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Some Properties of Interval Quadratic Programming Problem

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Abstract: For interval liner programming problems, Rohn proposed four equivalence relations regarding to the upper and lower bounds of the interval optimal value. In this paper, similar problems of interval quadratic programming problem have been discussed. Some interesting properties have been proved and an illustrative example and remarks are given to get an insight of the properties.

Keywords: Interval Quadratic Programming, Lower and Upper Bounds, Optimal Value

1. Introduction

Interval systems and interval optimal models are often used for modeling information systems and engineering problems, e.g. [1]. Over the past decades, the interval systems and interval mathematical programming (IvMP) have been studied by many authors, see e.g. [2-6, 20-27, 32-35] and the references therein. Some papers studied the problem of computing the range of optimal values of interval linear programming problems, see e.g., [12-17] among others. Some authors studied the problem of computing the range of optimal values of interval quadratic programs (IvQP) [7, 8, 18-19].

The other frequent problems is study the properties on upper and lower bounds of IvMP. There have been developed diverse methods for computing the lower and upper bounds of IvQP. Liu [18] and Li [19] described some methods to compute the lower and upper bounds of IvQP with inequality and nonnegative constraints. Hladik [7] focused on convex quadratic programming problems with interval data, the problem of computing the best case and the worst case optimal values was discussed for interval convex quadratic programming problems of certain forms, then he studied the method of the upper and lower bounds of interval-valued convex quadratic programming problems in a general form. For computing the upper bound, these methods described in [8, 18, 19] are based on the dual problem of IvQP (dual method for short), under the condition that the zero duality gap of a pair of primal and dual IvQP is specified. Recently, Li et al. [11] proposed a new method to compute the upper bound of optimal values of IvQP, in this new method, only primal program is taken into consideration, the dual problem is not required and thus the condition that the duality gap is zero is also removed, and then Li described the properties on the upper and lower bounds of interval quadratic programming [9-11]. However few was done on the relations among the upper and lower bounds. The relations among the upper and lower bounds of IvLP (interval linear programming) have been established in [4, 8]. In this paper, IvQP and several equivalent conditions for interval quadratic programming problem have been studied. First, some properties of interval quadratic program are formulated. Based on these results, some interesting and useful relations of interval quadratic program will be given, which give an insight into the corresponding problems.

2. Preliminaries

From notations from [4], an interval matrix is defined as

$$\boldsymbol{A}=[\underline{A},\overline{A}]=\{A\in R^{m\times n}\mid \underline{A}\le A\le \overline{A}\}$$

where $\underline{A},\overline{A}\in R^{m\times n}$, $\underline{A}\le \overline{A}$, and "$\le$" is understood componentwise. The center and the radius of matrix $\boldsymbol{A}$ is denoted by

$$A_c=\frac{1}{2}(\underline{A}+\overline{A}),A_\Delta=\frac{1}{2}(\overline{A}-\underline{A})$$

So $\boldsymbol{A}=[A_c-A_\Delta,A_c+A_\Delta]$ [4]. An interval vector $\boldsymbol{b}=[\underline{b},\overline{b}]=\{b\in R^m\mid \underline{b}\le b\le \overline{b}\}$ is understood as one-column interval matrix.

Let $\{\pm 1\}^m$ be the set of all $\{-1,1\}$ m-dimensional vectors, i.e.

$$\{\pm 1\}^m=\{y\in R^m \mid |y|=e\}$$

where $e=(1,\ldots,1)^T$ is the m-dimensional vector of all 1's.

For a given $y\in\{\pm 1\}^m$, let

$$T_y=diag(y_1,y_2,\ldots,y_m)$$

denote the corresponding diagonal matrix. For each $x\in R^n$, its sign vector sgn x is defined by

$$(\operatorname{sgn} x)_i=\begin{cases}1 & if\ x_i\ge 0\\ -1 & if\ x_i<0\end{cases}$$

where $i=1,2,\cdots,n.$ then $|x|=T_z x$, where $z=\operatorname{sgn} x\in\{\pm 1\}^n$.

Given an interval matrix $\boldsymbol{A}=[A_c-A_\Delta,A_c+A_\Delta]$, for each $y\in\{\pm 1\}^m$ and $z\in\{\pm 1\}^n$, the matrices $A_{yz}=A_c-T_yA_\Delta T_z$ are defined.

Similarly for an interval vector $b=[b_c-b_\Delta,b_c+b_\Delta]$ and for each $y\in\{\pm 1\}^m$, the vectors $b_y=b_c+T_yb_\Delta$ are defined.

Let $A\in R^{m\times n},B\in R^{k\times n},b\in R^m,c\in R^n,d\in R^k$ and $Q\in R^{n\times n}$, consider the quadratic programming problem $\min\frac{1}{2}x^TQx+c^Tx$ subject to $Ax\le b,Bx=d,x\ge 0,$ where Q is positive semidefinite. Briefly, the problem is rewritten as

$$\min\{\frac{1}{2}x^TQx+c^Tx\mid Ax\le b,Bx=d,x\ge 0\}. \qquad (1)$$

The Dorn dual problem[10] of the quadratic (1) is

$$\max\{-\frac{1}{2}u^TQu-b^Tv-d^Tw\mid Qu+A^Tv+B^Tw+c\ge 0,v\ge 0\} \qquad (2)$$

Let

$$f(A,B,b,c,d,Q)=\inf\{\frac{1}{2}x^TQx+c^Tx\mid Ax\le b,Bx=d,x\ge 0\},$$

and

$$g(A,B,b,c,d,Q)=\sup\{-\frac{1}{2}u^TQu-b^Tv-d^Tw\mid Qu+A^Tv+B^Tw+c\ge 0,v\ge 0\}.$$

denote the optimal values of (1) and (2), respectively.

The set of all m-by-n interval matrices will be denoted by $IR^{m\times n}$ and the set of all m-dimensional interval vectors by IR^m. Given $\boldsymbol{A}\in IR^{m\times n},\boldsymbol{B}\in IR^{k\times n},\boldsymbol{b}\in IR^m,\boldsymbol{c}\in IR^n,\boldsymbol{d}\in IR^k$ and $\boldsymbol{Q}\in IR^{n\times n}$, the interval convex quadratic program

$$\min\{\frac{1}{2}x^T\boldsymbol{Q}x+\boldsymbol{c}^Tx\mid \boldsymbol{A}x\le \boldsymbol{b},\boldsymbol{B}x=\boldsymbol{d},x\ge 0\} \qquad (3)$$

is the family of convex quadratic programs (1) with data satisfying $A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}.$ where Q is positive semidefinite for all $Q\in\boldsymbol{Q}$.

The lower and upper bounds of the optimal values are respectively defined as

$$\underline{f}=\inf\{f(A,B,b,c,d,Q)\mid A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}\},$$

$$\overline{f}=\sup\{f(A,B,b,c,d,Q)\mid A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}\}.$$

The upper bound of the optimal values of (2) is defined as

$$\overline{\varphi}=\sup\{g(A,B,b,c,d,Q)\mid A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}\}.$$

The following lemmas will be used in the proof of the main results.

Lemma 1.1. [11]

$$\underline{f}=\min\{\frac{1}{2}x^T\underline{Q}x+\underline{c}^Tx\mid \underline{A}x\le \overline{b},\underline{B}x\le \overline{d},\overline{B}x\ge \underline{d},x\ge 0\}$$

and

$$\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})=\sup_{y\in\{\pm 1\}^k} f(\overline{A},B_{ye},\underline{b},\overline{c},d_y,\overline{Q}).$$

Lemma 1.2. [9] Let $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})=-\infty,$ then there exists a $B_0\in\boldsymbol{B},$ such that

$$f(\underline{A},B_0,\overline{b},d,\underline{Q})\in\{-\infty,\infty\}$$

holds for each $d\in\boldsymbol{d}$.

Lemma 1.3. [10] It is hold that

$$\overline{\varphi}=\begin{cases}-\infty, & if\ \overline{f}=-\infty; \quad (a)\\ \overline{f}, & if\ \overline{f}\ is\ finite; \quad (b)\\ cannot\ dermine, & if\ \overline{f}=\infty. \quad (c)\end{cases}$$

Lemma 1.4. [9] Let $\underline{f}$ be finite and let x^* be an optimal solution of the problem (6) . then

$$\underline{f}=f(\underline{A},B_c-T_yB_\Delta,\overline{b},\underline{c},d_c+T_yd_\Delta,\underline{Q}),$$

where, for arbitrary $\alpha\in[-1,1]$

$$y_i=\begin{cases}\dfrac{(B_cx^*-d_c)_i}{(B_\Delta x^*+d_\Delta)_i} & if\ (B_\Delta x^*+d_\Delta)_i>0\\ \alpha & if\ (B_\Delta x^*+d_\Delta)_i=0\end{cases}\qquad (i=1,2,\ldots,k)$$

3. Some Properties of IvQP

For an interval linear programming problem (IvLP) with data $\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}$, Rohn proved that the following assertions are equivalent [4].

a. For each $A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c}$ the problem

$$\min\{c^Tx\mid Ax=b,x\geq 0\}$$

has an optimal solution.

b. Both $\underline{f}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ and $\overline{f}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ are finite.

c. Both $\underline{f}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ and $\overline{\varphi}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ are finite.

d. The system

$$\overline{A}^Tp_1-\underline{A}^Tp_2\leq\underline{c}$$

is feasible and $\overline{\varphi}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ is finite.

Similarly, the relationship of the following assertions is studied.

(a) For each $A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}$ the problem

$$\min\{\frac{1}{2}x^TQx+c^Tx\mid Ax\leq b,Bx=d,x\geq 0\} \qquad (4)$$

has an optimal solution.

(b) Both $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ and $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ are finite.

(c) Both $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ and $\overline{\varphi}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ are finite.

(d) The system

$$\underline{Q}u+\underline{A}^Tv_1+\underline{B}^Tv_2-\overline{B}^Tv_3+\underline{c}\geq 0,v_i\geq 0,i=1,2,3 \qquad (5)$$

is solvable, and $\overline{\varphi}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite.

Theorem 2.1. For an interval quadratic programming problem with data $A\in\boldsymbol{A},B\in\boldsymbol{B},b\in\boldsymbol{b},c\in\boldsymbol{c},d\in\boldsymbol{d},Q\in\boldsymbol{Q}$ there holds that

$$(a)\Rightarrow(b)\Leftrightarrow(c)\Rightarrow(d)$$

Proof. $(a)\Rightarrow(b)$: Since each problem (4) has an optimal solution, it must be $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})<\infty$. From Lemma 1.2, if $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})=-\infty$, then there exists a $B_0\in\boldsymbol{B}$, such that $f(\underline{A},B_0,\overline{b},d,\underline{Q})\in\{-\infty,\infty\}$, hold for each $d\in\boldsymbol{d}$, so the possibility of $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})=-\infty$ is precluded by Lemma 1.2. Hence $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite. From Lemma 1.4 and Lemma 1.1, which can be got is that $\underline{f}=f(\underline{A},B_c-T_yB_\Delta,\overline{b},\underline{c},d_c+T_yd_\Delta,\underline{Q})$ and $\overline{f}=\sup_{y\in\{\pm1\}^k}f(\overline{A},B_{ye},\underline{b},\overline{c},d_y,\overline{Q})$, in other words, $\overline{f}$ gets the maximum in a finite group generated by y , hence $\overline{f}(A,B,b,c,d,Q)$ is finite.

$(b)\Rightarrow(c)$: From Lemma 1.3, if $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite, $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})=\overline{\varphi}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$. Hence $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite implies that $\overline{\varphi}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite.

According to the process of above proof, something can be easily got that if $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite, then $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ is finite. Thus, the result of $(c)\Rightarrow(b)$ is obviously true, since that if $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ and $\overline{\varphi}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ are finite, $\underline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ and $\overline{f}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q})$ are obviously finite.

Thus, (b) and (c) are necessary and sufficient conditions to each other.

$(c)\Rightarrow(d)$: Since

$$\underline{f}=\inf\{\frac{1}{2}x^T\underline{Q}x+\underline{c}^Tx\mid\underline{A}x\leq\overline{b},\underline{B}x\leq\overline{d},\overline{B}x\geq\underline{d},x\geq 0\} \qquad (6)$$

The *Dorn* dual problem of (6) is

$$\max\{-\frac{1}{2}u^T\underline{Q}u-\overline{b}^Tv_1-\overline{d}^Tv_2+\underline{d}^Tv_3\mid\underline{Q}u+\underline{A}^Tv_1+\underline{B}^Tv_2-\overline{B}^Tv_3+\underline{c}\geq 0,v_i\geq 0,i=1,2,3\} \qquad (7)$$

If $\underline{f}(A,B,b,c,d,Q)$ is finite, then by the strong duality theory [9], the following formula is established

$$\underline{f} = \sup\{-\frac{1}{2}u^T \underline{Q}u - \overline{b}^T v_1 - \overline{d}^T v_2 + \underline{d}^T v_3 \mid \underline{Q}u + \underline{A}^T v_1 + \underline{B}^T v_2 - \overline{B}^T v_3 + \underline{c} \geq 0, v_i \geq 0, i = 1,2,3\} \tag{8}$$

so that (7) is finite, thus the system (5) is solvable.

4. An Illustrative Example

In this section, an illustrative example is given for Theorem 2.1, which helps us to understand Theorem 2.1.

Example 1 Consider the interval quadratic program

$$\min[\frac{1}{2},1]x_1 + x_2$$

$$\begin{cases} [1,2]x_1 - x_2 = [0,2] \\ [1,2]x_1 - x_2 = [-2,2] \\ x_1, x_2 \geq 0 \end{cases} \tag{9}$$

The corresponding interval matrices and vectors of (9) are

$$c = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} [1,2] & -1 \\ [1,2] & -1 \end{pmatrix} \quad d = \begin{pmatrix} [0,2] \\ [-2,2] \end{pmatrix}$$

The lower bound of the optimal values can be determined by convex quadratic program

$$\min \frac{1}{2}x_1 + x_2$$

$$\begin{cases} 2x_1 - x_2 \geq 0 \\ x_1 - x_2 \leq 2 \\ 2x_1 - x_2 \geq -2 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases} \tag{10}$$

It can be shown that $\underline{f} = 0$

Then the upper bound is computed by Lemma 1.1. This problem can be decomposed into four convex quadratic programs

$$f_1 = \min x_1 + x_2$$

$$\begin{cases} 2x_1 - x_2 = 0 \\ 2x_1 - x_2 = -2 \\ x_1, x_2 \geq 0 \end{cases} \tag{11}$$

$$f_2 = \min x_1 + x_2$$

$$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 - x_2 = 2 \\ x_1, x_2 \geq 0 \end{cases} \tag{12}$$

$$f_3 = \min x_1 + x_2$$

$$\begin{cases} x_1 - x_2 = 2 \\ 2x_1 - x_2 = -2 \\ x_1, x_2 \geq 0 \end{cases} \tag{13}$$

and

$$f_4 = \min x_1 + x_2$$

$$\begin{cases} x_1 - x_2 = 2 \\ x_1 - x_2 = 2 \\ x_1, x_2 \geq 0 \end{cases} \tag{14}$$

It is easy to see that the convex quadratic programs (11), (12) and (13) are infeasible and the convex quadratic program (14) is feasible, hence the optimal solution is exist in the convex quadratic program (14), it can be shown that optimal values of four convex quadratic programs are $f_1 = \infty, f_2 = \infty, f_3 = \infty, f_4 = 2$. Thus $\overline{f} = 2$.

Therefore $\overline{f}, \underline{f}$ are finite while the subproblem (11) has no optimal solution. This example shows that $(b) \not\Rightarrow (a)$ in Theorem 2.1.

5. Conclusion

In applications we are mostly interested in interval quadratic programming problems having finite optimal solutions. There for problems of interval optimization when all subproblems have optimal solutions is of particular interest. This paper discuss some interesting finite solution properties of interval quadratic programming problem with standard constraints. A topic is worth of further study is the properties of interval quadratic programming problem with mixed constraints.

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