

Strongly \mathcal{P} -projective Modules and \mathcal{P} -projective Complexes

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Abstract: In this paper we first study the properties of Strongly \mathcal{P} -projective modules, and obtain some equivalent conclusions about Strongly \mathcal{P} -projective modules, it is proved that a finitely generated right R -module N is strongly \mathcal{P} -projective if and only if $Ext^i(N, R) = 0$ for all $i \geq 1$ over left noetherian and right perfect ring, a \mathcal{P} -projective right R -module N is strongly \mathcal{P} -projective if and only if the first syzygy of N is strongly \mathcal{P} -projective. Then we extend the notion of \mathcal{P} -projective modules to that of \mathcal{P} -projective complexes. We study the relationships between \mathcal{P} -projective complexes and \mathcal{P} -projective modules, it is proved that a complex C is \mathcal{P} -projective if and only if every C^i is \mathcal{P} -projective for every integer i if and only if $Ext^1(C, P) = 0$ for every projective complex P if and only if for every exact sequence $0 \rightarrow A \rightarrow P \rightarrow C \rightarrow 0$ with P projective, $A \rightarrow P$ is a projective preenvelope of A . Some characterizations of \mathcal{P} -projective complexes also obtained.

Keywords: Strongly \mathcal{P} -projective Module, \mathcal{P} -projective Module, \mathcal{P} -projective Complex

1. Introduction

Throughout this paper, R is an associative ring with identity. By module we mean right R -module. \mathcal{C} will be an abelian category of complexes of right R -modules. This category has enough projectives and injectives. This can be seen from the fact that any complex of the form

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \dots$$

with M projective (injective) is projective (injective). For objects C and D of \mathcal{C} , $Hom(C, D)$ is the abelian group of morphisms from C to D in \mathcal{C} and $Ext^i(C, D)$ for $i \geq 0$ will denote the groups we get from the right derived functor of Hom .

In this paper, a complex

$$\dots \longrightarrow C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \dots$$

will be denoted C . We will use subscripts to distinguish complexes. So if $\{C^i\}_{i \in I}$ is a family of complexes, C_i will

be

$$\dots \longrightarrow C_i^{-1} \xrightarrow{\delta^{-1}} C_i^0 \xrightarrow{\delta^0} C_i^1 \xrightarrow{\delta^1} \dots$$

Given a module M , we will denote by \overline{M} the complex

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \dots$$

with the M in the -1 and 0th position. Also we mean by \underline{M} the complex with M in the 0th place and 0 in the other places. Given a complex C and an integer m , $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m \delta^{m+n}$. If C is a complex we let $Z(C)$ and $B(C)$ be the subcomplex of cycles and boundaries of C and we let $H(C) = Z(C)/B(C)$.

If $f : C \rightarrow D$ is a map of complexes, we can form $M(f)$, the mapping cone of f , $M(f)$ is a complex such that $M(f)^n = D^n \oplus C^{n+1}$ is mapped to $(\delta^n(x) + f(x), -\delta^{n+1}(y))$. It is easy to check that there is an exact sequence of complexes $0 \rightarrow D \rightarrow M(f) \rightarrow C[1] \rightarrow 0$.

$Ext^i(C, D)$ is the complex

$$\dots \longrightarrow Ext^i(C, D[n-1]) \longrightarrow Ext^i(C, D[n]) \longrightarrow Ext^i(C, D[n+1]) \longrightarrow \dots,$$

With boundary operator induced by the boundary operator of $D[9]$. let \mathcal{F} be a class of objects of abelian category \mathcal{C} . We denote by ${}^\perp\mathcal{F}$ the left orthogonal class of \mathcal{F} , where ${}^\perp\mathcal{F} = \{G : Ext^1(G, F) = 0, \text{ for all } F \in \mathcal{F}\}$, and denote by \mathcal{F}^\perp the right orthogonal class of \mathcal{F} , where $\mathcal{F}^\perp = \{G : Ext^1(F, G) = 0, \text{ for all } F \in \mathcal{F}\}$.

Let C be a object of \mathcal{C} , Recalled that a morphism $f : C \rightarrow F$ with $F \in \mathcal{F}$ is called an \mathcal{F} -preenvelope of C if for any morphism $g : C \rightarrow F'$ with $F' \in \mathcal{F}$, there is a morphism $\theta : F \rightarrow F'$ such that $\theta f = g$, moreover, when $F' = F$ and $g = f$ the only such θ are automorphisms of F , then $f : C \rightarrow F$ is called an \mathcal{F} -envelope of C , A monomorphism $f : C \rightarrow F$ is said to be a special \mathcal{F} -preenvelope of C if $coker(f) \in {}^\perp\mathcal{F}$. Dually, we have the concepts of (special) \mathcal{F} -precover and \mathcal{F} -cover. A pair $(\mathcal{F}, \mathcal{G})$ is called a cotorsion theory, if $\mathcal{F}^\perp = \mathcal{G}$ and ${}^\perp\mathcal{G} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{G})$ is called perfect if every object of \mathcal{C} has a \mathcal{F} -cover and a \mathcal{G} -envelope. A cotorsion theory $(\mathcal{F}, \mathcal{G})$ is called hereditary if \mathcal{G} is resolving.

Recalled that a complex P is projective if and only if it is exact and every $Ker\delta^i$ is a projective module for all $i \in \mathbb{Z}$, a complex E is called a $\#$ -projective complex if all terms E^i are projective.

A module M is said to be \mathcal{P} -projective if it is a coker of a projective preenvelope [3]. These modules were discovered when studying projective (pre)envelopes. We note that the notion of \mathcal{P} -projectivity is dual to that of copure injectiveness defined by Enochs and Jenda in [4].

A module M is said to be copure flat if it is flat with respect to the exact sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$ with B/A injective[5]. It is easy to see that M is copure flat if and only if $Tor_1(M, E) = 0$ for all injective right R -module. We will say that M is strongly copure flat if $Tor_i(M, E) = 0$ for all injective right R -module and all $i \geq 1$.

In section 2, we introduce the strongly \mathcal{P} -projective R -module, some characterizations of strongly \mathcal{P} -projective R -module are given. We also study \mathcal{P} -projective dimensions which is based on a similar idea due to [5].

The main purpose of section 3 is to extend the notions of \mathcal{P} -projective modules to that of copure injective \mathcal{P} -projective complexes . A complex C is said to be \mathcal{P} -projective if given any morphism $f : A \rightarrow B \rightarrow 0$ with $ker f$ projective and any morphism $g : C \rightarrow B$, there exists a homomorphism $h : C \rightarrow A$ such that the following diagram commutative

$$\begin{array}{ccc} C & & \\ | & \searrow g & \\ | & h & \\ \downarrow & & \\ A & \xrightarrow{f} & B \longrightarrow 0. \end{array} \quad (1)$$

We first obtain a number of characterizations of \mathcal{P} -projectivity of complexes. Then It is natural to consider the relationships of \mathcal{P} -projectivity of a complex C and \mathcal{P} -projectivity of all R -modules C^i for $i \in \mathbb{Z}$.

We prove that a complex C is \mathcal{P} -projective if and only if every C^i is \mathcal{P} -projective for $i \in \mathbb{Z}$. Some characterizations of \mathcal{P} -projective complexes also obtained.

2. Strongly \mathcal{P} -projective Modules

In this section we introduce the definitions of strongly \mathcal{P} -projective modules, and give some characterizations of strongly \mathcal{P} -projective modules.

A right R -module N is said to be \mathcal{P} -projective if it is a coker of projective (pre)envelope [3]. A right R -module N is \mathcal{P} -projective if and only if $Ext^1(N, P) = 0$ for any projective right R -module P . We shall say a right R -module N is strongly \mathcal{P} -projective if $Ext^i(N, P) = 0$ for any projective right R -module P and all $i \geq 1$.

Remark 2.1 (1)Projective module \Rightarrow Gorenstein projective module \Rightarrow (strongly) \mathcal{P} -projective module.

(2) The class of (strongly) \mathcal{P} -projective modules is closed under extensions, direct sums and direct summands.

(3) It is easy to see that N is \mathcal{P} -projective if and only if given any homomorphism $f : A \rightarrow B \rightarrow 0$ with $ker f$ projective and any homomorphism $g : N \rightarrow B$, there exists a homomorphism $h : N \rightarrow A$ such that the following diagram commutative

$$\begin{array}{ccc} N & & \\ | & \searrow g & \\ | & h & \\ \downarrow & & \\ A & \xrightarrow{f} & B \longrightarrow 0. \end{array} \quad (2)$$

Proposition 2.2 The following are equivalent for a right R -module N :

- (1) N is projective.
- (2) N is \mathcal{P} -projective and $pd(N) \leq 1$.
- (3) N is strongly \mathcal{P} -projective and $pd(N) \leq 1$.

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (1) Let N be a \mathcal{P} -projective module and $pd(N) \leq 1$. Then there exists an exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$. Note that K is projective since $pd(N) \leq 1$. So $Ext^1(N, K) = 0$, the above exact sequence splits. Thus N is projective.

(1) \Rightarrow (3) is trivial.

(3) \Rightarrow (1) is similar to (2) \Rightarrow (1).

Proposition 2.3 Let R be a left coherent and right perfect ring. Then the following results are true:

(1) Every (strongly) \mathcal{P} -projective right R -module is (strongly) copure flat.

(2)Every finitely presented (strongly) copure flat right R -module is (strongly) \mathcal{P} -projective.

Proof. (1) Let E be an injective left R -module, then E^+ is flat since R is left coherent ring, and so E^+ is projective since R is right perfect ring. Thus (1) follows from the standard isomorphism $Ext^1(N, M^+) \cong Tor_1(N, M)^+$, where N is any left R -module and M is any right R -module.

(2) Let N be a finitely presented (strongly) copure flat right R -module. Then there exists a standard isomorphism $Ext^i(N, P)^+ \cong Tor_i(M, P^+)$. Note that any projective right R -module P is flat since R is a right perfect ring, and so P^+ is injective. Thus the result holds.

Proposition 2.4 Let R be a left noetherian and right perfect ring. Then a finitely generated right R -module N is strongly \mathcal{P} -projective if and only if $Ext^i(N, R) = 0$ for all $i \geq 1$.

Proof. If N is strongly \mathcal{P} -projective, then $Ext^i(N, R) = 0$ for all $i \geq 1$ since R is projective as a right R -module. Conversely, Let P be a projective right R -module, P is flat since R is a right perfect ring. By Theorem 4.34 of [11], P is a direct limit of finitely generated free modules, $Ext^i(N, P) = 0$ by Lemma 3.1.16 of [6] since R is left noetherian and N is finitely generated.

Corollary 2.5 Let R be a left noetherian and right perfect ring. Then a finitely generated right R -module N is \mathcal{P} -projective if and only if $Ext^1(N, R) = 0$.

Let $\mathcal{P}_{\mathcal{P}}$ (resp., $S\mathcal{P}_{\mathcal{P}}$) denote the class of \mathcal{P} -projective (resp., strongly \mathcal{P} -projective) right R -modules. *Proposition 2.6* Let R be a left coherent and right perfect ring, then:

- (1) $(\mathcal{P}_{\mathcal{P}}, \mathcal{P}_{\mathcal{P}}^{\perp})$ is a perfect cotorsion theory.
- (2) $(S\mathcal{P}_{\mathcal{P}}, S\mathcal{P}_{\mathcal{P}}^{\perp})$ is a perfect hereditary cotorsion theory.

Proof. Since every projective right R -module P is pure injective over a left coherent and right perfect ring, $(\mathcal{P}_{\mathcal{P}}, \mathcal{P}_{\mathcal{P}}^{\perp})$ is a perfect cotorsion theory by Theorem 2.8 of [14], and $(S\mathcal{P}_{\mathcal{P}}, S\mathcal{P}_{\mathcal{P}}^{\perp})$ is a perfect cotorsion theory by Corollary 3.2.12 of [10].

Let \mathcal{P} denote the class of projective right R -module. We have the following : *Proposition 2.7* The following are equivalent:

- (1) R is a QF ring.
- (2) Every module is \mathcal{P} -projective.
- (3) Every quotient of a \mathcal{P} -projective module is \mathcal{P} -projective.
- (4) $(\mathcal{P}_{\mathcal{P}}, \mathcal{P})$ is a cotorsion theory.

Proof. (1) \Leftrightarrow (2) follows from the fact that R is a QF ring if and only if every projective module is injective (we also can see Remark 2. 3 of [3]).

(2) \Rightarrow (3) is trivial.

(3) \Rightarrow (2) We simply note that every module is a quotient of a \mathcal{P} -projective module since every projective module is \mathcal{P} -projective.

(1) \Rightarrow (4) is clear.

(4) \Rightarrow (1) R is a QF ring since every injective module is projective by (4).

An exact sequence $\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow N \rightarrow 0$ where $A_0 \rightarrow N, A_1 \rightarrow \ker(A_0 \rightarrow N), A_{n+1} \rightarrow \ker(A_n \rightarrow A_{n-1})$ are strongly \mathcal{P} -projective precovers is called a \mathcal{P} -projective resolution of N . If there exists a \mathcal{P} -projective resolution $0 \rightarrow A_n \rightarrow \cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow N \rightarrow 0$, we say that N has \mathcal{P} -projective dimension $(\mathcal{P}pd) \leq n$. Let $\mathcal{P}pd(R) = \sup\{\mathcal{P}pd(N) \mid N \text{ is a right } R\text{-module}\}$, we call $\mathcal{P}pd(R)$ the \mathcal{P} -projective dimension of R .

Proposition 2.8 Let R be a left coherent and right perfect ring. Then the following are equivalent for a right R -module:

- (1) $\mathcal{P}pd(N) \leq n$.

(2) $Ext^{n+i}(N, P) = 0$ for all projective right R -module P and all $i \geq 1$.

(3) Every n th syzygy of N is strongly \mathcal{P} -projective.

Proof. (1) \Leftrightarrow (2) Let $0 \rightarrow K \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow N \rightarrow 0$ be an exact sequence with $A_0, A_1, A_2, \dots, A_{n-1}$ strongly \mathcal{P} -projective. Then $Ext^i(K, P) \cong Ext^{n+i}(N, P)$. Thus the result follows.

(2) \Leftrightarrow (3) is similar to (1) \Leftrightarrow (2).

Remark 2.9 We note that the \mathcal{P} -projective dimension of a right R -module N can be considered as the largest positive integer n such that $Ext^n(N, P) \neq 0$ for some projective module P . Taking this as a definition of \mathcal{P} -projective dimension, we may drop the left coherent and right perfect conditions in Proposition above.

Corollary 2.10 Let R be a ring. Then a \mathcal{P} -projective right R -module N is strongly \mathcal{P} -projective if and only if the first syzygy of N is strongly \mathcal{P} -projective. *Corollary 2.11* if $pd(N) < \infty$, then $\mathcal{P}pd(N) = pd(N)$.

Proof. $\mathcal{P}pd(N) \leq pd(N)$ follows from Proposition 2.8 since projective modules are \mathcal{P} -projective. Suppose $pd(N) = n$, then there exists a right R -module A such that $Ext^n(N, A) \neq 0$. For right R -module A , there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$ with P projective. Applying $Hom(N, -)$ to this exact sequence, we get a long exact sequence $\cdots \rightarrow Ext^n(N, P) \rightarrow Ext^n(N, A) \rightarrow Ext^{n+1}(N, K) \rightarrow \cdots$, then $Ext^{n+1}(N, K) = 0$ since $pd(N) = n$. But $Ext^n(N, A) \neq 0$, so $Ext^n(N, P) \neq 0$. Thus $\mathcal{P}pd(N) \geq n = pd(N)$ by Remark 2.9. \square

Proposition 2.12 Let R be a left noetherian and right perfect ring, N a finitely generated right R -module and n a nonnegative integer. Then $\mathcal{P}pd(N) \leq n$ if and only if $Ext^{n+i}(N, R) = 0$ for all $i \geq 1$.

Proof. Similar to the proof of Proposition 2.4.

The following proposition is Dual to the equivalences (1)-(3) of [5].

Proposition 2.13 The following are equivalent for a left and right noetherian ring R :

- (1) R is n -Gorenstein.
- (2) $\mathcal{P}pd(N) \leq n$ for all R -modules (left and right) N .
- (3) Every n th syzygy of N is strongly \mathcal{P} -projective.

Corollary 2.14 The following are equivalent for a two-sided noetherian and perfect ring R :

- (1) R is 1-Gorenstein.
- (2) $\mathcal{P}pd(N) \leq 1$ for all R -module (left and right) N .
- (3) Every \mathcal{P} -projective R -module (left and right) is strongly \mathcal{P} -projective.

(4) Every submodule of a strongly \mathcal{P} -projective R -module (left and right) is strongly \mathcal{P} -projective.

Proof. (1) \Leftrightarrow (2) follows from the Proposition above.

(2) \Rightarrow (3) Let N be a \mathcal{P} -projective R -module. Then the first syzygy of N is strongly \mathcal{P} -projective by Proposition 2.8, and so N is strongly \mathcal{P} -projective by Corollary 2.10.

(3) \Rightarrow (2) Let N be an R -module. Then there exists an exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with P projective. Let $K \rightarrow P'$ be a projective preenvelope which is monic since K is a submodule of projective. Note that $L = \text{coker}(K \rightarrow$

P' is \mathcal{P} -projective by the definition of \mathcal{P} -projective modules, and so L is strongly \mathcal{P} -projective by (3). Hence K is strongly \mathcal{P} -projective, as desired.

(2) \Rightarrow (4) Let M be a submodule of a strongly \mathcal{P} -projective R -module N . Applying the functor $Hom(-, P)$ to the exact sequence $0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$, we obtain an exact sequence $0 = Ext^i(N, P) \rightarrow Ext^i(M, P) \rightarrow Ext^{i+1}$ for all $i \geq 1$. But the last term is zero by (2), so $Ext^i(M, P) = 0$ for all $i \geq 1$, as desired.

(4) \Rightarrow (2) is obvious.

3. \mathcal{P} -projective Complexes

Definition 3.1 A complex C is said to be \mathcal{P} -projective if given any morphism $f : A \rightarrow B \rightarrow 0$ with $ker f$ projective and any morphism $g : C \rightarrow B$, there exists a morphism $h : C \rightarrow A$ such that the following diagram commutative

$$\begin{array}{ccc}
 C & & \\
 \downarrow & \searrow g & \\
 A & \xrightarrow{f} & B \longrightarrow 0 \\
 \downarrow h & & \\
 A & &
 \end{array} \quad (3)$$

Proposition 3.2 The following are equivalent for a complex C :

- (1) C is \mathcal{P} -projective.
- (2) $Ext^1(C, P) = 0$ for every projective complex.
- (3) For every exact sequence $0 \rightarrow A \rightarrow P \rightarrow C \rightarrow 0$ with P projective, $A \rightarrow P$ is a projective preenvelope of A .
- (4) C is a coker of a projective preenvelope.
- (5) For any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and any projective P , the sequence $Hom(B, P) \rightarrow Hom(A, P) \rightarrow 0$ is exact.
- (6) $Hom(C, -)$ exact any short exact sequence $0 \rightarrow P \rightarrow A \rightarrow B \rightarrow 0$ with P projective.

Proof. (1) \Rightarrow (2) Let $0 \rightarrow P \rightarrow E \rightarrow N \rightarrow 0$ be a short injective resolution of projective complex P . Then we have a long exact sequence $0 \rightarrow Hom(C, P) \rightarrow Hom(C, E) \rightarrow Hom(C, N) \rightarrow Ext^1(C, P) \rightarrow Ext^1(C, E) = 0$. But $0 \rightarrow Hom(C, P) \rightarrow Hom(C, E) \rightarrow Hom(C, N) \rightarrow 0$ is exact by definition of \mathcal{P} -projective modules. So $Ext^1(C, P) = 0$.

(2) \Rightarrow (1) is straightforward.

(2) \Rightarrow (3) Let P' be projective, then there is an exact sequence $0 \rightarrow Hom(P, P') \rightarrow Hom(A, P') \rightarrow Ext^1(C, P') = 0$ by (2), so $A \rightarrow P$ is a projective preenvelope.

(3) \Rightarrow (4) is obvious.

(4) \Rightarrow (2) Let C be a coker of a projective preenvelope $f : A \rightarrow P$ with P projective. then there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$ with $K = im(f)$. For each projective module P' we have a long exact sequence $Hom(P, P') \rightarrow Hom(K, P') \rightarrow Ext^1(C, P') \rightarrow Ext^1(P, P') = 0$. But $Hom(P, P') \rightarrow Hom(K, P') \rightarrow 0$ is exact by (4). Hence $Ext^1(C, P') = 0$.

(2) \Rightarrow (5) is obvious.

(5) \Rightarrow (2) There exists an exact sequence $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$ with P projective. Then for any projective P' , we have the exact sequence $Hom(P, P') \rightarrow Hom(K, P') \rightarrow Ext^1(C, P') \rightarrow Ext^1(P, P') = 0$. But $Hom(P, P') \rightarrow Hom(K, P') \rightarrow 0$ is exact, so $Ext^1(C, P') = 0$.

(2) \Rightarrow (6) is obvious.

(6) \Rightarrow (2) Let P be a projective. Then there is an exact sequence $0 \rightarrow P \rightarrow E \rightarrow L \rightarrow 0$, and we have a long exact $0 \rightarrow Hom(C, P) \rightarrow Hom(C, E) \rightarrow Hom(C, L) \rightarrow Ext^1(C, P) \rightarrow Ext^1(C, E) = 0$. But $0 \rightarrow Hom(C, P) \rightarrow Hom(C, E) \rightarrow Hom(C, L) \rightarrow 0$ is exact by (6), so $Ext^1(C, P) = 0$. \square Remark 3.3 (1) Projective complex \Rightarrow Gorenstein projective complex \Rightarrow \mathcal{P} -projective complex.

(2) The class of \mathcal{P} -projective complexes is closed under extensions, direct sums and direct summands.

(3) If C is a \mathcal{P} -projective complex, then $C[i]$ is also a \mathcal{P} -projective complex for all $i \in \mathbb{Z}$.

It is natural to consider the relationships of \mathcal{P} -projectivity of a complex C and \mathcal{P} -projectivity of all modules C^i for $i \in \mathbb{Z}$. Next we give the following results.

Theorem 3.4 The following are equivalent for a complex C :

- (1) C is \mathcal{P} -projective.
- (2) $Ext^1(C, \overline{P}[n]) = 0$ for any projective module P .
- (3) Every C^i is \mathcal{P} -projective for $i \in \mathbb{Z}$ and $Hom(C, P)$ is exact for each projective complex P .
- (4) Every C^i is \mathcal{P} -projective for $i \in \mathbb{Z}$.

Proof. (1) \Rightarrow (2) For any projective module P , $\overline{P}[n]$ is projective, so $Ext^1(C, \overline{P}[n]) = 0$ by Proposition 3.2.

(2) \Rightarrow (1) Note that any projective complex is the direct product of complexes

$$\overline{P}_i[i] = \dots \longrightarrow 0 \longrightarrow P_i \xrightarrow{id} P_i \longrightarrow 0 \longrightarrow \dots$$

with P_i projective. It is easy to check that $Ext^1(C, P) = 0$ for any projective complex P . So C is \mathcal{P} -projective.

(1) \Rightarrow (3) Let $0 \rightarrow P \rightarrow A \rightarrow C^i \rightarrow 0$ be any exact sequence of modules with P projective. Then we get the following pullback diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & C^{i-2} & \xrightarrow{id} & C^{i-2} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P & \longrightarrow & B & \longrightarrow & C^{i-1} \longrightarrow 0 \\
 & & \downarrow id & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P & \longrightarrow & A & \longrightarrow & C^i \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & C^{i+1} & \xrightarrow{id} & C^{i+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array} \quad (4)$$

Then $Ext^1(C, \overline{P}[-i]) = 0$ since $\overline{P}[-i]$ is projective. So the above exact sequence of complexes is split. Thus the exact sequence $0 \rightarrow P \rightarrow A \rightarrow C^i \rightarrow 0$ is split, $Ext^1(C^i, P) = 0$, so C^i is \mathcal{P} -projective. For any projective complex P , the short exact sequence of complexes $0 \rightarrow P[n] \rightarrow M(f) \rightarrow C[1] \rightarrow 0$ is split for any $n \in \mathbb{Z}$ and any map $f : C \rightarrow p[n]$ by (1). So f is homotopic to zero by lemma 2.3.2 of [9]. It is easy to check that $Hom(C, P)$ is exact.

(3) \Rightarrow (4) is obvious.

(4) \Rightarrow (2) Let

$$0 \longrightarrow \overline{P}[n] \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

be any exact sequence of complexes, and we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & 0 & \longrightarrow & A^{-n-2} & \xrightarrow{g^{-n-2}} & C^{-n-2} \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow \delta_A^{-n-2} & & \downarrow \delta_C^{-n-2} \\
 0 & \longrightarrow & P & \xrightarrow{f^{-n-1}} & A^{-n-1} & \xrightarrow{g^{-n-1}} & C^{-n-1} \longrightarrow 0 \\
 \downarrow id & & \downarrow & & \downarrow \delta_A^{-n-1} & & \downarrow \delta_C^{-n-1} \\
 0 & \longrightarrow & P & \xrightarrow{f^{-n}} & A^{-n} & \xrightarrow{g^{-n}} & C^{-n} \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow \delta_A^{-n} & & \downarrow \delta_C^{-n} \\
 0 & \longrightarrow & 0 & \longrightarrow & A^{-n+1} & \xrightarrow{g^{-n+1}} & C^{-n+1} \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow \delta_A^{-n+1} & & \downarrow \delta_C^{-n+1} \\
 \vdots & & \vdots & & \vdots & & \\
 \end{array} \tag{5}$$

Note that $Ext^1(C^{-n}, P) = 0$ since C^{-n} is \mathcal{P} -projective. So $f^{-n} : P \rightarrow A^{-n}$ splits, there exists a morphism $h^{-n} : A^{-n} \rightarrow P$ such that $h^{-n} f^{-n} = id_P$. We define $h^{-n-1} : A^{-n-1} \rightarrow P$ as $h^{-n-1} = h^{-n} \delta_A^{-n-1}$ and $h^i = 0$ for $i \neq -n, -n - 1$. Then $h = \{h^i\}_{i \in \mathbb{Z}}$ is the morphism from A to $\overline{P}[n]$ such that $hf = id_{\overline{P}[n]}$. So the sequence

$$0 \longrightarrow \overline{P}[n] \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

splits, and so $Ext^1(C, \overline{P}[n]) = 0$.

Corollary 3.5 The following are equivalent for a module M :

- (1) M is \mathcal{P} -projective.
- (2) $\underline{M}[n]$ is \mathcal{P} -projective for all $n \in \mathbb{Z}$.
- (3) $\overline{M}[n]$ is \mathcal{P} -projective for all $n \in \mathbb{Z}$.

Corollary 3.6 The following are equivalent for a complex C :

- (1) C is \mathcal{P} -projective.
- (2) Every exact sequence $0 \rightarrow P \rightarrow A \rightarrow C \rightarrow 0$ with P projective splits.

(3) $Ext(C, P) = 0$ for any projective complex P .

Recalled that a complex C is said to be $\#$ -projective if every C^i is projective for all $i \in \mathbb{Z}$. We have the following Proposition.

Proposition 3.7 The following are equivalent for a complex C :

- (1) C is \mathcal{P} -projective.
- (2) $Ext^1(C, P) = 0$ for every $Hom(Q, -)$ exact bounded $\#$ -projective complex P whenever Q is a \mathcal{P} -projective module.
- (3) $Ext^1(C, P) = 0$ for every $Hom(Q, -)$ exact bounded above $\#$ -projective complex P whenever Q is a \mathcal{P} -projective module.

Proof.(1) \Rightarrow (3) Let P be a $Hom(Q, -)$ exact bounded above $\#$ -projective complex whenever Q is a \mathcal{P} -projective module, without loss of generality, we may assume that $P^i = 0$ for $i > 0$. Let

$$0 \longrightarrow P \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

be any exact sequence of complexes. Then we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P^{-2} & \xrightarrow{f^{-2}} & A^{-2} & \xrightarrow{g^{-2}} & C^{-2} \longrightarrow 0 \\
 \downarrow \delta_P^{-2} & & \downarrow & & \downarrow \delta_A^{-2} & & \downarrow \delta_C^{-2} \\
 0 & \longrightarrow & P^{-1} & \xrightarrow{f^{-1}} & A^{-1} & \xrightarrow{g^{-1}} & C^{-1} \longrightarrow 0 \\
 \downarrow \delta_P^{-1} & & \downarrow & & \downarrow \delta_A^{-1} & & \downarrow \delta_C^{-1} \\
 0 & \longrightarrow & P^0 & \xrightarrow{f^0} & A^0 & \xrightarrow{g^0} & C^0 \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow \delta_A^0 & & \downarrow \delta_C^0 \\
 0 & \longrightarrow & 0 & \longrightarrow & A^1 & \xrightarrow{g^1} & C^1 \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots & & \\
 \end{array} \tag{6}$$

We see that every exact sequence

$$0 \longrightarrow P^i \xrightarrow{f^i} A^i \xrightarrow{g^i} C^i \longrightarrow 0$$

is split since C^i is \mathcal{P} -projective by Theorem 3.4. So there exists $h^i : A^i \rightarrow P^i$ such that $h^i f^i = id_{P^i}$ for all $i \leq 0$. Now Let $\alpha^0 = h^0$.

Since $coker f^{-1} \cong C^{-1}$, it follows that

$$Hom(coker f^{-1}, P^{-1}) \rightarrow Hom(coker f^{-1}, P^0) \rightarrow 0$$

is exact. Note that $(\delta_P^{-1} h^{-1} - \alpha^0 \delta_A^{-1}) f^{-1} = \delta_P^{-1} h^{-1} f^{-1} - \alpha^0 \delta_A^{-1} f^{-1} = \delta_P^{-1} - \alpha^0 f^0 \delta_P^{-1} = \delta_P^{-1} - \delta_P^{-1} = 0$, so $\frac{\delta_P^{-1} h^{-1} - \alpha^0 \delta_A^{-1}}{\delta_P^{-1} h^{-1} - \alpha^0 \delta_A^{-1}} \in Hom(coker f^{-1}, P^0)$. Thus there exists $\gamma^{-1} \in Hom(coker f^{-1}, P^{-1})$ such that $\frac{\delta_P^{-1} h^{-1} - \alpha^0 \delta_A^{-1}}{\delta_P^{-1} h^{-1} - \alpha^0 \delta_A^{-1}} = \delta_P^{-1} \gamma^{-1}$, where $\gamma^{-1} \in Hom(A^{-1}, P^{-1})$ and $\gamma^{-1}(P^{-1}) = 0$. let $\alpha^{-1} = h^{-1} - \gamma^{-1}$, then $\delta_P^{-1} \alpha^{-1} = \delta_P^{-1} h^{-1} - \delta_P^{-1} \gamma^{-1} = \alpha^0 \delta_A^{-1}$, and $\alpha^{-1} f^{-1} = h^{-1} f^{-1} - \gamma^{-1} f^{-1} = h^{-1} f^{-1} = id_{P^{-1}}$ since $\gamma^{-1} f^{-1} = 0$.

It follows that

$$\text{Hom}(\text{coker } f^{-2}, P^{-2}) \rightarrow$$

$$\text{Hom}(\text{coker } f^{-2}, P^0)$$

is exact since $\text{coker } f^{-2} \cong C^{-2}$. Note that $(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})f^{-2} = \delta_P^{-2}h^{-2}f^{-2} - \alpha^{-1}\delta_A^{-2}f^{-2} = \delta_P^{-2} - \alpha^{-1}f^{-1}\delta_P^{-2} = \delta_P^{-2} - \delta_P^{-2} = 0$, then $(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2}) \in \text{Hom}(\text{coker } f^{-2}, P^{-1})$, but $\delta_P^{-1}(\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2}) = 0$, so there exists $\overline{\gamma^{-2}} \in \text{Hom}(\text{coker } f^{-2}, P^{-2})$ such that $\delta_P^{-2}\overline{\gamma^{-2}} = (\delta_P^{-2}h^{-2} - \alpha^{-1}\delta_A^{-2})$, where $\gamma^{-2} \in \text{Hom}(A^{-2}, P^{-2})$ and $\gamma^{-2}(P^{-2}) = 0$. Now let $\alpha^{-2} = h^{-2} - \gamma^{-2}$, then $\delta_P^{-2}\alpha^{-2} = \delta_P^{-2}h^{-2} - \delta_P^{-2}\gamma^{-2} = \alpha^{-1}\delta_A^{-2}$, and $\alpha^{-2}f^{-2} = h^{-2}f^{-2} - \gamma^{-2}f^{-2} = h^{-2}f^{-2} = \text{id}_{P^{-2}}$ since $\gamma^{-2}f^{-2} = 0$.

Similarly, we can obtain $\alpha^{-i} : A^{-i} \rightarrow P^{-i}$ such that $\delta_P^{-i}\alpha^{-i} = \alpha^{-i+1}\delta_A^{-i}$ and $\alpha^{-i}f^{-i} = \text{id}_{P^{-i}}$ for $i \geq 3$. Finally, let $\alpha^n = 0$ for $n \geq 1$. Then we obtain a morphism $\alpha = \{\alpha^i\}_{i \in \mathbb{Z}} : A \rightarrow P$ such that $\alpha f = \text{id}_P$. This implies that the exact sequence

$$0 \longrightarrow P \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$$

splits. Thus $\text{Ext}^1(C, P) = 0$.

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1) Let P be a projective module, $\text{Ext}^1(C, \overline{P}[n]) = 0$ by (2), so C is \mathcal{P} -projective by Theorem 3.4.

Proposition 3.8 The following are equivalent:

(1) R is a QF ring.

(2) Every complex is \mathcal{P} -projective.

(3) Every quotient of a \mathcal{P} -projective complex is \mathcal{P} -projective.

Proof. (1) \Rightarrow (2) Let C be a complex, then C^i is \mathcal{P} -projective by Proposition 2.7. So C is a \mathcal{P} -projective complex by Theorem 3.4.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1) Let M be a quotient of a \mathcal{P} -projective module N . Then \overline{M} is a quotient of a \mathcal{P} -projective complex \overline{N} , so \overline{M} is \mathcal{P} -projective, thus M is \mathcal{P} -projective by Theorem 3.4. So R is a QF ring by Proposition 2.7.

We conclude the paper with the following Remark. *Remark 3.9 (1) let R be a left coherent and right perfect ring. Then every right R -module has a \mathcal{P} -projective cover by Proposition 2.6. So every bounded above complex has a \mathcal{P} -projective precover by Proposition 5.3 of [12] since \mathcal{P} -projective complexes coincide with $\#\mathcal{P}$ -projective complexes by Theorem 3.4.*

(2) *If every complex has a projective envelope and we replace " \mathcal{P} -projective modules" with " \mathcal{P} -projective complexes" in [3], the results still hold by similar proofs.*

4. Conclusion

The relationship between module and complex is an important research content in homological algebra. We find a

new complex called \mathcal{P} -projective complex from \mathcal{P} -projective module, and reveal the relationship between them. Then we study the properties and homology dimension of \mathcal{P} -projective complex in different rings. This kind of work is very meaningful.

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