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# Absolutely New, Simple and Effective Theory of Movement Steadiness

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**Abstract:** The absolutely new, simple and effective theory is proposed which differs from the classical Liapunov's theory of the movement steadiness. This theory permits to simplify and to speed up the search for the stable movement many times. The classical theory is very complex for using in the engineer practice and one does not bring success in many cases. It was necessary to create a theory that would be devoid of all shortcomings of the classical theory. In this work, it is proposed exactly such theory. Instead of the very complex Liapunov's function we propose to use the variations calculation. This gives the invaluable winner in the speed and simplicity while searching for the stable movement.

**Keywords:** Dynamical Systems, Movement Steadiness, New Theory

## 1. Introduction

Investigations in the sphere of the movement steadiness began in the end of 19 century in several works of E. Raus, A. E. Jukovskii, A. M. Liapunov and later were continued by N. G. Chetaev, D. R. Merkin, N. N. Krasovskii and others. The most serious results were obtained in the dissertation of A. M. Liapunov in 1892. He proposed the general formulation for the stable motion problem.

Up to now, Liapunov's method of the positive-definite functions has remained the most common. However, this method has serious shortcomings. Liapunov's method is based on the sufficient conditions of the stable motion. But, it is known that the sufficient conditions almost always are less effective for searching the solution for any problems than the necessary conditions. And if by means of sufficient conditions it is impossible to solve the problem, then it does not mean that the solution does not exist. As the necessary conditions of the stable motion (in accordance with its definition) are regarded in assumption of existence of the stable motion, these conditions almost always permit to find this stable motion if one exists. Whereas by means of the sufficient conditions it often turns out impossible in principle to find the desirable solution for the problem.

In this paper, a new theory is proposed permitting to find the stable motion quite easily and more quickly. On the other hand, the classical Liapunov's theory often does not permit to find asymptotical steadiness even if one exists.

## 2. Method

### *2.1. Statement of the Problem*

Let there be given a process in $n$-space defined by the vectorial differential equation

$$\frac{dy}{dt} = Y(y(t),t), \tag{1}$$

where $Y(y,t)$ is the known vector-function satisfying requirements ensuring existence of the solution of the equation (1), $y(t) = (y_1(t),...,y_n(t)$ is the vector-function of the phase-coordinates $y_i(t), i=1,...n$. And let the partial derivatives $\dfrac{\partial^2 Y_i}{\partial y_k^2}, i,k=1,...,n$ exist and be continuous.

And let $z(t)$ be some solution of the equation (1). It is required to estimate the steadiness of $z(t)$ with respect to the small perturbations $x(t)$:

$$x(t) = y(t) - z(t).$$

Introduce this equality into equation (1) and rewrite it via coordinates

$$\frac{dz_i}{dt}+\frac{dx_i}{dt}=Y_i(z_1+x_1,...,z_n+x_n,t),\ i=1,...,n. \tag{1a}$$

The investigation of the steadiness can be performed directly on the basis of the equations (1a) or after expansion of these equations in rows on the small parameter $x(t)$ in a neighbourhood of solution $z(t)$:

$$\frac{dz_i}{dt}+\frac{dx_i}{dt}=Y_i(z,t)+(\sum_{k-1}^{n}\frac{\partial Y_i}{\partial x_k})_z x_k+\Delta Y_i,\ i=1,...,n,$$

where $\Delta Y_i$ is the sum of the members above the first order. As $z(t)$ is the solution of the equation

$$\frac{dz}{dt}=Y(z,t), \tag{2}$$

we receive the following perturbed equations

$$\frac{dx_i}{\mathrm{dt}}=\sum_{k=1}^{n}(\frac{\partial Y_i}{\partial x_k})_z x_k+\Delta Y_i,\ i=1,...,n \tag{3}$$

By means of urge towards $x(t)\to 0$ in the equations (3) or (1a) one can judge about the steadiness of the solution $z(t)$ of the equation (2).

Using only the linear terms of the equations (3) seldom brings success. The analysis of the (3) usually is done by means of the Liapunov's functions $V(x)$.

Already for more than 100 years the following Liapunov's theorem is used in the engineer practice [1, c. 37]:

"If for the differential equation of the perturbed motion it is possible to find a positive-definite function $V(x)>0$, such that the full derivative on time ($\dot{V}$) will be negative-definite ($\dot{V}<0$) or $\dot{V}\equiv 0$, then non-perturbed motion is stable."

If in Liapunov's theorem the last requirement $\dot{V}\equiv 0$ is excluded, then it defines the asymptotical steadiness. But the last requirement does not define stability from the practical point of view.

Among shortcomings of the classic theory [1-9], it is necessary to notice, first of all, that searching for the suitable function $V(x)$ is too much trouble. There is no guarantee to find the pair $(\mathrm{V(x)}>0,\ \dot{\mathrm{V}}(\mathrm{x})<0)$. Besides that, the elucidation of the sign-definite functions is also very complex.

In this paper, it is proposed to replace the very complex problem of searching for Liapunov's function with a very simple problem of searching maximum of function $\dot{V}$.

Before offering the new simple and effective theory of the motion stability, we shall demonstrate the principal problems of the classic Liapunov's method by the typical example.

Example 1. Let us consider the nonlinear differential equations of the perturbed motion [1, p. 54-55]:

$$\dot{x}_1=ax_1+bx_2^2,\quad \dot{x}_2=cx_1x_2+ex_2^3 \tag{4}$$

It is required to define restrictions of the parameters ($a,b,c,e$) of a real dynamic system that will ensure asymptotical steadiness of the zero-solution $x_1=x_2=0$ of the equations (4) with respect to small perturbations $(x_1,x_2)$.

In [1, p. 54-55] Liapunov's function is searched for in the form

$$V=\frac{1}{2}\ (\lambda x_1^2+2\mu x_1x_2+x_2^2) \tag{5}$$

where $\lambda$ and $\mu$ are chosen so that $V>0$ and $\dot{V}<0$. For the quadratic form to be positive-defined, it is necessary and sufficient that the diagonal minors of matrix

$$\begin{matrix}\lambda & \mu\\ \mu & 1\end{matrix}$$

be positive. It gives $(\lambda>0,\ \lambda>\mu^2)$.

The calculation of $\dot{V}$ for equations (4) gives

$$\dot{V}=\lambda ax_1^2+(\lambda b+c)x_1x_2^2+ex_2^4+$$
$$\mu(ax_1x_2+bx_2^3+cx_1^2x_2+ex_1x_2^3).$$

If $\mu\neq 0$ then the function $\dot{V}$ is sign-variable. In case $\mu=0$ we receive the quadratic form $\dot{V}$ for variables $(x_1,\ \overline{x_2}=x_2^2)$:

$$\dot{V}=\lambda ax_1^2+(\lambda b+c)x_1x_2^2+ex_2^4=\lambda ax_1^2+(\lambda b+c)x_1\overline{x_2}+e\overline{x_2^2}$$

and Silvestor's criterion for these variables leads to the following inequalities:

$$\lambda a<0,\ 4\lambda ae-(\lambda b+c)^2>0 \tag{6}$$

After definition of the roots of the square equation

$$4\lambda ae-(\lambda b+c)^2=0,$$

We have the following limitations of the parameters

$$a<0,\ e<0,\ bc<ae,\ \lambda_1<\lambda<\lambda_2 \tag{7}$$

where $\lambda_1$ and $\lambda_2$ are the positive numbers.

For parameters satisfying (7), the form $V$ will be the define-positive and the form $\dot{V}$ be define-negative, and, under Liapunov's theorem, the asymptotical steadiness of the zero-solution of the equations (4) takes place only about variables $x_1$ and $x_2^2$, while in [1] the attempt to find the asymptotical stability straight for the variables $(x_1 x_2)$ failed.

Notice, though, that if in (5) we take $\lambda = 1, \mu = 0$ then we receive a different Liapunov's function S= $\frac{1}{2}(x_1^2 + x_2^2)$ which is also the define-positive ( $S > 0$ ) and the function $\dot{S} < 0$ in relation to $(x_1, x_2^2)$ as it follows from (6). And consequently in this case, we also receive the asymptotical stability in relation to $(x_1, x_2^2)$, but not in relation to $(x_1, x_2)$. And now we find absolutely different values of parameters:

$$a<0,\ e<0,\ (b+c)^2 < 4ea. \tag{8}$$

So, absolutely different stability conditions ((7) and (8)) were found on the base of the different Liapunov's functions.

This example demonstrates the serious difficulties which appear while searching for the asymptotical stability in relation to the desirable variables $(x_1, x_2)$.

### *2.2. The New Theory of Stability of Dynamic Systems*

Consider a really different approach to the problem of steadiness based on the variational calculation [10] without shortcomings of the classic theory of Liapunov functions.

Let in $n$ -space $X$ be defined half-metrics $S = \frac{1}{2}\sum_{k=1}^{n} x_k^2$ and a small quantity $\varepsilon > 0$. Consider the solution of equation (3) (or (1a)) in a small $\varepsilon$ -environment of zero in $X$ :

$$S = \frac{1}{2}\sum_{k=1}^{n} x_k^2 \le \varepsilon. \tag{9}$$

Definition 1. We say that a solution of the equation (1) is $\varepsilon$ -stable if there exists the small quantity $\varepsilon > 0$ and a moment $t_1$ $(0 < t_1 < \infty)$ that for all $t > t_1$ the trajectory $x(t)$ remains in the sphere (9). And we say that a movement is asymptotically stable if, for any small quantity $\varepsilon > 0$, the trajectory $x(t)$ aspires to zero and reaches value $x(t) \equiv 0$ in (9).

Assertion 1. Suppose that the problem of the movement steadiness has a positive solution. As it follows from Definition 1, the object moves inside sphere (9) in the space $X$ In this case, it is obvious that $\dot{S}(x)$ =(grad S , $\dot{x}$) $\le 0$ and function $\dot{S}(x)$ reaches its maximum in some point $x$ in (9) for $t > t_1$, and we talk in this case about $\varepsilon$ -stable. If this maximum is reached in the point $x = 0$ we talk about the asymptotical stability.

The Assertion 1, in essence, proves the following theorem.

Theorem 1. For a solution of the dynamic system to be $\varepsilon$ - stable to the small perturbations it is necessary that the full derivative on time $\dot{S}(x)$ of the function $S(x)$, calculated with regard to the differential equations (3) or (1a), reach maximum in the sphere (9), and in the case of the asymptotical stability it is necessary that maximum $\dot{S}(x)$ be reached in the point $x = 0$.

Consequence 1. For the asymptotical stability of zero-solution of the equations (3) or (1a) relative to the coordinate $x_i(t)$, it is necessary that, in the small environment of the point $x = 0$ and in this point, the following conditions take place:

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} \le 0,\ [10], \text{p. 35-36]},\ \frac{\partial \dot{S}}{\partial x_i} = 0, (i = 1, ..., n).$$

Consequence 2. For the asymptotical stability of zero-solution of the equations (3) or (1a) relative to the coordinate $x_i$ , it is necessary and sufficient that, in the small environment of the point $x = 0$ and in this point, the following conditions be satisfied:

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} < 0,\ [10, \text{p. 35-36]},\ \frac{\partial \dot{S}}{\partial x_i} = 0, (i = 1, ..., n)$$

The theorem 1 allows to decide the steadiness problem for the nonlinear dynamic systems far simpler and quicker than Liapunov's method. The proposed new method does not require very complex searching for Liapunov's functions and one reduces the stability problem to a very simple problem searching for $\max_{x \in R^n} \dot{S}(x)$ . Notice that in Theorem 1, the character of urge $x \to 0$ does not matter, therefore Theorem 1 contains in itself Krasovskii's theorems, [1, p. 42-46].

### *2.3. Demonstration of Effectiveness of This Method*

Demonstrate now by Example 1 the possibilities of the proposed new theory. We shall search for conditions of the asymptotical stability about the natural variables $(x_1, x_2)$, which were not found in [1, p. 54-55].

At the beginning, we define function $\dot{S}$ :

$$\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2 = ax_1^2 + x_1x_2^2(b+c) + ex_2^4 \tag{10}$$

and calculate the first particular derivative defining extremals in the considered problem:

$$\frac{\partial \dot{S}}{\partial x_1} = 2ax_1 + (b+c)x_2^2 = 0, \tag{11}$$

$$\frac{\partial \dot{S}}{\partial x_2} = 2x_1x_2(b+c) + 4ex_2^3 = 0. \tag{12}$$

In consequence of assumption about existence of maximum of the function $\dot{S}$, the second partial derivative must be non-positive [10, p. 35-36]:

$$\frac{\partial^2 \dot{S}}{\partial x_1^2} = 2a \le 0, \tag{13}$$

$$\frac{\partial^2 \dot{S}}{\partial x_2^2} = 2[x_1(b+c) + 6ex_2^2] \ \le \ 0. \qquad (14)$$

From (13), we have $a \le 0$. And the substitution of the extremal (11) into (14) points out that for small $x$ and in the point $x = 0$ attitude is fulfilled

$$\frac{\partial^2 \dot{S}}{x_2^2} = x_2^2[6e - \frac{(b+c)^2}{2a}] \ \le \ 0, \qquad (15)$$

Hence, it follows

$$12ae \ \ge (b+c)^2. \qquad (16)$$

If also to substitute the second extremal (12) in (14), we receive

$$8ex_2^2 \ \le \ 0,$$

and hence $e \le 0$. Also notice that, from the extremals (11) and (12), it follows $4ae=(b+c)^2$.

Thus, on the base of the proposed new method, the asymptotical stability was found very simply in relation to the desirable variables $(x_1, x_2)$, Compare the offered new theory with the classic Liapunov's theory on two more examples.

Example 2 [1, p. 46 ]. Consider the following equations of the perturbed motion:

$$\dot{x}_1 = - \frac{2x_1}{(1+x_1^2)^2} + 2x_2,$$
$$\dot{x}_2 = - \frac{2x_1}{(1+x_1^2)^2} - \frac{2x_2}{(1+x_1^2)^2} \ .$$

So as to search for the asymptotical steadiness, a very exotic Liapunov's function was found in [1, p. 46] probably, with difficulty

$$V = \frac{x_1^2}{(1+x_1^2)} + x_2^2,$$

However, without taking much effort in search of the suitable Liapunov's function, the conditions of stability of this system can be found very easily due to the proposed variational method.

Indeed, the asymptotical stability of the state $x_1 = x_2 = 0$ follows from inequalities

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} = -4 < 0, \ i=1,2.$$

Example 3, [1, p. 67-68]. Let the body be in rest in viscous surroundings and let then the body be troubled by a vector-moment

$$M = -b\omega^{a-1}\omega,$$

where $a$ and $b$ are parameters and $\omega$ is a speed of rotation of the perturbed body. In this case, the dynamic equations are following

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z - \frac{b}{I_x}\omega^{a-1}\omega_x,$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z - \frac{b}{I_y}\omega^{a-1}\omega_y, \qquad (17)$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_x\omega_y - \frac{b}{I_z}\omega^{a-1}\omega_z,$$

Supposing $S=\frac{1}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2) = \frac{1}{2}\omega^2$, we receive the function

$$\dot{S}=\omega_x\dot{\omega}_x+\omega_y\dot{\omega}_y+ \omega_z\dot{\omega}_z = \frac{I_Y - I_z}{I_x}\omega_x\omega_y\omega_z - \frac{b}{I_x}\omega^{a-1}\omega_x^2 + \frac{I_z - I_x}{I_{y-}}\omega_x\omega_y\omega_z - \frac{b}{I_y}\omega^{a-1}\omega_y^2 + \frac{I_x - I_y}{I_z}\omega_x\omega_y\omega_z - \frac{b}{I_z}\omega^{a-1}\omega_z^2$$

Calculate the first and second particular derivatives about the variables $(\omega_x, \omega_y, \omega_z)$. As this problem is symmetric for these variables so it is sufficient to calculate, for example, the derivative only about $\omega_x$. We receive the following extremal for $\omega_x$:

$$\frac{\partial \dot{S}}{\partial \omega_x} = \frac{I_y - I_z}{I_x}\omega_y\omega_z - \frac{b}{I_x}[(a-1)\omega^{a-3}\omega_x^3 + 2\omega^{a-1}\omega_x] + \frac{I_z - I_x}{I_y}\omega_y\omega_z - \frac{b}{I_y}[(a-1)\omega_y^2\omega^{a-3}\omega_x + \frac{I_x - I_y}{I_z}\omega_y\omega_z - \frac{b}{I_z}[(a-1)\omega_z^2\omega^{a-3}\omega_x = 0$$

The second partial derivative about $\omega_x$ is following

$$\frac{\partial^2 \dot{S}}{\omega_x^2} = -\frac{b}{I_x}[(a-1)(a-3)\omega^{a-5}\omega_x^4 + 3(a-1)\omega^{a-3}\omega_x^2 + 2(a-1)\omega^{a-3}\omega_x^2 + 2\omega^{a-1}] - \frac{b}{I_y}[(a-1)(a-3)\omega^{a-5}\omega_x^2\omega_y^2 + (a-1)\omega^{a-3}\omega_y^2]$$
$$-\frac{b}{I_z}[(a-1)(a-3)\omega^{a-5}\omega_x^2\omega_z^2 + (a-1)\omega^{a-3}\omega_z^2]$$

Hence, we see that, for $a > 3$, in a small neighbourhood of $\omega = 0$ the second derivative on $\omega_x$ is negative ($\frac{\partial^2 \dot{S}}{\partial \omega_x^2} < 0$). Therefore, the function $\dot{S}$ reaches maximum in the point $\omega = 0$. In consequence of the full symmetry of movement equations, the same result takes place also for other variables ($\omega_y, \omega_z$). So, in this problem, there is the asymptotical stability of the state $\omega = 0$ in case of $a > 3$. Notice, however, that, by means of the Liapunov's method in [1, c. 67-68], it was found asymptotical stability for $a > 1$, but it was a mistake.

Notice. The proposed new method does not require expanding of the perturbed differential equations in rows and permits to receive the function $\dot{S}$ directly on the basis of the equations

$$\frac{dx_i}{dt} = Y_i(z_1 + x_1, \ldots, z_n + x_n, t) - \frac{dz_i}{dt}, \; i = 1, \ldots, n.$$

## 3. Result

The absolutely new, simple and effective theory is worked out which differs from the classical Liapunov's theory and from all known theories [1-9] of the movement steadiness.

## 4. Discussion

This theory permits to simplify and to speed up the search for the stable movement many times. It was demonstrated on the examples that the proposed new theory of the movement stability is far simpler and more effective than the classic Liapunov's theory and all its known modifications and improvements [1-9].

## 5. Conclusions

This new theory gives the invaluable winner in the speed and simplicity searching the stable movement. By means of this new theory, engineers now can for some minutes or hours to define the asymptotical stability of any nonlinear dynamical systems, while up to now this work required often many hours, weeks or months, because the searching Liapunov's function was very complex.

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