International Journal of Theoretical and Applied Mathematics
2017; 3(2): 64-69
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20170302.13

Dot Products and Matrix Properties of 4×4 Strongly Magic Squares

Neeradha. C. K.[1], V. Madhukar Mallayya[2]

[1]Dept. of Science & Humanities, Mar Baselios College of Engineering & Technology, Thiruvananthapuram, Kerala, India
[2]Department of Mathematics, Mohandas College of Engineering & Technology, Thiruvananthapuram, Kerala, India

Email address:
ckneeradha@yahoo.co.in (Neeradha. C. K.), mmallayyav@gmail.com (V. M. Mallayya)

To cite this article:
Neeradha. C. K., V. Madhukar Mallayya. Dot Products and Matrix Properties of 4×4 Strongly Magic Squares. *International Journal of Theoretical and Applied Mathematics.* Vol. 3, No. 2, 2017, pp. 64-69. doi: 10.11648/j.ijtam.20170302.13

Received: November 4, 2016; **Accepted**: December 27, 2016; **Published**: February 13, 2017

Abstract: Magic squares have been known in India from very early times. The renowned mathematician Ramanujan had immense contributions in the field of Magic Squares. A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. The matrix properties of 4×4 strongly magic squares dot products and different properties of eigen values and eigen vectors are discussed in detail.

Keywords: Strongly Magic Square (SMS), Dot Products of SMS, Eigen Values of SMS, Rank and Determinant of SMS

1. Introduction

Magic squares date back in the first millennium B. C. E in China [1], developed in India and Islamic World in the first millennium C. E, and found its way to Europe in the later Middle Ages [2] and to sub-Saharan Africa not much after [3]. Magic squares generally fall into the realm of recreational mathematics [4, 5], however a few times in the past century and more recently, they have become the interest of more-serious mathematicians. Srinivasa Ramanujan had contributed a lot in the field of magic squares. Ramanujan's work on magic squares is presented in detail in Ramanujan's Notebooks [6]. A normal magic square is a square array of consecutive numbers from $1 \dots n^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [7]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

2. Notations and Mathematical Preliminaries

2.1. Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an n[th] order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho \; for \; i = 1,2, \dots n \qquad (1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho \; for \; i = 1,2, \dots n \qquad (2)$$

$$\sum_{i=1}^{n} a_{ii} = \rho, \sum_{i=1}^{n} a_{i,n-i+1} = \rho, for \; j = 1,2, \dots n \qquad (3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [8]

2.2. Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

2.3. Strongly Magic Square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \; for \; i = 1,2, \ldots .. n^2 \quad (4)$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \; for \; i = 1,2, \ldots .. n^2 \quad (5)$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho, \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \quad (6)$$

$$\sum_{l=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \; for \; i,j = 1,2, \ldots .. n^2 \quad (7)$$

$where\ the\ subscripts\ are\ congruent\ modulo\ n^2$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n^{th} order sub square sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

2.4. Column/Row Dot Product of Two Magic Squares

Let C and C' or (R and R') be any two columns or (rows) of two magic squares A and A' of order n. If $a_1, a_2, \ldots a_n$ and ${a_1}', {a_2}', \ldots {a_n}'$ are the elements of C and C' or (R and R') respectively, then the dot product of C and C' or (R and R') denoted by $C.C'$ or $(R.R')$ is defined as

$$\mathrm{C.C'\ or\ R.R'} = \sum_{i=1}^{n} a_i . a_i' \quad (8)$$

For example

Two magic squares A and B are given in such a way that

$$A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix} \text{ and } A' = \begin{bmatrix} 3 & 13 & 2 & 16 \\ 10 & 8 & 11 & 5 \\ 15 & 1 & 14 & 4 \\ 6 & 12 & 7 & 9 \end{bmatrix}$$

Then the column dot products of A and A$'$ are given by

$$C_1.C'_1 = (16 \times 3) + (2 \times 10) + (13 \times 15) + (3 \times 6) = 281$$

$$C_1.C'_2 = (16 \times 13) + (2 \times 8) + (13 \times 1) + (3 \times 12) = 273$$

Also the row dot products of A and A$'$ are given by

$$R_3.R'_4 = (13 \times 6) + (8 \times 12) + (1 \times 7) + (12 \times 9) = 289$$

3. Propositions and Theorems

3.1. Dot Products of 4×4 Strongly Magic Squares

3.1.1.

If $A = [a_{ij}]$ be an SMS of order 4 and if R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns of SMS respectively, then

$i)\ R_1R_2 = R_3R_4\ ii)\ R_1R_4 = R_2R_3\ and\ iii)\ C_1C_2 = C_3C_4\ iv) C_1C_4 = C_2C_3$ or in general $R_iR_{i+1} = R_{i+2}R_{i+3}$ and $C_iC_{i+1} = C_{i+2}C_{i+3}$ where $i = 1,2,3,4$ and the subscripts should be taken modulo 4

Proof

The general form of a 4x4 SMS is given by

$$\begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix} \quad (9)$$

$where\ \rho\ is\ the\ magic\ constant\ and\ a,b,c,d\ \in R.$ [9]

$$R_3R_4 = \rho^2 + 2bd + ac - c^2 - 2\rho b - 2\rho d + cd + bc$$

$$R_1R_2 = (a+c)\rho + 2bd + ac - c^2 + cd + bc - b\rho - \rho d$$

$$= [\rho - (b+d)]\rho + 2bd + ac - c^2 + cd + bc - b\rho - \rho d$$

$$= \rho^2 + 2bd + ac - c^2 - 2\rho b - 2\rho d + cd + bc$$

i. Hence $R_1R_2 = R_3R_4$

ii. $R_1R_4 = ac - a^2 - \frac{a\rho}{2} + \frac{3b\rho}{2} - b^2 - bc - \frac{\rho c}{2} + \frac{3\rho d}{2} - cd - d^2 = R_2R_3$

iii. $C_1C_2 = 2ab + cd + ac - bc - c^2 + \frac{\rho b}{2} + \frac{5\rho c}{2} - \frac{3\rho^2}{2} - \frac{3a\rho}{2} + \frac{\rho d}{2} = C_3C_4$

3.1.2.

If $A = [a_{ij}]$ be an SMS of order 4 and if R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns of SMS respectively, then $i) R_1C_1 = R_3C_3\ ii) R_1C_2 = R_3C_4\ iii) R_1C_3 = R_3C_1\ iv) R_1C_4 = R_3C_2\ v) R_2C_1 = R_4C_3$
$vi) R_2C_2 = R_4C_4\ vii) R_2C_3 = R_4C_1\ viii) R_2C_4 = R_4C_2$ or in general $R_iC_j = R_{i+2}C_{j+2}$ where $i,j = 1,2,3,4$ and the subscripts should be taken modulo 4

Proof

From the general form of a 4x4 SMS as in 3.1.1

$$i)\ R_1C_1 = a^2 - c^2 + b\rho + \frac{\rho c}{2} + cd - ad - \frac{\rho d}{2}$$

$$R_3C_3 = a^2 - c^2 + \frac{\rho c}{2} + cd - ad - \frac{\rho(a+c)}{2} + \frac{\rho b}{2} - \rho d + \rho^2/2$$

$$= a^2 - c^2 + \frac{\rho c}{2} + cd - ad - \frac{\rho(\rho - (b+d))}{2} + \frac{\rho b}{2} - \rho d + \rho^2/2$$

$$= a^2 - c^2 + b\rho + \frac{\rho c}{2} + cd - ad - \frac{\rho d}{2}$$

$$ii)\ R_1C_2 = ab + bc - b\rho + \frac{c\rho}{2} - 2cd + \frac{3\rho d}{2}$$

$$R_3C_4 = ab + bc - 2cd + \frac{\rho^2}{2} + \rho d - \frac{\rho a}{2} - \frac{3\rho b}{2}$$

$$= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \frac{\rho d}{2} - \frac{\rho a}{2} - \frac{\rho b}{2} - \rho b$$

$$= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \rho/2(a+b+d) - \rho b$$

$$= ab + bc - 2cd + \frac{\rho^2}{2} + \frac{3\rho d}{2} - \rho/2(\rho - c) - \rho b$$

$$= ab + bc - b\rho + \frac{c\rho}{2} - 2cd + \frac{3\rho d}{2} = R_1C_2$$

$$iii) R_1C_3 = ab - bc + b\rho - \frac{\rho d}{2} + \frac{\rho c}{2}$$

$$R_3C_1 = ab - bc + \frac{b\rho}{2} - \frac{a\rho}{2} + \frac{\rho^2}{2} - \rho d$$

$$= ab - bc + b\rho - \frac{\rho(a+b)}{2} + \frac{\rho^2}{2} - \rho d$$

$$= ab - bc + b\rho - \frac{\rho\big(\rho - (c+d)\big)}{2} + \frac{\rho^2}{2} - \rho d$$

$$= ab - bc + b\rho - \frac{\rho d}{2} + \frac{\rho c}{2} = R_1C_3$$

$$iv) R_3C_2 = ad + b^2 - d^2 - cd + \frac{\rho^2}{2} + \rho d - \frac{a\rho}{2} - 3\rho b/2$$

$$R_1C_4 = ad + b^2 - d^2 - cd - b\rho + \frac{c\rho}{2} + 3\rho d/2$$

$$= ad + b^2 - d^2 - cd - b\rho + \frac{\rho\big(\rho - (a+b+d)\big)}{2} + 3\rho d/2$$

$$= ad + b^2 - d^2 - cd + \frac{\rho^2}{2} + \rho d - \frac{a\rho}{2} - \frac{3\rho b}{2} = R_3C_2$$

$$v) R_4C_3 = 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \frac{\rho^2}{2} - \rho c/2 - b\rho + \rho d/2$$

$$R_2C_1 = 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \rho d + \frac{a\rho}{2} - b\rho/2$$

$$= 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \rho d + \rho\big(\rho - (b+c+d)\big) - \frac{b\rho}{2}$$

$$= 2c^2 - 2ac + bc + 2a\rho - 2c\rho - ab + \frac{\rho^2}{2} - \frac{\rho c}{2} - b\rho + \frac{\rho d}{2} = R_4C_3$$

$$vi) R_2C_2 = 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho b}{2} - 3\rho d + \frac{a\rho}{2}$$

$$R_4C_4 = 3cd + d^2 - ad - b^2 - 2bc - \frac{7\rho d}{2} + 3\rho b + \frac{\rho^2}{2} - \frac{\rho c}{2}$$

$$= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho(b-d)}{2} - \frac{\rho(b+c)}{2} + \frac{\rho^2}{2}$$

$$= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho(b-d)}{2} - \frac{\rho\big(\rho - (a+d)\big)}{2} + \frac{\rho^2}{2}$$

$$= 3cd + d^2 - ad - b^2 - 2bc + \frac{7\rho b}{2} - 3\rho d + \frac{a\rho}{2} = R_2C_2$$

$$vii) R_4C_1 = 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \frac{\rho^2}{2} - \rho b - \frac{\rho c}{2} + \frac{\rho d}{2}$$

$$R_2C_3 = 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \rho d + \frac{\rho a}{2} - \frac{\rho b}{2}$$

$$= 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \rho d + \frac{\rho\big(\rho - (b+c+d)\big)}{2} - \frac{\rho b}{2}$$

$$= 2\rho c + 2ac - c^2 - a^2 + ad - cd - 2a\rho + \frac{\rho^2}{2} - \rho b - \frac{\rho c}{2} + \frac{\rho d}{2} = R_4C_1$$

$$viii) R_2C_4 = \rho d - \frac{b\rho}{2} + \frac{\rho a}{2} + bc - ab - \frac{\rho^2}{2}$$

$$R_4C_2 = \frac{\rho^2}{2} - \rho b + \frac{\rho d}{2} - \frac{\rho c}{2} + bc - ab$$

$$= \frac{\rho^2}{2} - \frac{\rho b}{2} - \frac{\rho(b+c)}{2} + \frac{\rho d}{2} + bc - ab$$

$$= \frac{\rho^2}{2} - \frac{\rho b}{2} - \frac{\rho(\rho - (a+d))}{2} + \frac{\rho d}{2} + bc - ab$$

$$= \rho d - \frac{b\rho}{2} + \frac{\rho a}{2} + bc - ab - \frac{\rho^2}{2} = R_2C_4$$

3.1.3.

If $A = [a_{ij}]$ be an SMS of order 4 and if R_1, R_2, R_3, R_4 and C_1, C_2, C_3, C_4 be the rows and columns of SMS respectively, then $i) R_iR_i = R_{i+2}R_{i+2}$ and $ii) C_iC_i = C_{i+2}C_{i+2}$ i.e. $i.1)\ R_1R_1 = R_3R_3$ $i.2) R_2R_2 = R_4R_4$ $ii.1) C_1C_1 = C_3C_3$ $ii.2) C_2C_2 = C_4C_4$

Proof

From the general form of a 4x4 SMS as in 3.1.1

$$i.1)\ R_1R_1 = a^2 + b^2 + c^2 + d^2$$

$$R_3R_3 = \left(\frac{\rho}{2} - c\right)^2 + \left(\frac{\rho}{2} - d\right)^2 + \left(\frac{\rho}{2} - a\right)^2 + \left(\frac{\rho}{2} - b\right)^2$$

$$= a^2 + b^2 + c^2 + d^2 = R_1R_1$$

$$i.2)\ R_2R_2 = (\rho)^2 + (c + d - \rho)^2 + (a - c + \rho)^2 + (b + c - \rho)^2$$

$$= a^2 + b^2 + d^2 + 3c^2 + 2bc - 2ac + 2cd + 4\rho^2 - 6\rho c + 2\rho a - 2\rho b - 2\rho d$$

$$R_4R_4 = a^2 + b^2 + d^2 + 3c^2 + 2bc - 2ac + 2cd + 5\rho^2 - 7\rho c + \rho a - 3\rho b - 3\rho d$$

$$= R_2R_2 - 2\rho a + \rho^2 - \rho c - \rho b - \rho d + \rho a$$

$$= R_2R_2 - 2\rho a + \rho^2 - \rho c - \rho(\rho - (a+c)) + \rho a = R_2R_2$$

$$ii.1)\ C_1C_1 = 2a^2 + \frac{3}{2}\rho^2 + 2c^2 - 2\rho c - 2ac + a\rho = C_3C_3$$

$$ii.2)\ C_2C_2 = 2b^2 + 2c^2 + 2d^2 + \frac{7}{2}\rho^2 + 2cd - 5c\rho - 3\rho d + 2bc - 3\rho b = C_4C_4$$

3.2. Eigen Values of 4×4 Strongly Magic Squares

3.2.1.

The eigen values of 4x4 SMS are $\rho, 0, \varphi, -\varphi, where\ \varphi^2 = [(a-d)^2 - (b+c)^2 + 4(\rho b + cd \pm \rho d)]$.

Proof

The general form of a 4x4 SMS is given by

$$\begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix}$$

$where\ \rho\ is\ the\ magic\ constant\ and\ a, b, c, d \in R.$ [9]

The characteristic polynomial of A is given by $|A - \lambda I| = 0$

i.e,

$$\begin{vmatrix} a-\lambda & b & c & d \\ \rho & c+d-\rho-\lambda & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a-\lambda & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d-\lambda \end{vmatrix} = 0 \tag{10}$$

Simplifying (10) the characteristic polynomial can be written as

$$\lambda^4 - \lambda^3\rho - \lambda^2\varphi^2 + \lambda\rho\varphi^2 = 0; where\ \varphi^2 = [(a-d)^2 - (b+c)^2 + 4(\rho b + cd \pm \rho d)]$$

or in case of factorized form expressed by $\lambda(\lambda-\rho)(\lambda-\varphi)(\lambda+\varphi) = 0$.

This completes the proof

3.2.2.

Sum of the eigen values of a SMS of order $n^2 \times n^2$ is the magic constant.

Proof

Let λ be the eigen value of a SMS.

Then $\sum \lambda_i = trace\ of\ A; i = 1,2 \ldots n^2$

$$\sum \lambda_i = \rho\ (Since\ trace\ of\ A\ is\ the\ magic\ constant)$$

Thus sum of the eigen values of a SMS is the magic constant ρ.

3.2.3.

$(1,1,1,\ldots.1)^T$ is the eigen vector corresponding to the eigen value ρ of a strongly magic square A.

Proof

The eigen vector X of a matrix A with eigen value λ is given by AX = λX.

By using the fact that the one of the eigen value is ρ and the row sum is also ρ; we have

$(1,1,1,\ldots.1)^T$ as the eigen vector corresponding to eigen value ρ. Clarifies the proof have to illustration in the following form

The particular 4x4 SMS Sri Rama Chakra is given by

$$A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix}$$

Assume $(1,1,1,1)^T$ be the eigen vector, X.

Then $AX = \lambda X$ gives

$$\begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}34\\34\\34\\34\end{bmatrix} = 34\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Remark: 1. $(1,1,1,\ldots 1)^T$ is the eigen vector for a SMS and its transpose

2. It can be observed that the eigenvalues except for ρ and 0 of strongly magic square will be either 0 or $\pm\lambda$

Corollary:

The eigen values of a magic square cannot be all positive.

Proof

From the Remark 2, the result is obtained

3.3. Determinant and Rank of 4×4 Strongly Magic Squares

3.3.1.

The determinant of a 4x4 SMS is always 0.

Proof

The general form of a 4×4 SMS is given by

$$A = \begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix}$$

$where\ \rho\ is\ the\ magic\ constant\ and\ a, b, c, d \in R.$ [9]

It can be verified that $|A| = 0$.

3.3.2.

The rank of a 4×4 strongly magic square is always 3.

Proof

The general form of a 4x4 SMS is given by

$$A = \begin{bmatrix} a & b & c & d \\ \rho & c+d-\rho & a-c+\rho & b+c-\rho \\ \frac{\rho}{2}-c & \frac{\rho}{2}-d & \frac{\rho}{2}-a & \frac{\rho}{2}-b \\ c-a-\frac{\rho}{2} & \frac{3\rho}{2}-b-c & -\rho/2 & \frac{3\rho}{2}-c-d \end{bmatrix}$$

$$where\ \rho\ is\ the\ magic\ constant\ and\ a, b, c, d\ \in R.$$

It can be verified using matlab which is the rank given above SMS A is 3.

3.3.3.

The rank of a 4x4 SMS A and $AA^T are\ equal.$

Proof

By taking the general form as in 3.3.2

$$AA^T = [b_{ij}]\ where\ i, j = 1,2,3,4\ such\ that$$

$$b_{11} = a^2 + b^2 + c^2 + d^2$$

$$b_{12} = a\rho + c(a-c+\rho) + b(c+d-\rho) + d(b+c-\rho)$$

$$b_{13} = -a\left(c-\frac{\rho}{2}\right) - c\left(a-\frac{\rho}{2}\right) - b\left(d-\frac{\rho}{2}\right) - d\left(b-\frac{\rho}{2}\right)$$

$$b_{14} = -a(a - c + \rho/2) - (c\rho)/2 - b(b + c - (3\rho)/2) - d(c + d - (3\rho)/2)$$

$$b_{21} = a\rho + c(a - c + \rho) + b(c + d - \rho) + d(b + c - \rho)$$

$$b_{22} = (a - c + \rho)^2 + (b + c - \rho)^2 + (c + d - \rho)^2 + \rho^2$$

$$b_{23} = -\rho(c - \rho/2) - (a - \rho/2)(a - c + \rho) - (b - \rho/2)(b + c - \rho) - (d - \rho/2)(c + d - \rho)$$

$$b_{24} = \rho(a-c+\rho/2) - (b+c-\rho)(c+d-(3\rho)/2) - (b+c-(3\rho)/2)(c+d-\rho) - (\rho*(a-c+\rho))/2$$

$$b_{31} = -a(c-\frac{\rho}{2}) - c(a-\frac{\rho}{2}) - b(d-\frac{\rho}{2}) - d(b-\frac{\rho}{2})$$

$$b_{32} = -\rho(c - \rho/2) - (a - \rho/2)(a - c + \rho) - (b - \rho/2)(b + c - \rho) - (d - \rho/2) * (c + d - \rho)$$

$$b_{33} = \left(a-\frac{\rho}{2}\right)^2 + \left(b-\frac{\rho}{2}\right)^2 + \left(c-\frac{\rho}{2}\right)^2 + \left(d-\frac{\rho}{2}\right)^2$$

$$b_{34} = \left(c-\frac{\rho}{2}\right)\left(a - c + \frac{\rho}{2}\right) + \frac{\rho\left(a-\frac{\rho}{2}\right)}{2} + \left(b-\frac{\rho}{2}\right)\left(c + d - \frac{3\rho}{2}\right) + (d - \rho/2)(b + c - (3\rho)/2$$

$$b_{41} = -a(a - c + \rho/2) - (c\rho)/2 - b(b + c - (3\rho)/2) - d(c + d - (3\rho)/2)$$

$$b_{42} = -\rho(a-c+\rho/2) - (b+c-\rho)(c+d-(3\rho)/2) - (b+c-(3\rho)/2)(c+d-\rho) - (\rho(a-c+\rho))/2$$

$$b_{43} = (c-\rho/2)(a-c+\rho/2) + (\rho(a-\rho/2))/2 + (b-\rho/2)(c+d-(3\rho)/2) + (d-\rho/2)(b+c-(3\rho)/2$$

$$b_{44} = \left(b + c-\frac{3\rho}{2}\right)^2 + \left(c + d-\frac{3\rho}{2}\right)^2 + \left(a - c + \frac{\rho}{2}\right)^2 + \rho^2/4$$

Using Matlab it can be easily verified that rank of A and AA^T is 3

4. Conclusion

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described. Despite the fact that magic squares have been studied for a long time, they are still the subject of research projects. These include pure mathematical research, much of which is connected with the algebra and combinatorial geometry of polyhedra (see, for example, [10]). Physical application of magic squares is still a new topic that needs to be explored more. There are many interesting ideas for research in this field.

Acknowledgement

We express sincere gratitude for the valuable suggestions given by Dr. Ramaswamy Iyer, Former Professor in Chemistry, Mar Ivanios College, Trivandrum, in preparing this paper.

References

[1] Schuyler Cammann, Old Chinese magic squares. Sinologica 7 (1962), 14–53.

[2] Andrews, W. S. Magic Squares and Cubes, 2nd rev. ed. New York: Dover, 1960.

[3] Claudia Zaslavsky, Africa Counts: Number and Pattern in African Culture. Prindle, Weber & Schmidt, Boston, 1973.

[4] Paul C. Pasles. Benjamin Franklin's numbers: an unsung mathematical odyssey. Princeton UniversityPress, Princeton, N. J., 2008.

[5] C. Pickover. The Zen of Magic Squares, Circles and Stars. Princeton University Press, Princeton, NJ, 2002.

[6] Bruce C. Berndt, Ramanujan's Notebooks Part I, Chapter 1 (pp 16-24), Springer, 1985.

[7] T. V. Padmakumar “Strongly Magic Square”, Applications Of Fibonacci Numbers Volume 6 Proceedings of The Sixth International Research Conference on Fibonacci Numbers and Their Applications, April 1995.

[8] Charles Small, “Magic Squares Over Fields” The American Mathematical Monthly Vol. 95, No. 7 (Aug. - Sep., 1988), pp. 621-625.

[9] Neeradha. C. K, Dr. V. Madhukar Mallayya “Generalized Form Of A 4x4 Strongly Magic Square” IJMMS, Vol. 12, No. 1 (January-June; 2016), pp 79-84.

[10] A. Mudgal, Counting Magic Squares, Undergraduate thesis, IIT Bombay, 2002.