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Partial Averaging of Fuzzy Hyperbolic Differential Inclusions

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Abstract: In this article, we considered the fuzzy hyperbolic differential inclusions (fuzzy Darboux problem), introduced the concept of R-solution and proved the existence of such a solution. Also the substantiation of a possibility of application of partial averaging method for hyperbolic differential inclusions with the fuzzy right-hand side with the small parameters is considered.

Keywords: Hyperbolic Differential Inclusion, Fuzzy, Averaging, R-solution

1. Introduction

In 1990 J.P. Aubin [6] and V.A. Baidosov [7, 8] introduced differential inclusions with the fuzzy right-hand side. Their approach is based on usual differential inclusions. E. Hüllermeier [20, 21] introduced the concept of R-solution similarly how it has been done in [34]. Later, the various properties of fuzzy solutions of differential inclusions, and their use in modeling various natural science processes were considered (see [1, 4, 5, 17, 18, 26, 27] and the references therein).

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. After the systematic researches done by N. M. Krylov, N. N. Bogoliubov, Yu. A. Mitropolsky etc, in 1930s, the averaging method gradually became one of the classical methods in analyzing nonlinear oscillations (see [10, 25, 40, 42] and references therein). In works [36-39], the possibility of application of schemes of full and partial averaging for fuzzy differential inclusions with a small parameter was proved.

In papers [2, 3, 9, 11, 13, 18, 22, 30, 32, 33, 35], authors investigate classical models of partial differential equations with uncertain parameters, considering the parameters as fuzzy numbers. It was an obvious step in the mathematical modeling of physical processes. Study of fuzzy partial differential equations means the generalization of partial differential equations in fuzzy sense. While doing modelling of real situation in terms of partial differential equation, we see that the variables and parameters involve in the equations are uncertain (in the sense that they are not completely known or inexact or imprecise). Many times common initial or boundary condition of ambient temperature is a fuzzy condition since ambient temperature is prone to variation in a range. We express this impreciseness and uncertainties in terms of fuzzy numbers. So we come across with fuzzy partial differential equations. Also obviously, these equations can be written in as fuzzy partial differential inclusions.

In this work we consider fuzzy hyperbolic differential inclusions (fuzzy Darboux problem) and introduce the concept of R-solution similarly how it has been done in [36, 40, 50, 52, 53]. Also we ground the possibility of application of partial averaging method for fuzzy Darboux problem. This result generalize the results of A. N. Vityuk [40, 52] for the ordinary hyperbolic differential inclusions and M. Kiselevich [23], D. G. Korenevskii [24] for the ordinary hyperbolic differential equations.

2. Preliminaries

Let $comp(R^n)(conv(R^n))$ be a family of all nonempty (convex) compact subsets from the space R^n with the Hausdorff metric

$$h(A,B)=\min_{r\geq 0}\{B\subset S_r(A), A\subset S_r(B)\},$$

where $A,B\in comp(R^n)$, $S_r(A)$ is r-neighborhood of set A.

Let E^n be a family of all $u:R^n\to[0,1]$ such that u satisfies the following conditions:

1) u is normal, i.e. there exists an $x_0\in R^n$ such that $u(x_0)=1$;
2) u is fuzzy convex, i.e. $u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\}$ for any $x,y\in R^n$ and $0\leq\lambda\leq 1$;
3) u is upper semicontinuous, i.e. for any $x_0\in R^n$ and $\varepsilon>0$ where exists $\delta(x_0,\varepsilon)>0$ such that $u(x)<u(x_0)+\varepsilon$ whenever $\|x-x_0\|<\delta(x_0,\varepsilon), x\in R^n$;
4) the closure of the set $\{x\in R^n:u(x)>0\}$ is compact.

If $u\in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space.

Definition 1. The set $\{x\in R^n:u(x)\geq\alpha\}$ is called the α*-level* $[u]^\alpha$ of a fuzzy number $u\in E^n$ for $0<\alpha\leq 1$. The closure of the set $\{x\in R^n:u(x)>0\}$ is called the 0-level $[u]^0$ of a fuzzy number $u\in E^n$.

It is clearly that the set $[u]^\alpha\in conv(R^n)$ for all $0\leq\alpha\leq 1$.

Theorem 1. (Stacking Theorem [31]) If $u\in E^n$ then

1) $[u]^\alpha\in conv(R^n)$ for all $\alpha\in[0,1]$;
2) $[u]^{\alpha_2}\subset[u]^{\alpha_1}$ for all $0\leq\alpha_1\leq\alpha_2\leq 1$;
3) if $\{\alpha_k\}$ is a nondecreasing sequence converging to $\alpha>0$, then $[u]^\alpha=\bigcap_{k\geq 1}[u]^{\alpha_k}$.

Conversely, if $\{A_\alpha:\alpha\in[0,1]\}$ is the family of subsets of R^n satisfying conditions 1) - 3) then there exists $u\in E^n$ such that $[u]^\alpha=A_\alpha$ for $0<\alpha\leq 1$ and $[u]^0=\overline{\bigcup_{0<\alpha\leq 1}A_\alpha}\subset A_0$.

Let θ be the fuzzy number defined by $\theta(x)=0$ if $x\neq 0$ and $\theta(0)=1$.

Define $D:E^n\times E^n\to[0,\infty)$ by the relation

$$D(u,v)=\sup_{0\leq\alpha\leq 1}h\left([u]^\alpha,[v]^\alpha\right).$$

Then D is a metric in E^n. Further we know that [41]:

i). (E^n,D) is a complete metric space,

ii). $D(u+w,v+w)=D(u,v)$ for all $u,v,w\in E^n$,

iii). $D(\lambda u,\lambda v)=|\lambda|D(u,v)$ for all $u,v\in E^n$ and $\lambda\in R$.

3. Fuzzy Hyperbolic Differential Inclusion. R-solution

Consider the fuzzy hyperbolic differential inclusion (or in other words, fuzzy Darboux problem)

$$\begin{gathered}u_{xy}(x,y)\in F(x,y,u(x,y)),\\ u(x,0)=\varphi(x), x\in[0,a],\\ u(0,y)=\psi(y), y\in[0,b], \varphi(0)=\psi(0),\end{gathered}\tag{1}$$

where $u\in R^n, u_{xy}(x,y)=\dfrac{\partial^2 u(x,y)}{\partial x\partial y}, F:[0,a]\times[0,b]\times R^n\to E^n, \varphi:[0,a]\to R^n, \psi:[0,b]\to R^n$.

We interpret fuzzy Darboux problem (1) as a family of set-valued Darboux problems

$$\begin{aligned}u_{xy}^\alpha(x,y)&\in[F(x,y,u^\alpha(x,y))]^\alpha,\\ u^\alpha(x,0)&=\varphi(x), x\in[0,a],\\ u^\alpha(0,y)&=\psi(y), y\in[0,b],\\ \varphi(0)&=\psi(0), \alpha\in[0,1].\end{aligned}\tag{2}$$

Qualitative properties and structure of the set of solutions of the set-valued Darboux problem have been studied by many authors, for instance [12, 14-16, 28, 29, 40, 44-53] and references therein.

Definition 2 [28, 43]. A function $u:[0,a]\times[0,b]\to R^n$ is said to be absolutely continuous on $[0,a]\times[0,b]$ ($u(\cdot,\cdot)\in AC([0,a]\times[0,b])$) if there exist absolutely continuous functions $\varphi:[0,a]\to R^n$ and $\psi:[0,b]\to R^n$, and Lebesgue integrable function $g:[0,a]\times[0,b]\to R^n$ such that

$$u(x,y)=\varphi(x)+\psi(y)-c+\int_0^x\int_0^y g(\xi,\varsigma)d\varsigma d\xi, \varphi(0)=\psi(0)=c.$$

Definition 3. An $\alpha-$solution $u^\alpha(\cdot,\cdot)$ of (1) is understood to be an absolutely continuous function $u^\alpha:[0,a]\times[0,b]\to R^n$ that satisfies (2) for almost every $(x,y)\in[0,a]\times[0,b]$ and the boundary conditions for any $x\in[0,a]$ and $y\in[0,b]$.

Let U^α denote the $\alpha-$ solution set of (2) and $U^\alpha(x,y)=\{u^\alpha(x,y):u^\alpha(\cdot,\cdot)\in U^\alpha\}$. Clearly a family of subsets $U(x,y)=\{U^\alpha(x,y):\alpha\in[0,1]\}$ may not satisfy to conditions of Theorem 1, i.e. $U(x,y)\not\in E^n$. For example,

$U^\alpha(x,y)\in comp(R^n)$ and $U^\alpha(x,y)\not\in conv(R^n)$ for any $\alpha\in[0,1]$. Therefore, we introduce the definition of R-solutions for fuzzy Darboux problem (1).

Definition 3. The upper semicontinuous fuzzy mapping $R:[0,a]\times[0,b]\to E^n$ that satisfies to the following system

$$\sup_{\alpha\in[0,1]} h\Big([R(t+\sigma,s+\eta)]^\alpha, \bigcup_{u\in[R(t,s)]^\alpha}\Big\{v(t+\sigma)+\tau(s+\eta)-u+\int_t^{t+\sigma}\int_s^{s+\eta}[F(\xi,\nu,u)]^\alpha d\nu\, d\xi\Big\}\Big)=o(\sigma\eta), \tag{3}$$

is called the R-solution of fuzzy Darboux problem (1), where $v(x)\in AC(t,t+\sigma)$, $\tau(y)\in AC(s,s+\eta)$, $v(x)\in[R(x,s)]^\alpha$, $x\in[t,t+\sigma]$, $\tau(y)\in[R(t,y)]^\alpha$, $y\in[s,s+\eta]$, $v(t)=\tau(s)=u$, $\lim\limits_{\substack{\sigma\to0_+\\ \eta\to0_+}}\dfrac{o(\sigma\eta)}{\sigma\eta}=0.$

Now we are interested in the following question: Under what conditions, there exists a unique R-solution to (1). In the next theorem we find the existence result for a unique R-solution of fuzzy Darboux problem (1).

Theorem 2. Suppose the following conditions hold:

1) fuzzy mapping $F(\cdot,\cdot,u)$ is measurable, for all $u\in R^n$;

2) there exists $\lambda>0$ such that for all $u',u''\in R^n$

$$D(F(x,y,u'),F(x,y,u''))\le\lambda\|u'-u''\|$$

for every $(x,y)\in[0,a]\times[0,b]$;

3) there exists $\gamma>0$ such that $D(F(x,y,u),\theta)\le\gamma$ for every $(x,y,u)\in[0,a]\times[0,b]\times R^n$;

4) for all $\beta\in[0,1], u',u''\in R^n$ and every $(x,y)\in[0,a]\times[0,b]$

$$\beta F(x,y,u')+(1-\beta)F(x,y,u'')\subset F(x,y,\beta u'+(1-\beta)u'');$$

5) functions $\varphi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous functions on $[0,a]$ and $[0,b]$.

Then there exists a unique R-solution $R(\cdot,\cdot)$ of fuzzy Darboux problem (1) defined on the set $[0,a]\times[0,b]$.

Proof. By [29,49], every set-valued Darboux problem of family (2) has solution $u^\alpha(\cdot,\cdot)$ on the set $[0,a]\times[0,b]$, i.e. $U^\alpha(x,y)\ne\varnothing$ for every $(x,y)\in[0,a]\times[0,b]$ and $\alpha\in[0,1]$.

Also by [40] and [50], $U^\alpha(x,y)\in comp(R^n)$ for every $(x,y)\in[0,a]\times[0,b]$ and $\alpha\in[0,1]$.

By [40] and [51], if $u^\alpha(\cdot,\cdot)\in U^\alpha$ then

$$u^\alpha(x_2,y_2)-u^\alpha(x_2,y_1)-u^\alpha(x_1,y_2)+u^\alpha(x_1,y_1)\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,u^\alpha(x,y))]^\alpha dydx$$

for every $[x_1,x_2]\times[y_1,y_2]\subset[0,a]\times[0,b]$.

Consider any solutions $u_1^\alpha(\cdot,\cdot),u_2^\alpha(\cdot,\cdot)\in U^\alpha$ and any $\beta\in[0,1]$. Let $u_\beta^\alpha(\cdot,\cdot)$ be such that

$$u_\beta^\alpha(x,y)=\beta u_1^\alpha(x,y)+(1-\beta)u_2^\alpha(x,y)$$

for every $(x,y)\in[0,a]\times[0,b]$.

Then

$$u_\beta^\alpha(x_2,y_2)-u_\beta^\alpha(x_2,y_1)-u_\beta^\alpha(x_1,y_2)+u_\beta^\alpha(x_1,y_1)=$$

$$=\beta u_1^\alpha(x_2,y_2)-\beta u_1^\alpha(x_2,y_1)-\beta u_1^\alpha(x_1,y_2)+\beta u_1^\alpha(x_1,y_1)+$$

$$+(1-\beta)u_2^\alpha(x_2,y_2)-(1-\beta)u_2^\alpha(x_2,y_1)-(1-\beta)u_2^\alpha(x_1,y_2)+(1-\beta)u_2^\alpha(x_1,y_1)\in$$

$$\in\beta\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,u_1^\alpha(x,y))]^\alpha dydx+(1-\beta)\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,u_2^\alpha(x,y))]^\alpha dydx\subset$$

$$\subset\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,\beta u_1^\alpha(x,y)+(1-\beta)u_2^\alpha(x,y))]^\alpha dydx=\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,u_\beta^\alpha(x,y))]^\alpha dydx$$

i.e. $u_\beta^\alpha(x_2,y_2)-u_\beta^\alpha(x_2,y_1)-u_\beta^\alpha(x_1,y_2)+u_\beta^\alpha(x_1,y_1)\in \int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,u_\beta^\alpha(x,y))]^\alpha dydx$ for every $(x,y)\in[0,a]\times[0,b]$ and $\beta\in[0,1]$.

By [40] and [51], function $u_\beta^\alpha(\cdot,\cdot)$ is solution of set-valued Darboux problem (2), i.e. $u_\beta^\alpha(x,y)\in U^\alpha(x,y)$ for every $(x,y)\in[0,a]\times[0,b]$. Consequently $U^\alpha(x,y)\in conv(R^n)$ for every $(x,y)\in[0,a]\times[0,b]$ and $\alpha\in[0,1]$.

Since, $[F(x,y,u)]^{\alpha_2}\subset[F(x,y,u)]^{\alpha_1}$ for all $0\le\alpha_1<\alpha_2\le 1$ and $(x,y,u)\in[0,a]\times[0,b]\times R^n$, then $U^{\alpha_2}(x,y)\subset U^{\alpha_1}(x,y)$ for all $0\le\alpha_1<\alpha_2\le1$ and $(x,y)\in[0,a]\times[0,b]$.

By [50, 52], every Darboux problem of family (2) has one R-solution $R^\alpha(\cdot,\cdot)$ on the set $[0,a]\times[0,b]$ and we have $R^\alpha(x,y)=U^\alpha(x,y)$ for every $\alpha\in[0,1]$ and $(x,y)\in[0,a]\times[0,b]$.

By [20, 53], we get that a family of subsets $R(x,y)=\{R^\alpha(x,y):\alpha\in[0,1]\}$ satisfies to conditions of Theorem 1, i.e. $R(x,y)\in E^n$ for every $(x,y)\in[0,a]\times[0,b]$. This concludes the proof.

4. The Method of Partial Averaging

Now consider fuzzy Darboux problem with the small parameters

$$\begin{gathered} u_{xy}(x,y)\in\varepsilon_1\varepsilon_2F(x,y,u(x,y)),\\ u(x,0)=\varphi(x),x\in R_+,\\ u(0,y)=\psi(y),y\in R_+,\varphi(0)=\psi(0), \end{gathered} \tag{4}$$

where $\varepsilon_1>0,\varepsilon_2>0$ - small parameters, $R_+=[0,+\infty)$.

In this work, we associate with the problem (4) the following full averaged fuzzy Darboux problem

$$\begin{gathered} z_{xy}(x,y)\in\varepsilon_1\varepsilon_2G(x,y,z(x,y)),\\ z(x,0)=\varphi(x),x\in R_+,\\ z(0,y)=\psi(y),y\in R_+,\varphi(0)=\psi(0), \end{gathered} \tag{5}$$

where $G:R^n\to E^n$ such that

$$\lim_{\substack{T_1\to\infty\\T_2\to\infty}} D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}F(x,y,z)dydx,\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}G(x,y,z)dydx\right)=0. \tag{6}$$

The main theorem of this section is on averaging for fuzzy Darboux problem with the small parameters. It establishes nearness of R-solutions of (4) and (5), and reads as follows.

Theorem 3. Let in the domain $Q=\{(x,y,u):x\in R_+,y\in R_+,u\in B\subset R^n\}$ the following conditions hold:

1) fuzzy mappings $F(\cdot,\cdot,u)$ and $G(\cdot,\cdot,u)$ is continuous on $R_+\times R_+$;
2) fuzzy mappings $F(x,y,\cdot)$ and $G(x,y,\cdot)$ satisfy a Lipschitz condition

$$D(F(x,y,u'),F(x,y,u''))\le\lambda\|u'-u''\|,$$
$$D(G(x,y,u'),G(x,y,u''))\le\lambda\|u'-u''\|$$

with a Lipschitz constant $\lambda>0$;
3) there exists $\gamma>0$ such that

$$D(F(x,y,u),\theta)\le\gamma,\quad D(G(x,y,u),\theta)\le\gamma$$

for every $(x,y)\in R_+\times R_+$ and every $u\in R^n$;
4) for all $\beta\in[0,1],u',u''\in R^n$ and every $(x,y)\in R_+\times R_+$

$$\beta F(x,y,u')+(1-\beta)F(x,y,u'')\subset F(x,y,\beta u'+(1-\beta)u''),$$

$$\beta G(x,y,u')+(1-\beta)G(x,y,u'')\subset G(x,y,\beta u'+(1-\beta)u'');$$

5) limit (6) exists uniformly with respect to u in the domain B;
6) functions $\varphi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous functions on R_+ and $\varphi(x)\in B',\psi(y)\in B'$ for all $x,y\in R_+$, where $B'+S_\rho(0)\subset B$;
7) the R-solution of the Darboux problem

$$\begin{gathered} u_{xy}^1(x,y)\in\varepsilon_1\varepsilon_2[F(x,y,u^1(x,y))]^1,\\ u^1(x,0)=\varphi(x),x\in[0,\infty),\\ u^1(0,y)=\psi(y),y\in[0,\infty),\varphi(0)=\psi(0), \end{gathered}$$

together with a $\rho-$neighborhood belong to the domain B for $\varepsilon_1,\varepsilon_2\in(0,\bar\varepsilon]$.

Then for any $\eta\in(0,\rho]$ and L > 0 there exists $\varepsilon_0(\eta,L)\in(0,\bar\varepsilon]$ such that for all $\varepsilon_1,\varepsilon_2\in(0,\varepsilon_0]$ and $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ the following inequality holds

$$D(R(x,y),\bar R(x,y))<\eta \tag{7}$$

where $R(\cdot,\cdot),\bar R(\cdot,\cdot)$ are the R-solutions of initial and partial averaged Darboux problems.

Proof. By theorem 2, we have unit R-solution of Darboux problem (4) on $[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ and unit R-solution of

Darboux problem (5) on $[0, L\varepsilon_1^{-1}]\times[0, L\varepsilon_2^{-1}]$.

Let $\delta_1 = L\varepsilon_1^{-1}$, $\delta_2 = L\varepsilon_2^{-1}$, $K_n = \left\{ (x_i, y_j) : x_i = ih, y_j = jl,\ i, j = 0,1,...,n,\ h = \frac{\delta_1}{n},\ l = \frac{\delta_2}{l} \right\}$,

$K_n^{ij} = [x_i, x_{i+1}]\times[y_j, y_{j+1}]$ and $K = [0, L\varepsilon_1^{-1}]\times[0, L\varepsilon_2^{-1}]$. We denote fuzzy mappings $P^m(\cdot,\cdot)$ and $Q^m(\cdot,\cdot)$ such that

$$[P^n(x,y)]^\alpha = \bigcup_{u\in[P^n(x_i,y_j)]^\alpha} \left\{ v_{ij}(x) + \tau_{ij}(y) - u + \varepsilon_1\varepsilon_2 \int_{x_i}^{x}\int_{y_j}^{y} [F(\varsigma,\zeta,u)]^\alpha d\varsigma\, d\zeta \right\},$$

$$[Q^n(x,y)]^\alpha = \bigcup_{z\in[Q^n(x_i,y_j)]^\alpha} \left\{ \overline{v}_{ij}(x) + \overline{\tau}_{ij}(y) - z + \varepsilon_1\varepsilon_2 \int_{x_i}^{x}\int_{y_j}^{y} [G(\varsigma,\zeta,z)]^\alpha d\varsigma\, d\zeta \right\},$$

$$P^n(x,0) = Q^n(x,0) = \varphi(x), P^n(0,y) = Q^n(0,y) = \psi(y),$$

where $v_{ij}(x)\in AC(x_i, x_{i+1})$, $\tau_{ij}(y)\in AC(y_j, y_{j+1})$, $v_{ij}(x)\in[P^n(x,y_j)]^\alpha$, $x\in[x_i, x_{i+1}]$, $\tau_{ij}(y)\in[P^n(x_i,y)]^\alpha$, $y\in[y_j, y_{j+1}]$, $v_{ij}(x_i) = \tau_{ij}(y_j) = u$, $\overline{v}_{ij}(x)\in AC(x_i, x_{i+1})$, $\overline{\tau}_{ij}(y)\in AC(y_j, y_{j+1})$, $\overline{v}_{ij}(x)\in[Q^n(x,y_j)]^\alpha$, $x\in[x_i, x_{i+1}]$, $\overline{\tau}_{ij}(y)\in[Q^n(x_i,y)]^\alpha$, $y\in[y_j, y_{j+1}]$, $\overline{v}_{ij}(x_i) = \overline{\tau}_{ij}(y_j) = z$.

By [52], it follows that the sequences $\{[P^n(\cdot,\cdot)]^\alpha\}_{n=1}^\infty$, and $\{[Q^n(\cdot,\cdot)]^\alpha\}_{n=1}^\infty$ are equicontinuous and fundamental and their limits are $\alpha-$ levels of R-solutions $[R(\cdot,\cdot)]^\alpha$ and $[\overline{R}(\cdot,\cdot)]^\alpha$ of the problems (4) and (5).

Consequently, the sequences $\{P^n(\cdot,\cdot)\}_{n=1}^\infty$ and $\{Q^n(\cdot,\cdot)\}_{n=1}^\infty$ meet by $R(\cdot,\cdot)$ and $\overline{R}(\cdot,\cdot)$.

By [52], for any $\eta_1 > 0$ there exists $0 < \varepsilon_0 \le \overline{\varepsilon}$ such that

$$h([P^n(x,y)]^\alpha, [Q^n(x,y)]^\alpha) \le \eta_1\lambda^{-1}\exp(\lambda L^2), \tag{8}$$

$$h([R(x,y)]^\alpha, [P^n(x,y)]^\alpha) \le \frac{3\gamma L^2}{n}(1+\exp(\lambda L^2)), \tag{9}$$

$$h([Q^n(x,y)]^\alpha, [\overline{R}(x,y)]^\alpha) \le \frac{3\gamma L^2}{n}(1+\exp(\lambda L^2)) \tag{10}$$

for any $\alpha\in[0,1]$, $(x,y)\in K$ and $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0]$.

Combining (8), (9) and (10), choosing $n \ge \frac{9\gamma L^2(1+\exp(\lambda L^2))}{\eta}$ and $\eta_1 < \frac{\eta\lambda}{3\exp(\lambda L^2)}$ we obtain

$$D(R(x,y), \overline{R}(x,y)) \le$$

$$\le D(R(x,y), P^n(x,y)) + D(P^n(x,y), Q^n(x,y)) + D(Q^n(x,y), \overline{R}(x,y)) =$$

$$= \sup_{\alpha\in[0,1]} h([R(x,y)]^\alpha, [P^n(x,y)]^\alpha) + \sup_{\alpha\in[0,1]} h([P^n(x,y)]^\alpha, [Q^n(x,y)]^\alpha) +$$

$$+ \sup_{\alpha\in[0,1]} h([Q^n(x,y)]^\alpha, [\overline{R}(x,y)]^\alpha) < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$

The theorem is proved.

5. Conclusion.

We conclude with a few remarks.

Remark 1. In this work, we considered the fuzzy differential inclusion, when fuzzy mapping $F(\cdot,\cdot,u)$ is measurable on $[0,a]\times[0,b]$. If $F(\cdot,\cdot,u)$ is continuous on $[0,a]\times[0,b]$ then instead of the equation (1) it is possible to consider the following more simple equation

$$\sup_{\alpha\in[0,1]} h\Big([R(t+\sigma,s+\eta)]^{\alpha}, \bigcup_{u\in[R(t,s)]^{\alpha}} \left\{v(t+\sigma)+\tau(s+\eta)-u+\sigma\eta[F(t,s,u)]^{\alpha}\right\}\Big)=o(\sigma\eta), \tag{11}$$

and similarly we can prove all results received earlier.

Remark 2. If the condition 4) of Theorem 3 is not true, then the R-solutions can not exist. But there are valid the following conditions:

1) for any α -solution $u^{\alpha}(\cdot,\cdot)$ of inclusion (4) there exists a α -solution $z^{\alpha}(\cdot,\cdot)$ of inclusion (5) such that $\left\|u^{\alpha}(x,y)-z^{\alpha}(x,y)\right\|<\eta$ for all $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ and $\alpha\in[0,1]$;
2) for any α -solution $z^{\alpha}(\cdot,\cdot)$ of inclusion (5) there exists a α -solution $u^{\alpha}(\cdot,\cdot)$ of inclusion (4) such that $\left\|u^{\alpha}(x,y)-z^{\alpha}(x,y)\right\|<\eta$ for all $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ and $\alpha\in[0,1]$.

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