dominance,

- (SD_2) strict stochastic dominance implies (SD_3) strict stochastic dominance.
- (SD_5) strict stochastic dominance implies (SD_6) strict stochastic dominance.

2.4. P-boxes

Definition: 2.4.1

A probability box or P-box for short, $(\underline{F}, \bar{F})$ is the set of cumulative distribution functions bounded between two finitely additive distribution function $\underline{F} \leq \bar{F}$. We shall refer to $\underline{F}$ as the lower distribution function and to $\bar{F}$ as the upper distribution function of the P-box.

Note: 2.4.2

Let $\underline{F}, \bar{F}$ need not be cumulative distribution functions, and as such they need not belong to the set $(\underline{F}, \bar{F})$; they are only required to be finitely additive distribution functions.

Note: 2.4.3

Let $\mathcal{F}$ be set of cumulative distribution functions, its associated P- boxes $(\underline{F}, \bar{F})$ is determined by, $\underline{F}(x) = {}_{F\in\mathcal{F}}^{min} F_X(x)_\alpha^L$, $\bar{F}(x) = {}_{F\in\mathcal{F}}^{max} F_X(x)_\alpha^U$, $\forall x \in [0,1]$.

3. Imprecise Stochastic Dominance

Definition: 3.1

A fuzzy probability box or P-box for short, $(\underline{F}_X, \bar{F}_X)$ is the set of cumulative distribution functions bounded between two finitely additive distribution function $\underline{F}_X \leq \bar{F}_X$ We shall refer to $\underline{F}_X$ as the lower distribution function and to $\bar{F}_X$ as the upper distribution function of the P-box.

Note: 3.2

Let $\underline{F}_X, \bar{F}_X$ need not be cumulative distribution functions, and as such they need not belong to the set $(\underline{F}_X, \bar{F}_X)$; they are only required to be finitely additive distribution functions.

Definition: 3.3

Let $\mathcal{F}_X$ be set of cumulative distribution functions, its associated P- boxes $(\underline{F}_X, \bar{F}_X)$ is determined by,

$\underline{F}_x(x) = \min\{ {}_{F\in\mathcal{F}}^{inf} F_X(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_X(x)_\alpha^U \}$, $\bar{F}_X(x) = \max\{ {}_{F\in\mathcal{F}}^{inf} F_X(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_X(x)_\alpha^U \}$.

Proposition: 3.4

Let $\mathcal{F}_1, \mathcal{F}_2$ be two sets of fuzzy cumulative distribution functions anddenote by $(\underline{F}_1, \bar{F}_1)$ and $(\underline{F}_2, \bar{F}_2)$ the P- boxes they induce.

i. $(\mathcal{F}_1 \geq_{SD1} \mathcal{F}_2 \Leftrightarrow \bar{F}_{1X}(x) \leq \underline{F}_{2X}(x)$.

ii. $\mathcal{F}_1 \geq_{SD2} \mathcal{F}_2 \Leftrightarrow \underline{F}_{1X}(x) \leq \underline{F}_{2X}(x)$.

iii. $\mathcal{F}_1 \geq_{SD3} \mathcal{F}_2 \Leftrightarrow \underline{F}_{1X}(x) \leq \underline{F}_{2X}(x)$.

iv. $\mathcal{F}_1 \geq_{SD4} \mathcal{F}_2 \Leftrightarrow \underline{F}_{1X}(x) \leq \bar{F}_{2X}(x)$

v. $\mathcal{F}_1 \geq_{SD5} \mathcal{F}_2 \Leftrightarrow \bar{F}_{1X}(x) \leq \bar{F}_{2X}(x)$.

vi. $\mathcal{F}_1 \geq_{SD6} \mathcal{F}_2 \Leftrightarrow \bar{F}_{1X}(x) \leq \bar{F}_{2X}(x)$.

Where, $\underline{F}_{1X}(x) = {}_{F_1\epsilon\mathcal{F}_1}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \}$.

$$\bar{F}_{1X}(x) = {}_{F_1\epsilon\mathcal{F}_1}^{\max}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \}.$$

$$\underline{F}_{2X}(x) = {}_{F_2\epsilon\mathcal{F}_2}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}.$$

$$\bar{F}_{2X}(x) = {}_{F_2\epsilon\mathcal{F}_2}^{\max}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}.$$

Proof

(i) Note that $\mathcal{F}_1 \geq_{SD1} \mathcal{F}_2$ if and only if for every $F_{1X}(x)\epsilon\mathcal{F}_1$, $F_{2X}(x)\epsilon\mathcal{F}_2$, and this equivalent to

$$\bar{F}_{1X}(x) = {}_{F_1\epsilon\mathcal{F}_1}^{\max}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \} \leq \underline{F}_{2\,X}(x) = {}_{F_2\epsilon\mathcal{F}_2}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}$$

$$\Rightarrow \bar{F}_{1X}(x) \leq \underline{F}_{2\,X}(x).$$

(ii) $\mathcal{F}_1 \geq_{SD2} \mathcal{F}_2$ if $\underline{F}_{1X}(x) \leq \underline{F}_{2X}(x)$

$\Rightarrow \underline{F}_{1X}(x) \leq \underline{F}_{2X}(x)$, for every $F_{1X}(x)\epsilon\mathcal{F}_1$, $F_{2X}(x)\epsilon\mathcal{F}_2$

$$\Rightarrow {}_{F_1\epsilon\mathcal{F}_1}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \} \leq {}_{F_2\epsilon\mathcal{F}_2}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}$$

$$\Rightarrow \underline{F}_{1X}(x) \leq \underline{F}_{2X}(x).$$

(iii) By the Hypothesis, For every $F_{2X}(x)\epsilon\mathcal{F}_2$ there is some $F_{1X}(x)\epsilon\mathcal{F}_1$,

such that $F_{1X}(x) \leq F_{2X}(x)$ As a consequence, $\underline{F}_{1X}(x) \leq F_{2X}(x) \forall F_{2X}(x)\epsilon\mathcal{F}_2$

$$\Rightarrow \underline{F}_{1X}(x) \leq inf_{F_2\epsilon\mathcal{F}_2}$$

There fore $F_{1X}(x) = \underline{F}_{2X}(x)$

$$\Rightarrow {}_{F_1\epsilon\mathcal{F}_1}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \} \leq {}_{F_2\epsilon\mathcal{F}_2}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}$$

$$\Rightarrow F_{1X}(x) \leq F_{2X}(x).$$

(iv) If there are $F_{1X}(x)\epsilon\,\mathcal{F}_1$ and $F_{2X}(x)\epsilon\,\mathcal{F}_2$ such that $F_{1X}(x) \leq F_{2X}(x)$ then

$$\Rightarrow \underline{F}_1 \leq F_1 \leq F_2 \leq \bar{F}_2$$

$$\Rightarrow {}_{F_1\epsilon\mathcal{F}_1}^{\min}\{ {}_{F\in\mathcal{F}}^{inf} F_{1X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{1X}(x)_\alpha^U \} \leq F_{1X}(x) \leq F_{2X}(x) \leq {}_{F_2\epsilon\mathcal{F}_2}^{\max}\{ {}_{F\in\mathcal{F}}^{inf} F_{2X}(x)_\alpha^L, {}_{F\in\mathcal{F}}^{sup} F_{2X}(x)_\alpha^U \}$$

Hence, $\underline{F_1}_X(x) \leq F_{1_X}(x) \leq F_{2_X}(x) \leq \bar{F}_2$.

There fore $\underline{F_1}_X(x) \leq \bar{F}_{2_X}(x)$.

(v) If there are $F_{1_X}(x)\epsilon\mathcal{F}_1$ and $F_{2_X}(x)\epsilon\,\mathcal{F}_2$ such that $F_{1_X}(x) \leq F_{2_X}(x)$ then $\bar{F}_{1_X}(x) \leq \bar{F}_{2_X}(x)$

$$\Rightarrow \max_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \} \leq \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \}$$

$$\bar{F}_{1_X}(x) \leq F_{1_X}(x) \leq F_{2_X}(x) \leq \bar{F}_{2_X}(x)$$

There fore $\bar{F}_{1_X}(x) \leq F_{2_X}(x)$.

(v) If for every $F_{1_X}(x)\epsilon\mathcal{F}_1$ there is some $F_{2_X}(x)\epsilon\,\mathcal{F}_2$ such that $F_{1_X}(x) \leq F_{2_X}(x)$

then, $\bar{F}_{1_X}(x) = \max_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \}$

$$\bar{F}_{1_X}(x) = \sup_{F_1\epsilon\mathcal{F}_1} F_{1_X}(x)$$

$$\bar{F}_{2_X}(x) = \sup_{F_2\epsilon\mathcal{F}_2} F_{2_X}(x)$$

$$\bar{F}_{2_X}(x) = \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \}$$

Hence $\bar{F}_{1X}(x) \leq \bar{F}_{2X}(x)$.

Proposition: 3.5

If$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sets of fuzzy cumulative distribution functions then

(i) $\underline{F_1}_X(x)\epsilon\mathcal{F}_1 \Rightarrow \{\mathcal{F}_1 \geq_{SD_2} \mathcal{F}_2 \Leftrightarrow \mathcal{F}_2 \geq_{SD_3} \mathcal{F}_2\}$

(ii) $\bar{F}_{2_X}(x)\epsilon\mathcal{F}_2 \Rightarrow \{\mathcal{F}_1 \geq_{SD_5} \mathcal{F}_2 \Leftrightarrow \mathcal{F}_2 \geq_{SD_6} \mathcal{F}_2\}$

Proof: (i)

To see the first statement, use that $\mathcal{F}_1 \geq_{SD_1} \mathcal{F}_2 \geq_{SD_2} \mathcal{F}_3 \geq_{SD_3} \mathcal{F}_4 \geq_{SD_4} \mathcal{F}_5 \geq_{SD_5} \mathcal{F}_6$

$\Rightarrow \mathcal{F}_1 \geq_{FSD_3} \mathcal{F}_2$.

More over $\mathcal{F}_1 \geq_{FSD_3} \mathcal{F}_2$ if and only if $F_{2_X}(x)\epsilon\,\mathcal{F}_2$ there is $F_{1_X}(x)\epsilon\mathcal{F}_1$

such that $F_{1_X}(x) \leq F_{1_X}(x)$

In particular, since $\min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \} = \underline{F_1}_X(x)$

So, $\underline{F_1}_X(x) \leq F_{1_X}(x)$ for every $F_{1_X}(x)\epsilon\mathcal{F}_1$.

It holds that, $\min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \} \leq F_{2_X}(x)$ for every $F_{2_X}(x)\epsilon\mathcal{F}_2$ and

Consequently,

as $\min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \}\epsilon\mathcal{F}_1$, that$\mathcal{F}_1 \geq_{SD_3} \mathcal{F}_2$.

(iii) To see the second statement, we use that

$$\mathcal{F}_1 \geq_{SD_1} \mathcal{F}_2 \geq_{SD_2} \mathcal{F}_3 \geq_{SD_3} \mathcal{F}_4 \geq_{SD_4} \mathcal{F}_5 \geq_{SD_5} \mathcal{F}_6$$

$$\Rightarrow \mathcal{F}_1 \geq_{FSD_5} \mathcal{F}_2 \Rightarrow \mathcal{F}_1 \geq_{FSD_6} \mathcal{F}_2.$$

Moreover, $\mathcal{F}_1 \geq_{FSD_6} \mathcal{F}_2$ if and only of for every $F_{2_X}(x)\epsilon\mathcal{F}_2$ there if $F_{1_X}(x)\epsilon\mathcal{F}_1$

such that, $F_{1_X}(x) \leq F_{2_X}(x)$.

In particular, since $\bar{F}_{2_X}(x) = \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \} \leq F_{1_X}(x)$

$\Rightarrow \bar{F}_{2_X}(x) \leq F_{1_X}(x)$ for every $F_{1_X}(x)\epsilon\mathcal{F}_1$, it holds that $\bar{F}_{2_X}(x) \leq F_{2_X}(x)$

For every $F_{2_X}(x)\epsilon\mathcal{F}_2$ and consequently, as $F_{2_X}(x)\epsilon\mathcal{F}_2$, that $\mathcal{F}_1 \geq_{SD_5} \mathcal{F}_2$.

Corollary:3.6

Let $\mathcal{F}_1, \mathcal{F}_2$ be two sets of fuzzy cumulative distribution functions andlet $\left(\underline{F_1}_X(x), \bar{F}_{1_X}(x)\right)$ and $\left(\underline{F_2}_X(x), \bar{F}_{2_X}(x)\right)$ denotetheirassociatedp-boxes. If $\underline{F_1}_X(x), \bar{F}_{1_X}(x)\,\epsilon\mathcal{F}_1$ and $\bar{F}_{2_X}(x), \underline{F_2}_X(x)\,\epsilon\mathcal{F}_2$, then

(i) $\mathcal{F}_1 \geq_{SD_1} \mathcal{F}_2 \Leftrightarrow \bar{F}_{1_X}(x) \leq \underline{F_2}_X(x)$.

(ii) $\mathcal{F}_1 \geq_{SD_2} \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1 \geq_{SD_3} \mathcal{F}_2 \Leftrightarrow \underline{F_1}_X(x) \leq F_{2_X}(x)$.

(iii) $\mathcal{F}_1 \geq_{SD_4} \mathcal{F}_2 \Leftrightarrow \underline{F_1}_X(x) \leq \bar{F}_{2_X}(x)$.

(iv) $\mathcal{F}_1 \geq_{SD_5} \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1 \geq_{SD_6} \mathcal{F}_2 \Leftrightarrow \bar{F}_{1_X}(x) \leq \bar{F}_{2_X}(x)$.

Proof:

(i) If $\underline{F_2}_X \geq \underline{F_1}_X(x)\epsilon\,\mathcal{F}_1$

$$\Rightarrow \min_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \} \geq \min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \}\epsilon\mathcal{F}_1$$

If there is some $F_1(x)\epsilon\mathcal{F}_1$ such that, $F_1(x) \leq \mathcal{F}_{2_X}(x)$ for all $F_2(x)\epsilon\mathcal{F}_2$ and as a consequence $\mathcal{F}_1 \geq_{FSD_5} \mathcal{F}_2$.

(ii) If $\underline{F_1}_X(x)\epsilon\,\mathcal{F}_1$,$\bar{F}_{2_X}(x)\epsilon\,\mathcal{F}_2$ and

$$\Rightarrow \min_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \} \geq \min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \}$$

$\Rightarrow \underline{F_1}_X(x) \leq \bar{F}_{2_X}(x)$, then there exist, $F_1(x)\epsilon\mathcal{F}_1$ and $F_2(x)\epsilon\mathcal{F}_2$ such that $F_1(x) \leq \mathcal{F}_{2_X}(x)$

Hence $\mathcal{F}_1 \geq_{FSD4} \mathcal{F}_2$.

(iii) If

$$\max_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1X}(x)_\alpha^U \} \leq \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2X}(x)_\alpha^U \}$$

Then since $\bar{F}_{2_X}(x)\epsilon\mathcal{F}_2$ then there is some $F_2(x)\epsilon\mathcal{F}_2$ such that $F_1(x) \leq \mathcal{F}_{2_X}(x)$ for every $F_1(x)\epsilon\mathcal{F}_1$, because $F_1(x) \leq \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \}$ for any $F_1(x)\epsilon\mathcal{F}_1$

Finally we can get,

$$\min_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1X}(x)_\alpha^U \} \leq \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2X}(x)_\alpha^U \}$$

$\Rightarrow \underline{F_1}_X(x) \leq \bar{F}_{2_X}(x)$.

Let $\mathcal{F}_1 \geq_{FSD_5} \mathcal{F}_2 \Rightarrow \mathcal{F}_1 \geq_{FSD_6} \mathcal{F}_2$, if and only if $F_2\epsilon\mathcal{F}_2$ and $F_1\epsilon\mathcal{F}_1$.

If every $F_1(x)\epsilon\mathcal{F}_1$ there is some $F_2(x)\epsilon\mathcal{F}_2$ such that

$$F_1(x) \leq F_{2_X}\text{t}\,(x)$$

$\bar{F}_{1_X}(x) = \max_{F_1\epsilon\mathcal{F}_1}\{ \inf_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{1_X}(x)_\alpha^U \}$ and

$$\bar{F}_{2_X}(x) = \max_{F_2\epsilon\mathcal{F}_2}\{ \inf_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^L, \sup_{F\in\mathcal{F}} F_{2_X}(x)_\alpha^U \}$$

Hence by the Dominating, $F_1(x) \leq F_{2_X}(x)$.

Lemma: 3.7

Let $\mathcal{F}_1{}'$ and $\mathcal{F}_2{}'$ be two sets of finitely additive fuzzy distribution functions with associated P-boxes $\left(\underline{F}_{1_X}(x), \bar{F}_{1_X}(x)\right), \left(\underline{F}_{2_X}(x), \bar{F}_{2_X}(x)\right)$. Assume that If $\bar{F}_{1_X}(x), \underline{F}_{1_X}(x) \epsilon \mathcal{F}_1$ and $\underline{F}_{2_X}(x), \bar{F}_{2_X}(x) \epsilon \mathcal{F}_2$, then

(i) $\mathcal{F}_1{}' \geq_{FSD_1} \mathcal{F}_2{}' \Leftrightarrow \bar{F}_{1_X}(x) \leq \underline{F}_{2_X}(x)$.

(ii) $\mathcal{F}_1{}' \geq_{FSD_2} \mathcal{F}_2{}' \Leftrightarrow \underline{F}_{1_X}(x) \leq \underline{F}_{2_X}(x)$.

(iii) $\mathcal{F}_1{}' \geq_{FSD_3} \mathcal{F}_2{}' \Leftrightarrow \underline{F}_{1_X}(x) \leq \underline{F}_{2_X}(x)$

(iv) $\mathcal{F}_1{}' \geq_{FSD_4} \mathcal{F}_2{}' \Leftrightarrow \underline{F}_{1_X}(x) \leq \underline{F}_{2_X}(x)$.

(v) $\mathcal{F}_1{}' \geq_{FSD_5} \mathcal{F}_2{}' \Leftrightarrow \bar{F}_{1_X}(x) \leq \bar{F}_{2_X}(x)$.

(vi) $\mathcal{F}_1{}' \geq_{FSD_6} \mathcal{F}_2{}' \Leftrightarrow \bar{F}_{1_X}(x) \leq \bar{F}_{2_X}(x)$.

Proof:

Let $\mathcal{F}_1{}'$ and $\mathcal{F}_2{}'$ be two sets of finitely additive fuzzy distribution functions with associated P-boxes $\left(\underline{F}_{1_X}(x), \bar{F}_{1_X}(x)\right)$ and $\left(\underline{F}_{2_X}(x), \bar{F}_{2_X}(x)\right)$.

Let the lower and upper distribution functions of the associated P-box belong to our set of cumulative distribution functions.

Note that $\mathcal{F}_1{}' \geq_{FSD_1} \mathcal{F}_2{}'$ if and only if $F_{1_X}(x) \leq F_{2_X}(x)$ for every $F_{1_X}(x) \ \epsilon \ \mathcal{F}_1{}'$,$F_{2_X}(x) \epsilon \mathcal{F}_2{}'$

This equivalent to,

$$\bar{F}_{1_X}(x) = \max_{F_1 \epsilon \mathcal{F}_1} \{ \inf_{F_1 \epsilon \mathcal{F}_1} F_{1_X}(x)_\alpha^L, \sup_{F_1 \epsilon \mathcal{F}_1} F_{1_X}(x)_\alpha^L \} \epsilon \mathcal{F}_1{}' \text{ and}$$

$$\bar{F}_{2_X}(x) = \max_{F_2 \epsilon \mathcal{F}_2} \{ \inf_{F_2 \epsilon \mathcal{F}_2} F_{2_X}(x)_\alpha^L, \sup_{F_2 \epsilon \mathcal{F}_2} F_{2_X}(x)_\alpha^L \} \epsilon \mathcal{F}_2{}'$$

There fore $\bar{F}_{1_X}(x) \epsilon \mathcal{F}_1{}' \leq \ \underline{F}_{2_X}(x) \epsilon \ \mathcal{F}_2{}'$

Hence $\mathcal{F}_1{}' \geq_{FSD_1} \mathcal{F}_2{}' \Leftrightarrow \bar{F}_{1_X}(x) \leq \underline{F}_{2_X}(x)$.

Similarly the other stochastic dominance hold under the same condition regardless of whether we work with finitely or σ- additive probability measure.

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