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# Global Stability of Critical Points for Type SIS Epidemiological Model

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**Abstract:** The construction of mathematical models is one of the tools used today for the study of problems in Medicine, Biology, Physiology, Biochemistry, Epidemiology, and Pharmacokinetics, among other areas of knowledge; its primary objectives are to describe, explain and predict phenomena and processes in these areas. The simulation, through mathematical models, allows exploring the impact of the application of one or several control measures on the dynamics of the transmission of infectious diseases, providing valuable information for decision-making with the objective of controlling or eradicating them. The mathematical models in Epidemiology are not only descriptive but also predictive, helping to prevent pandemics (epidemics that spread through large areas and populations) or by intervening in vaccination and drug acquisition policies. In this article we study the existence of periodic orbits and the general stability of the equilibrium points for a susceptible-infected-susceptible model (SIS), with a non-linear incidence rate. This type of model has been studied in many articles with a very particular incidence rate, here the novelty of the problem is that the aforementioned incidence rate is very general, in this sense this research provides a solution to an open problem. The methodology used is the Dulac technique, proceeding by reduction to the absurdity of the statement to the main test. It shows that the only point of equilibrium is asymptotically stable global. It can be noted that this problem may be subject to discretion or for equations in timescales. This can generate other research.

**Keywords:** Periodic Orbits, Global Stability, Equilibrium Points

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## 1. Introduction

There are different factors of a disease that make us unable to study all in the same way, such as: the mode of transmission, infectious agents, the affected population and the states through which an individual can pass, the latter are: susceptible (S), healthy individuals who can get the disease; exposed (E), those infected but who cannot spread the disease; infected (I), infected individuals and who can infect others; resistant (R), those that are resistant to the disease, have usually overcome it or have been vaccinated and carriers (M), individuals who carry the disease, but may never suffer from it [1].

For the study of this area of research, it is assumed that individuals are in one of several possible states. The models studied are: SIR model, SI model and SIS model. En este artículo trabajamos el modelo SIS, el cual amplía el modelo

SI; ya que en este caso los individuos pueden recuperarse, pero pasan otra vez a ser susceptibles. En este caso se toman las variables S(t), susceptibles, e I(t), infecciosas, de forma que  $S(t) + I(t) = 1$ . Lo cual se representa en el esquema:  $S \xrightarrow{\alpha I} I \xrightarrow{\beta} S$  [2].

In the SIS model a new variable is added to the SI model, this is the  $V(t)$  variable, which describes the rate at which individuals recover from the disease or become susceptible, indicating the number that has been effectively vaccinated in every moment Therefore we have the following equations:

$$\frac{dS}{dt} = -\alpha SI + \beta I - V(t) \quad (1)$$

$$\frac{dI}{dt} = \alpha SI - \beta I \quad (2)$$

where  $S(t)$  and  $I(t)$  represent the susceptible and infectious,

and  $\alpha$  and  $\beta$  the contagion rate and recovery rate, respectively [1-4].

SIS models are used for diseases where there is no immunity, then once that infected people recover, they become susceptible again. Hence his name; since the progression of the disease, from the point of view of an individual, is susceptible-infected-susceptible (SIS) [5]. These models are epidemiological models of compartments, which are defined from classes and subclasses. The transition rates between classes for these models are estimated from qualitative and evident knowledge in the natural history of the disease. There are two ways to achieve the overall effectiveness of a disease control program: a retrospective analysis of data and simulation using mathematical models. Several researches have shown that the use of mathematical models to describe the transmission dynamics of infectious diseases is a necessary tool to perform a good cost-effectiveness analysis in the application of control measures in the spread of a disease [6]. Recent history and events show that contagious diseases pose a threat to the world's population. Countries are increasingly interconnected thanks to modern transport systems. New diseases arise or old ones reappear that can cross borders and spread quickly through several populations. This was demonstrated by the AH1N1 influenza pandemic; by whose outbreak was detected in April 2009 in Mexico and after a month had spread to 24 countries [7]. From these and other experiences can be concluded that a determining factor in the spread of a disease is the intense and rapid mobility of the population [8]. The spread of infectious diseases and their control measures have been the subject of several studies; most of them have done so in models for the dynamics in a population. The objective of this work is to use susceptible-infected-susceptible models (SIS). These models allow to know the influence of the migratory flows in the propagation of a disease and to understand the characteristics of the propagation in subpopulations, each with its own dynamics but connected by the movement of people with each other. In this sense, a study of the existence of periodic orbits and stability of equilibrium points for susceptible-infected-removed-susceptible (SIRS) type system, with incidence rate  $H(I, S) = kI^{p-1}S^q$ ,  $k > 0, p \geq 1, q > 0$ , is considered. It should be noted that this study is limited to a particular rate for  $p = 1$  and  $q = 1$ ; thus, the rate loses a considerable degree of generality [9]. However, the ideas in this article are very relevant, and are used in this work for the achievement of the study to be developed in it. Other researchers have conducted studies of an SIS type model and a SIRS type model through the Liouville formula, which determines the characteristic multipliers, which induces the system to admit a periodic solution that is stochastically stable orbital [10-13]. In our article, we aim to present a technique that involves more geometric elements related to the theory of equations and precise with a technique different from that of the other authors, the asymptotic uniformity of the stability of the solutions, which predicts if the population will be controlled to avoid outbreaks of disease.

## 2. Preliminary

In this section, we give some basic results that will be used in this article. Details, demonstrations and further developments are scattered in some references [14, 15].

Consider the system:

$$X'(t) = f(X) \tag{3}$$

$$X(0) = x_1 \tag{4}$$

Assume that  $f$  is  $C^1(W, R^n)$  class and  $W$  is an open subset of  $R^n$ . The system (3) will be called autonomous and function  $f$  is a vector field.

Consider the matrix  $A = f'(x_0)$ , where  $x_0$  is an equilibrium point of the system (3)-(4), the matrix  $A$  is called the Jacoby matrix of the function  $f$ .

Theorem 2.1

The real part of the eigen values of matrix  $A$  are all negative if and only if  $x_0$  is locally asymptotically stable.

## 3. Routh-Hurwitz Criterion

We consider a polynomial  $P$  as follows:

$$P(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

Let us define the matrix  $H_p$  as:

$$H_p = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 \dots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_n \end{bmatrix}_{n \times n}$$

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \dots, \dots,$$

$$\Delta_n = \begin{vmatrix} a_1 & 1 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & \dots & 0 \\ a_5 & a_4 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{vmatrix}$$

It can be seen that  $\Delta_n = \Delta_{n-1} a_n, n > 1$

Theorem 3.1. (Routh-Hurwitz).

The roots of a polynomial have negative real part if and only if  $\Delta_i > 0$ , for all  $i = 1, 2, 3, \dots, n$ .

### 4. Dulac Criterion

Theorem 4.1

- a. Let  $\Omega \subset \mathfrak{R}^2$  be a simply-connected set, F1 and F2 two continuously differentiable functions in  $\Omega$ , associated with the system of differential equations:

$$I' = F_1(I, S) \tag{5}$$

$$S' = F_2(I, S) \tag{6}$$

- b.  $\Omega$  contains at least an equilibrium point of the system (5), (6).

If there is a function  $B: \Omega \rightarrow \mathfrak{R}_0^+$ , for which the function  $\varphi$  defined by:

$$\varphi(I, S) = (B(I; S).F_1(I, S))_I + (B(I, S).F_2(I, S))_S,$$

preserves the same sign in  $\Omega$ , then  $\Omega$  does not contain periodic solutions.

Notation:  $(B(I, S).F_1(I, S))_I$  and  $(B(I, S).F_2(I, S))_S$  denote the partial derivatives with respect to  $I$  and  $S$ , respectively.

### 5. SIS Type Model

The SIS (susceptible-infected-susceptible) models are epidemiological models of compartments, which are defined from classes and subclasses. Transition rates between classes for these models are estimated from the qualitative and evident knowledge in the natural history of the disease [8]. For this study, a general form was chosen for the incidence rate of the population-based goal of SIS disease transmission. In this type of model, the population is divided into two groups of people: those who have been infected by the disease and are infectious and those who are susceptible to being infected by the disease.

SIS models are used for diseases where there is no immunity, because once infected people recover, they become susceptible again. Therefore, the progression of the disease from an individual's point of view is susceptible-infected-susceptible.

The SIS model used in this case is a variant of one already proposed as well as the treatment application [16, 17].

In this model, the population was divided into two patches that may represent cities, geographic regions or communities, which are interconnected by immigration and emigration processes [18].

Specifically, the SIS type model is proposed to be studied as follows:

$$S' = -I.H(I, S) + \gamma I + B(N) - \mu S \tag{7}$$

$$I' = I.H(I, S) - (\gamma + \mu).I \tag{8}$$

Let us denote by  $H(I, S) = kI^{p-1}S^q$ , with  $k > 0, p \geq 1, q > 0$ , the incidence rate for affected individuals, where

$I$ : individuals capable of transmitting the disease.

$S$ : individuals susceptible to the disease.

$B$ : nonnegative function of class  $C^1$  and represents the

birth rate.

$N$ : total number of individuals or population size

$\mu$ : per capita mortality rate.

$\gamma$ : recovery rate per capita of infected individuals (retirement rate).

If we add equations (7) and (8), we obtain:

$$N' = B(N) - \mu N \tag{9}$$

This equation describes the behavior of the total population. We note that since  $B$  is of class  $C^1$ , then the solutions of this equation exist and in addition they are unique locally.

We will assume henceforth that exist  $N_0$  such that:

$B(N_0) - \mu N_0 = 0$ ; that is, the equation has a critical point  $N_0 > 0$  asymptotically stable. This assumption is reasonable, since the absence of a asymptotically stable equilibrium point, would be an unstoppable population growth or its extinction. In which case, both situations would not reflect a reasonable biological situation.

In the current system, we just study the behavior of the solutions on the straight line  $I + S = N_0$ . Without loss of generality, we can assume that  $N_0 = 1$ ; since otherwise normalized coordinates  $I, S$  by:  $\hat{S} = S / N_0; \hat{I} = I / N_0$ , So that  $\hat{S} + \hat{I} = 1$  and so the system is reduced to:

$$\hat{I}' = \hat{I}.\hat{H}(\hat{I}, \hat{S}) - (\gamma + \mu).\hat{I} \tag{10}$$

$$\hat{S}' = -\hat{I}.\hat{H}(\hat{I}, \hat{S}) + \gamma.\hat{I} + B(N) - \mu.\hat{S} \tag{11}$$

where,  $\hat{H}(\hat{I}, \hat{S}) = H(\hat{S}.N_0, \hat{I}.N_0)$ .

From now on we will assume that the changes were already made and omit the caret (^) on the system variables. Specifically, our system will be represented by:

$$I' = I.H(I, S) - (\gamma + \mu).I \tag{12}$$

$$S' = -I.H(I, S) + \gamma.I + B(N) - \mu.S \tag{13}$$

$$S + I = 1$$

The following lemma of enough interest, is enunciate it in order to show that the relevant SIS dynamic model is concentrated in the first quadrant bounded subset of  $\mathfrak{R}^2$ . This shows that the problem is biologically well posed [19].

Lemma 6.1. The set  $\Omega = \{(I, S) \in \mathfrak{R}^2 / S \geq 0, I \geq 0, S + I \leq 1\}$  is positively invariant under the flow through the induced system (12)-(13).

### 6. SIS Model with $H(I, S) = k.I^{p-1}.S^q$ , and $k > 0, p \geq 1, q > 0$

$$I' = k.I^p S^q - (\gamma + \mu)I = F_1(I, S) \tag{14}$$

$$S' = -k.I^p S^q + \gamma.I + B(N) - \mu S = F_2(I, S) \tag{15}$$

$$S + I = 1$$

In what follows, when we talk about trivial equilibrium,

we refer to the points that are on the boundary of  $\Omega$ . When they are inside  $\Omega$ , we refer to the nontrivial equilibrium.

For  $F_1(I,S) = 0$  y  $F_2(I,S) = 0$ , we find that the only point of trivial equilibrium is  $A_1=(0,1)$  and nontrivial equilibrium points are obtained from the algebraic equation:

$h(I) = p^{-1} \cdot (I-1)^q = C$ , where  $C = (\gamma + \mu)/k, C > 0, 0 < I < 1$ . (see Figure 1)

We note that we can have one or two or any nontrivial critical point, depending on the value of  $C$ .

An analysis is made to determine when we can obtain explicitly, from the equation  $h(I) = C$ , the equilibrium points of the system, as follows:

- a) For  $q = p-1$ , the algebraic equation  $h(I) = C$ , is reduced to  $I \cdot (I-1) = C^{1/q}$ , and the incidence rate on the system would be represented in the form  $H(I,S) = k \cdot I^q \cdot S^q$ . We note that the order of generality is high. In this case, there are two nontrivial critical points as in the case  $p=q=1$ . It can be seen the number of cases to be analyzed.
- b) For  $p = 1, q > 0$ , the algebraic equation reduces to  $(I-1) = C^{1/q}$ , with incidence rate  $H(I,S) = kS^q$ . We note that the degree of generality is very high too, but lower than the previous.

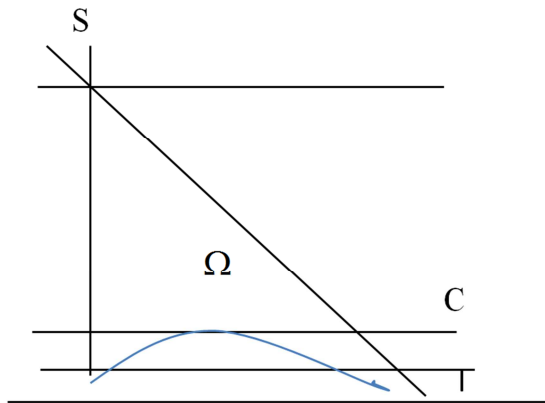


Figure 1. Graph of equilibrium points.

We limit this article to the study case (b) in order to illustrate the asymptotic globality of equilibrium points in each case.

The existence of equilibrium points for SIS Type Model se presenta a continuación.

Proposition 7.1.

- i. If  $C > 1$ , then there is only one equilibrium point, the trivial one  $A_1 = (0,1)$ .
- ii. If  $0 < C < 1$ , then there is a single point of equilibrium, the nontrivial  $A_2 = (I^*, S^*)$ .

### 7. Local Dynamic for SIS Type Model, When $p = 1, q > 0$

Theorem 8.1.

- i. If  $C > 1$ , then the point  $A_1$  is locally asymptotically stable.
- ii. If  $0 < C < 1$ , then the point  $A_2$  is locally asymptotically

stable.

Proof. Let us consider the linearized system  $Y' = (F_x(A_i)) \cdot Y$  with  $i = 1, 2$ ; where  $A_i$  denotes equilibrium points system.

(12), (13) y  $F = (F_1, F_2)^T, X = (I, S)^T$ .

It is easy to see that:

$$F_x = \begin{bmatrix} kS^q - (\gamma + \mu) & kqIS^{q-1} \\ -kS^q + \gamma & -kqIS^{q-1} - \mu \end{bmatrix}$$

$$F_x(0,1) = \begin{bmatrix} k - (\gamma + \mu) & 0 \\ -k + \gamma & -\mu \end{bmatrix}$$

We note that the eigenvalues are:  $\lambda_1 = k - (\gamma + \mu)$  y  $\lambda_2 = -\mu < 0$

So, if  $\lambda_1 < 0$ , then  $A_1$  is locally asymptotically stable, if  $\lambda_1 > 0$ , then  $A_1$  is a saddle point.

On the other hand,

$$F_x(A_2) = \begin{bmatrix} kC - (\gamma + \mu) & kqI^* \frac{C}{S^*} \\ -kC + \gamma & -kqI^* \frac{C}{S^*} - \mu \end{bmatrix}, \text{ and remember}$$

that  $(S^*)^q = C$ .

In this case we have:

$$\Delta_1 = (\mu - kC) + kqI^* \frac{C}{S^*} + \gamma + \mu = a_1 > 0, \text{ if } (\mu - kC) > 0$$

$$\Delta_2 = a_1 a_2 > 0, \text{ with } a_2 = (\mu - kC) \left( kqI^* \frac{C}{S^*} + \mu \right) + \mu\gamma + (kC)^2 q \frac{I^*}{S^*} > 0$$

Then,  $A_2$  is locally asymptotically stable (Theorem 2.1).

### 8. Existence of Periodic Orbits for a SIS Type Model

Theorem 9.1. The SIS type model for the case  $p = 1, q > 0$  does not have periodic orbits.

Proof. Applying the Dulac's theorem (Theorem 4.1) and observing the shape of the system, we notice that we must find a function  $B(I, S) = F \cdot S^q$ , with arbitrary  $x$  and  $y$ , for which the function  $\phi$ , described in the theorem, preserve the sign in  $\Omega$ .

Doing some calculations and taking the values  $x=-I$  and  $y=0$ , we find that the shape of  $\phi$  is:

$$\phi(I, S) = -(kS^q I^{-2} + qkI^{-1} S^{q-1} + \mu I^{-1}) < 0,$$

which is negative in  $\Omega$ . Therefore, the SIS model, in this case, does not support periodic orbits.

### 9. Global Dynamics Equilibrium Points for a SIS Type Model When $p = 1, q > 0$

Theorem 10.1

- i. If  $\frac{\mu}{k} > C > I$ , then the point  $A_1$  is globally asymptotically stable.
- ii. If  $0 < C < \frac{\mu}{k} < I$ , then the point  $A_2$  is globally asymptotically stable.

Proof. To prove (i) is sufficient to see that the function  $\phi(t) = A_1$ , for all  $t$  in  $\mathbb{R}$ , is globally asymptotically stable.

That is to say:

1.  $T$  is stable.
2. For all  $p$  en  $\dot{\Omega}$ ,  $\Psi(t, p) \rightarrow \phi(t), t \rightarrow \infty$ , where  $\Psi(., p)$ , denotes the unique solution of the system (10), (11), such that  $\Psi(0, p) = p$ .

Suppose that  $\phi$  is not globally asymptotically stable (by reduction to the absurd). Then, it exists  $p^*$  in  $\dot{\Omega}$ , such that  $\Psi(t, p) \rightarrow \phi(t), t \rightarrow +\infty$ . Let us denote by  $\gamma_{p^*}$ , the periodic orbit corresponding to the solution  $\Psi(., p^*)$  (theorem of Poincare-Bendixson [20]). As  $\Omega$  is compact and positively invariant, then  $\gamma_{p^*}^+$  is bounded and, therefore,  $\omega_{p^*} \neq \phi$  compact, connected and

$dist(\Psi(t, p^*), \omega_{p^*}) \rightarrow 0, t \rightarrow +\infty$ . It is clear that  $A_1$  does not belong to  $\omega_{p^*}$ , and by the theorem of Poincare-Bendixson [20]  $\omega_{p^*}$  is a periodic orbit system (14), (15); on the other hand, according to Dulac's theorem,  $\Omega$  has no periodic orbits, which is a contradiction.

To prove part (ii), we proceed similarly; ie, we suppose that  $A_2$  is not globally asymptotically stable. That is, there exists  $p^*$  in  $\Omega$ , and a solution  $\Psi(., p^*)$  with  $\Psi(0, p^*) = p^*$ , such that  $\Psi(t, p) \rightarrow \phi(t) = A_2, t \rightarrow +\infty$ . Let  $\gamma_{p^*}$  be the orbit corresponding to the solution  $\Psi(0, p^*)$ . As  $\Omega$  is compact and positively invariant, then  $\omega_{p^*} \neq \phi$  is compact, connected and  $dist(\Psi(t, p^*), \omega_{p^*}) \rightarrow 0, t \rightarrow +\infty$ .

Again, by Bendixson theorem  $\omega_{p^*}$  is a periodic orbit in  $\Omega$ , for which  $A_2$  does not belong to  $\omega_{p^*}$ , by hypothesis [20]. By Dulac's theorem, as in  $\Omega$  there are no periodic orbits, and this is a contradiction.

## 10. Conclusions

This article presents a new version of the test of the Global Stability of Critical Points for Type SIS Epidemiological Model, through the Dulac criteria, via reduction to the absurd. It can be noted that this problem can be raised discreetly or for equations in timescales. It should be noted that epidemiological models are made up of a wide variety of problems and the aforementioned approaches can generate a wide range of publications, this of course gives rise to a considerable number of articles, the results of which can be obtained with new and possibly independent techniques of the one used in this article, which is the subject of other studies.

For the construction of models that represent the majority of sexually transmitted diseases (STDs), the SIS model is very useful; since only a small number of STDs confer immunity after infection; that is, individuals, for the most

part, are susceptible to infection again.

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