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Boundedness and Asymptotic Behaviour of the Solutions for a Third- Order Fuzzy Difference Equation

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Abstract: Our aim in this paper is to investigate the dynamics of a third-order fuzzy difference equation. By using new iteration method for the more general nonlinear difference equations and inequality skills as well as a comparison theorem for the fuzzy difference equation, some sufficient conditions which guarantee the existence, unstability and global asymptotic stability of the equilibriums for the nonlinear fuzzy system are obtained. Moreover, some numerical solutions of the equation describing the system are given to verify our theoretical results.

Keywords: Fuzzy Difference Equation, Boundedness, Existence, Uniqueness, Asymptotic Behavior

1. Introduction

Nonlinear difference equation is an important mathematical models which describe the relationship between the real world phenomenons. It not only enriched the theory of mathematics, but also solved the practical problem, such as the fields of the number of population structure analysis, economic, genetic, biology etc. (see, eg., [1-5] and the references therein). In recent years, research on discrete systems has become a hot problem. (see, eg., [6, 7]).

However, the traditional mathematics theories and methods seem to be inadequate in the face of fuzzy phenomenon. Based on this background, the fuzzy difference systems are a powerful tool which can be used to study better some uncertain phenomenons. In recent decades, the fuzzy mathematics theory and its applications have achieved fruitful results. In view of the fact, the fuzzy difference equation system has attracted more and more interest which further enriches the research of the difference system.

In 1998, DeVault et al [8] discussed the existence, boundedness, oscillation behavior of the positive solutions and the global asymptotic behavior of the equilibrium points for the nonlinear difference equation.

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, n = 0,1,\ldots, \tag{1}$$

Where A, x_{-1}, x_0 are positive numbers.

Similarly, In 1998, Papaschinopoulos and Schinas [9] studied the following difference equations.

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, y_{n+1} = A + \frac{x_n}{y_{n-p}}, n = 0,1,\ldots, \tag{2}$$

Where $A, x_{-p}, x_{-p+1}, \ldots, x_0, y_{-q}, y_{-q+1}, \ldots, y_0$ are positive real numbers p, q are positive integers. There are similar conclusions that if $p = q = 1, x_n = y_n, n = -1, 0, \ldots,$ then x_n is both a solution of Eq. (2) and the solution of Eq. (1).

Besides, Zhang et al [10] researched the following nonlinear fuzzy difference equation.

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-1}}, n = 0, 1, 2, \ldots, \tag{3}$$

Where $\{x_n\}$ is a sequence of positive fuzzy number, a, b, A and the initial values x_{-1}, x_0 are positive fuzzy numbers. The existence and boundedness of the positive solutions asymptotic behavior of the equilibrium points for the difference equation are discussed. Moreover, In 2014, Zhang et al [11] continuously proved similar conclusion for the follow first-order fuzzy difference equation

$$x_{n+1}=\frac{A+x_n}{B+x_n}, n=0,1,2,\ldots, \tag{4}$$

Where $\{x_n\}$ is a sequence of positive fuzzy numbers, A,B and the initial value x_0 are positive fuzzy numbers.

More recently, Wang et al [12] investigate the existence and uniqueness of the positive solutions and the asymptotic behavior of the equilibrium points of the following fuzzy difference equation.

$$x_{n+1}=\frac{Ax_{n-1}x_{n-2}}{D+Bx_{n-3}+Cx_{n-4}}, n=0,1,2,\ldots, \tag{5}$$

Where $\{x_n\}$ is a sequence of positive fuzzy numbers, the parameters A,B,C,D and the initial conditions $x_{-4},x_{-3},x_{-2},x_{-1},x_0$ are positive fuzzy numbers. For closely related papers in this research area, see, for example, [13-30] and the references therein.

Motivated by the discussions above, the purpose of this paper is to discuss the existence and uniqueness of the positive solutions and the asymptotic behavior of the equilibrium points for the following third-order fuzzy difference equation.

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_{n-2}x_{n-1}x_n}, n=0,1,2,\ldots, \tag{6}$$

Where $\{x_n\}$ is a sequence of positive fuzzy numbers, the parameters A,B,C and the initial conditions x_{-2},x_{-1},x_0 are positive fuzzy numbers.

This paper is arranged as follows: in Section 2, some definitions and preliminary results are given. The main results and related proofs are obtained in Section 3. Finally, some numerical examples are used to illustrate our theoretical results.

2. Preliminaries and Notations

For the convenience of readers, some definitions and preliminary results related to the theoretical proof of this paper are given，see [31-35].

Definition 2.1 For a set B we denote by $\overline{B}$ the closure of B. We say that a function $A:R\to[0,1]$ is a fuzzy number if the follow conclusions are true:

(i) A is normal, i.e., there exists $x\in R$ such that $A(x)=1$;

(ii) A is a fuzzy convex set, i.e.,

$A(tx_1+(1-t)x_2)\geq\min\{A(x_1),A(x_2)\},\forall t\in[0,1],x_1,x_2\in R;$

(iii) A is upper semicontinuous on R;

(iv) The support of A, i.e., supp $A=\overline{\cup_{\alpha\in(0,1]}[A]_\alpha}=\overline{\{x\in R:A(x)>0\}}$ is compact.

where the $\alpha-cuts$ of A are closed intervals, define as $[A]_\alpha=\{x\in R:A(x)\geq\alpha,\}$ if supp A $\subset(0,\infty)$ then fuzzy number A is obviously positive.

Definition 2.2 Let A,B be fuzzy numbers which satisfy $[A]_\alpha=[A_{l,\alpha},A_{r,\alpha}],[B]_\alpha=[B_{l,\alpha},B_{r,\alpha}],\ \alpha\in(0,1]$, the follow metric is denoted

$$D(A,B)=\sup\max\{\,|A_{l,\alpha}-B_{l,\alpha}|,|A_{r,\alpha}-B_{r,\alpha}|\},$$

Where sup is taken for all $\alpha\in(0,1]$. Then (R_f,D) is a complete metric space. For the convenience of application in the future, we define $\hat{0}\in R_f$ as

$$\hat{0}(x)=\begin{cases}1, x=0,\\ 0, x\neq 0.\end{cases}$$

Thus, $[\hat{0}]_\alpha=[0,0],0<\alpha<1$.

Definition 2.3 Persistence (resp. boundedness) of fuzzy numbers is defined, if there exist positive real number $M(resp.N)$, such that the following conclusions are true $\sup p\, x_n\subset[M,\infty](resp.\sup p\, x_n\subset(0,1]),n=1,2,\ldots,$ where $\{x_n\}$ is a sequence of positive fuzzy numbers.

Further, $\{x_n\}$ is bounded and persistent if there exist positive real numbers $M,N,$ such that $\sup p\, x_n\subset[M,N],n=1,2,\ldots.$.

Lemma 2.1 Let I_x,I_y be some intervals of real numbers and let $f:I_x^{k+1}\times I_y^{l+1}\to I_x$, $g:I_x^{k+1}\times I_y^{l+1}\to I_y$ be continuously differential functions. Then for every set of initial conditions $(x_i,y_j)\in I_x\times I_y,(i=-k,-k+1,\ldots,0,j=-l,-l+1,\ldots,0)$, the following system of difference equations

$$\begin{cases}x_{n+1}=f(x_n,x_{n-1},\ldots,x_{n-k},y_n,y_{n-1},\ldots,y_{n-l}),\\ y_{n+1}=g(x_n,x_{n-1},\ldots,x_{n-k},y_n,y_{n-1},\ldots,y_{n-l}),\end{cases} n=0,1,2,\ldots, \tag{7}$$

Has a unique solution $\{(x_i,y_j)\}_{i=-k,j=-l}^{+\infty,+\infty}$.

Definition 2.4 A point $(\overline{x},\overline{y})\in I_x\times I_y$ is called an equilibrium point of system (7) if $\overline{x}=f(\overline{x},\overline{x},\ldots,\overline{x},\overline{y},\overline{y},\ldots,\overline{y})$, $\overline{y}=g(\overline{x},\overline{x},\ldots,\overline{x},\overline{y},\overline{y},\ldots,\overline{y})$, that is, $(x_n,y_n)=(\overline{x},\overline{y})$ for $n\geq 0$ is the solution of difference system (7), or equivalently, $(\overline{x},\overline{y})$ is a fixed point of the vector map (f,g).

Definition 2.5 Suppose that $(\overline{x},\overline{y})$ be an equilibrium point of the system (7), then we have

(i) $(\bar{x},\bar{y})$ is called locally stable if for every $(\bar{x},\bar{y})$ there exists $\delta>0$ such that for any initial conditions $(x_i,y_i)\in I_x\times I_y (i=-k,\ldots,0, j=-l,\ldots,0)$ with $\sum_{i=-k}^{0}|x_i-\bar{x}|<\delta, \sum_{j=-l}^{0}|y_i-\bar{y}|<\delta,$ we have $|x_n-\bar{x}|<\varepsilon,$ $|y_n-\bar{y}|<\varepsilon$ for any $n>0.$

(ii) $(\bar{x},\bar{y})$ is called attractor if $\lim_{n\to\infty} x_n=\bar{x}, \lim_{n\to\infty} y_n=\bar{y},$ for any initial conditions $(x_i,y_j)\in I_x\times I_y,$ $(i=-k,-k+1,\ldots,0, j=-l,-l+1,\ldots,0).$

(iii) $(\bar{x},\bar{y})$ is called asymptotically stable if it is stable, and $(\bar{x},\bar{y})$ is also attractor.

(iv) $(\bar{x},\bar{y})$ is called unstable if it is not locally stable.

Definition 2.6 Let $(\bar{x},\bar{y})$ be an equilibrium point of the vector map $F=(f,x_n,\ldots,x_{n-k},g,y_n,\ldots,y_{n-l}),$ where f and g are continuously differential functions at $(\bar{x},\bar{y})$ The linearized system of (7) about the equilibrium point $(\bar{x},\bar{y})$ is $X_{n+1}=F(X_n)=F_J\cdot X_n$, where F_J is the Jacobian matrix of the system (7) about $(\bar{x},\bar{y})$ and $X_n=(x_n,\ldots,x_{n-k},y_n,\ldots,y_{n-l})^T.$

Definition 2.7 let p,q,s,t be four nonnegative integers such that $p+q=n, s+t=m,$ splitting $x=(x_1,x_2\ldots,x_n)$ into $x=([x]_p,[x]_q)$ and $y=(y_1,y_2\ldots,y_m)$ into $y=([y]_s,[y]_t),$ where $[x]_\sigma$ denotes a vector with σ -components of x. We say that the function $f=(x_1,x_2\ldots,x_n,y_1,y_2\ldots,y_m)$ possesses a mixed monotone property in subsets $I_x^n\times I_y^m$ of $R^n\times R^m$ if $f=([x]_p,[x]_q,[y]_s,[y]_t)$ is monotone non-decreasing in each component of $([x]_p,[y]_s)$, and is monotone non-increasing in each component of $([x]_q,[y]_t)$ for $(x,y)\in I_x^n\times I_y^m$. In particular, if $q=0, t=0,$ then it is said to be monotone non-decreasing in $I_x^n\times I_y^m$.

Lemma 2.2 Assume that $X_{n+1}=F(X_n), n=0,1,\ldots,$ is a system of difference equations and $\bar{X}$ is the equilibrium point of this system, i.e. $F(\bar{X})=\bar{X}.$ Then we have.

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda|<1$, then $\bar{X}$ is locally asymptotically stable.

(ii) If one of eigenvalues of the Jacobian J_F matrix about $\bar{X}$ has norm greater than one, then $\bar{X}$ is unstable.

3. Main Results

The following lemmas are applied to study the existence and uniqueness of the positive solutions of equations (6).

Lemma 3.1 [36] Let f be a continuous function from $R^+\times R^+\times R^+\to R^+$ into R^+ and A,B,C be fuzzy numbers, then

$$[f(A,B,C)]_\alpha=f([A]_\alpha,[B]_\alpha,[C]_\alpha), \alpha\in(0,1].$$

Lemma 3.2 [37] Let $u\in R_f$ write $[u]_\alpha=[u_{l,\alpha},u_{r,\alpha}], \alpha\in(0,1].$ Then $u_{l,\alpha}$ and $u_{r,\alpha}$ can be regarded as functions on $(0,1]$ which satisfy

(i) $u_{l,\alpha}$ is nondecreasing and left continuous,

(ii) $u_{r,\alpha}$ is nonincreasing and left continuous,

(iii) $u_{l,\alpha}\le u_{r,\alpha}.$

Conversely for any functions $a(\alpha)$ and $b(\alpha)$ defined on $(0,1]$ which satisfy the above (i)-(iii), there exists a unique $u\in R_f$ such that $u(\alpha)=[a(\alpha),b(\alpha)]$ for any $\alpha\in(0,1].$

Theorem 3.1 Consider equation (6), where A,B,C are positive fuzzy numbers. Then for any positive fuzzy numbers x_{-2},x_{-1},x_0, there exist a unique positive solution x_n of Eq. (6).

Proof. Suppose that there exists a sequence of fuzzy numbers $\{x_n\}$ satisfying Eq. (6) with initial conditions x_{-2},x_{-1},x_0. Consider the $\alpha-cuts, \alpha\in(0,1],$

$$\begin{aligned}&[A]_\alpha=[A_{l,\alpha},A_{r,\alpha}],[B]_\alpha=[B_{l,\alpha},B_{r,\alpha}],\\&[C]_\alpha=[C_{l,\alpha},C_{r,\alpha}],[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}], n=-2,-1,\cdots.\end{aligned}\tag{8}$$

The following results are true from (6), (8) and Lemma 3.1

$$[x_{n+1}]_\alpha=[L_{n+1,\alpha},R_{n+1,\alpha}]=\left[\frac{Ax_{n-2}}{B+Cx_{n-2}x_{n-1}x_n}\right]_\alpha=\frac{[Ax_{n-2}]_\alpha}{[B+Cx_{n-2}x_{n-1}x_n]_\alpha}$$

$$=\frac{[A_{l,\alpha},A_{r,\alpha}][L_{n-2,\alpha},R_{n-2,\alpha}]}{[B_{l,\alpha},B_{r,\alpha}]+[C_{l,\alpha},C_{r,\alpha}][L_{n-2,\alpha},R_{n-2,\alpha}][L_{n-1,\alpha},R_{n-1,\alpha}][L_{n,\alpha},R_{n,\alpha}]}$$

$$=\frac{[A_{l,\alpha}L_{n-2,\alpha},A_{r,\alpha}R_{n-2,\alpha}]}{[B_{l,\alpha}+C_{l,\alpha}L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha},B_{r,\alpha}+C_{r,\alpha}R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}]}$$

$$=\left[\frac{A_{l,\alpha}L_{n-2,\alpha}}{B_{r,\alpha}+C_{r,\alpha}R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}},\frac{A_{r,\alpha}R_{n-2,\alpha}}{B_{l,\alpha}+C_{l,\alpha}L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}}\right].$$

Hence, for $\alpha\in(0,1], n=-2,-1,\cdots,$ according to the above result, it follows that

$$\begin{aligned} L_{n+1`,\alpha} &= \frac{A_{l,\alpha}L_{n-2,\alpha}}{B_{r,\alpha}+C_{r,\alpha}R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}}, \\ R_{n+1,\alpha} &= \frac{A_{r,\alpha}R_{n-2,\alpha}}{B_{l,\alpha}+C_{l,\alpha}L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}}. \end{aligned} \tag{9}$$

Then, for any initial conditions $(L_{j,\alpha},R_{j,\alpha}), j=-2,-1,0, \alpha\in(0,1],$ it is obvious from Lemma 2.1 that exists a unique solution $(L_{n,\alpha},R_{n,\alpha})$ of the systems (9).

Conversely, it is proved that the positive solution $\{x_n\}$ of the equation (6) is determined by $[L_{n,\alpha},R_{n,\alpha}],\ \alpha\in(0,1]$ is true, Where $(L_{n,\alpha},R_{n,\alpha})$ is the solution of the system (9) with initial conditions $(L_{j,\alpha},R_{j,\alpha}),\ j=-2,-1,0,$ and $\{x_n\}$ satisfy the following conditions.

$$[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\alpha\in(0,1],n=-2,-1,\cdots. \tag{10}$$

For any $\alpha_1,\alpha_2\in(0,1], \alpha_1<\alpha_2,$ according to Lemma 3.2, the following conclusions can be obtained,

$$\begin{aligned} &0<A_{l,\alpha_1}\le A_{l,\alpha_2}\le A_{r,\alpha_2}\le A_{r,\alpha_1}, 0<B_{l,\alpha_1}\le B_{l,\alpha_2}\le B_{r,\alpha_2}\le B_{r,\alpha_1}, \\ &0<C_{l,\alpha_1}\le C_{l,\alpha_2}\le C_{r,\alpha_2}\le C_{r,\alpha_1}, 0<L_{j,\alpha_1}\le L_{j,\alpha_2}\le R_{j,\alpha_2}\le R_{j,\alpha_1}, j=-2,-1,0. \end{aligned} \tag{11}$$

Where $A,B,C,x_j(j=-2,-1,0)$ are positive fuzzy numbers.

Next, the following conclusions is proved to be true by mathematical induction.

$$0<L_{n,\alpha_1}\le L_{n,\alpha_2}\le R_{n,\alpha_2}\le R_{n,\alpha_1}, n=1,2,\cdots. \tag{12}$$

From (11), the conclusions (12) are true for $n=-2,-1,0.$ Suppose that (12) are true for $n\le k,\ \ k\in\{1,2,\cdots\},$ then from (9)-(12), it follows that for $n=k+1$.

$$\begin{aligned} L_{k+1`,\alpha 1} &= \frac{A_{l,\alpha_1}L_{k-2,\alpha_1}}{B_{r,\alpha_1}+C_{r,\alpha_1}R_{k-2,\alpha_1}R_{k-1,\alpha_1}R_{k,\alpha_1}} \\ &\le \frac{A_{l,\alpha_2}L_{k-2,\alpha_2}}{B_{r,\alpha_2}+C_{r,\alpha_2}R_{k-2,\alpha_2}R_{k-1,\alpha_2}R_{k,\alpha_2}}=L_{k+1,\alpha_2} \\ &\le \frac{A_{r,\alpha_2}R_{k-2,\alpha_2}}{B_{l,\alpha_2}+C_{l,\alpha_2}L_{k-2,\alpha_2}L_{k-1,\alpha_2}L_{k,\alpha_2}}=R_{k+1,\alpha_2} \\ &\le \frac{A_{r,\alpha_1}R_{k-2,\alpha_1}}{B_{l,\alpha_1}+C_{l,\alpha_1}L_{k-2,\alpha_1}L_{k-1,\alpha_1}L_{k,\alpha_1}}=R_{k+1,\alpha_1}. \end{aligned}$$

Hence (12) are true

Thus, from (9), it holds that

$$L_{1`,\alpha}=\frac{A_{l,\alpha}L_{-2,\alpha}}{B_{r,\alpha}+C_{r,\alpha}R_{-2,\alpha}R_{-1,\alpha}R_{0,\alpha}}, R_{1,\alpha}=\frac{A_{r,\alpha}R_{-2,\alpha}}{B_{l,\alpha}+C_{l,\alpha}L_{-2,\alpha}L_{-1,\alpha}L_{0,\alpha}}, \alpha\in(0,1]. \tag{13}$$

Moreover, in view of $A,B,C,x_j,(j=-2,-1,0)$ are positive fuzzy numbers, then $A_{l,\alpha},A_{r,\alpha},B_{l,\alpha},B_{r,\alpha}B_{l,\alpha},C_{l,\alpha},$ $C_{r,\alpha},L_{-2,\alpha},R_{-2,\alpha},L_{-1,\alpha},R_{-1,\alpha},L_{0,\alpha},R_{0,\alpha}$ are left continuous from Lemma 3.2. So, it is obvious that $L_{1,\alpha},R_{1,\alpha}$ are also left continuous from (13). Finally, it is obtained by mathematical induction that $L_{n,\alpha},R_{n,\alpha},n=1,2,\cdots,$ are left continuous.

Now, we prove that the support of x_n, i.e., supp $x_n=\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}$ is compact. It is easy to verify that the above result is true, that is

$$\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}\subset(0,\infty), n=1,2,\cdots. \tag{14}$$

Thus, we just prove that $\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}$ is bounded. Let $n=1,$ there exists constants $M_i,N_i>0, i=1,2,3$ such that for all $\alpha\in(0,1]$

$$\begin{aligned} &[A_{l,\alpha},A_{r,\alpha}]\subset[M_1,N_1],[B_{l,\alpha},B_{r,\alpha}]\subset[M_2,N_2], \\ &[C_{l,\alpha},C_{r,\alpha}]\subset[M_3,N_3],[L_{j,\alpha},R_{j,\alpha}]\subset[M_j,N_j], j=-2,-1,0, \end{aligned} \tag{15}$$

Where $A,B,C,x_j(j=-2,-1,0)$ are positive fuzzy numbers. It follows from (13) and (15) that

$$[L_{1`,\alpha} R_{1,\alpha}] \subset \left[\frac{M_1 M_{-2}}{N_2 + N_3 N_{-2} N_{-1} N_0}, \frac{N_1 N_{-2}}{M_2 + M_3 M_{-2} M_{-1} M_0}\right], \alpha \in (0,1].$$

Moreover, one has

$$\cup_{\alpha\in(0,1]}[L_{1`,\alpha} R_{1,\alpha}] \subset \left[\frac{M_1 M_{-2}}{N_2 + N_3 N_{-2} N_{-1} N_0}, \frac{N_1 N_{-2}}{M_2 + M_3 M_{-2} M_{-1} M_0}\right], \alpha \in (0,1]. \quad (16)$$

So, from (16), $\overline{\cup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact. Therefore, it is evident that $\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact by mathematical induction method.

Therefore from Lemma 3.2, relation (12), (14), and $L_{n,\alpha}, R_{n,\alpha}$ are left continuous, it means that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers $\{x_n\}$ such that Eq. (6) holds.

Furthermore, it is proved that $\{x_n\}$ is the solution of equation (6) with initial conditions x_{-2}, x_{-1}, x_0. Since for all $\alpha \in (0,1]$

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}]$$

$$= \left[\frac{A_{l,\alpha} L_{n-2,\alpha}}{B_{r,\alpha} + C_{r,\alpha} R_{n-2,\alpha} R_{n-1,\alpha} R_{n,\alpha}}, \frac{A_{r,\alpha} R_{n-2,\alpha}}{B_{l,\alpha} + C_{l,\alpha} L_{n-2,\alpha} L_{n-1,\alpha} L_{n,\alpha}}\right]$$

$$= \left[\frac{A x_{n-2}}{B + C x_{n-2} x_{n-1} x_n}\right]_\alpha.$$

It is obvious that $\{x_n\}$ is the solution of Eq. (6) with initial conditions x_{-2}, x_{-1}, x_0.

Assume that there exists another solution $\{x_n^*\}$ of Eq. (6) with initial conditions x_{-2}, x_{-1}, x_0, then

$$\left[x_n^*\right]_\alpha = \left[L_{n,\alpha}, R_{n,\alpha}\right], \alpha \in (0,1], n = 0, 1, \cdots. \quad (17)$$

It follows from (10) and (17) that

$$\left[x_n\right]_\alpha = \left[x_n^*\right], \alpha \in (0,1], n = -2, -1, \cdots,$$

From which it holds $x_n = x_n^*, \alpha \in (0,1], n = -2, -1, \cdots,$ and then the proof is completed.

Next, the primary goal is to investigate the asymptotic behavior of the equilibrium point of Eq. (6). From the above proof process, x_n is the unique positive solution of Eq. (6) with the initial values x_{-2}, x_{-1}, x_0, thus, it holds that

$$\left[x_n\right]_\alpha = \left[L_{n,\alpha}, R_{n,\alpha}\right], \alpha \in (0,1], n = 0, 1, \cdots.$$

Hence, it is obvious that $(L_{n,\alpha}, R_{n,\alpha})$ satisfies the systems (9). To conveniently study the asymptotic behavior of Eq. (6), according to the systems (9), the corresponding linearized form of the system is constructed as follows

$$y_{n+1} = \frac{a y_{n-2}}{f + d z_{n-2} z_{n-1} z_n}, z_{n+1} = \frac{b z_{n-2}}{e + c y_{n-2} y_{n-1} y_n}, n = 0, 1, \cdots, \quad (18)$$

Where the parameters a, b, c, d, e, f and initial conditions $y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0$ are positive real constants. Obviously, the systems (18) have a unique solution (y_n, z_n) from Lemma 2.1.

Now, the primary purpose is to prove that the solution (y_n, z_n) of the system (18) is bounded.

Theorem 3.2 let (y_n, z_n) be positive solution of system (18), then for all $k \ge 0$,

$$(i)\ 0 \le y_n \le \begin{cases} (\frac{a}{f})^{k+1} y_{-2}, n = 3k+1, \\ (\frac{a}{f})^{k+1} y_{-1}, n = 3k+2, \\ (\frac{a}{f})^{k+1} y_0, n = 3k+3, \end{cases} \quad (ii)\ 0 \le y_n \le \begin{cases} (\frac{b}{e})^{k+1} z_{-2}, n = 3k+1, \\ (\frac{b}{e})^{k+1} z_{-1}, n = 3k+2, \\ (\frac{b}{e})^{k+1} z_0, n = 3k+3. \end{cases} \quad (19)$$

Proof. Using the induction method to prove (19). This assertion is true for $k = 0$. Suppose that (19) is true for $k = m$, then it follows that for $k = m+1$

$$y_n=\begin{cases} y_{3(m+1)+1}\le \frac{a}{f}y_{3(m+1)-2}=\frac{a}{f}y_{3m+1}\le\frac{a}{f}\cdot(\frac{a}{f})^{m+1}y_{-2}, n=3(m+1)+1,\\ y_{3(m+1)+2}\le \frac{a}{f}y_{3(m+1)+1-2}=\frac{a}{f}y_{3m+2}\le\frac{a}{f}\cdot(\frac{a}{f})^{m+1}y_{-1}, n=3(m+1)+2,\\ y_{3(m+1)+3}\le \frac{a}{f}y_{3(m+1)+2-2}=\frac{a}{f}y_{3m+3}\le\frac{a}{f}\cdot(\frac{a}{f})^{m+1}y_{0}, n=3(m+1)+3,\end{cases}$$

And

$$z_n=\begin{cases} z_{3(m+1)+1}\le \frac{b}{e}y_{3(m+1)-2}=\frac{b}{e}y_{3m+1}\le\frac{b}{e}\cdot(\frac{b}{e})^{m+1}y_{-2}, n=3(m+1)+1,\\ z_{3(m+1)+2}\le \frac{b}{e}y_{3(m+1)+1-2}=\frac{b}{e}y_{3m+2}\le\frac{b}{e}\cdot(\frac{b}{e})^{m+1}y_{-1}, n=3(m+1)+2,\\ z_{3(m+1)+3}\le \frac{b}{e}y_{3(m+1)+2-2}=\frac{b}{e}y_{3m+3}\le\frac{b}{e}\cdot(\frac{b}{e})^{m+1}y_{0}, n=3(m+1)+3.\end{cases}$$

The proof is completed.

Furthermore, it is easy to know that the systems (18) has two equilibrium points

$$\overline{X}_1=(\overline{y}_1,\overline{z}_1)=(0,0),\overline{X}_2=(\overline{y}_2,\overline{z}_2)=((\frac{b-e}{c})^{1/3},(\frac{a-f}{d})^{1/3}).$$

For the two equilibrium points, the following results are shown clearly.

Theorem 3.3 For the equilibrium point $\overline{X}_1=(0,0)$ of Eq. (18), it holds that

(i) If $a<f, b<e,$ then the equilibrium point $\overline{X}_1$ of the system (18) is locally asymptotically stable,

(ii) If $a>f$ or $b>e$, then the equilibrium point $\overline{X}_1$ of the system (18) is unstable.

Proof. (i) Let $F:(R^+)^3\to R^+, G:(R^+)^3\to R^+$ be multivariate function defined by

$$F(y_{n-2},z_{n-2},z_{n-1},z_n)=\frac{ay_{n-2}}{f+dz_{n-2}z_{n-1}z_n}, G(z_{n-2},y_{n-2},y_{n-1},y_n)=\frac{bz_{n-2}}{e+cy_{n-2}y_{n-1}y_n},$$

Thus, one has

$$\begin{aligned} &F_{y_{n-2}}=\frac{a}{f+dz_{n-2}z_{n-1}z_n}, F_{z_{n-2}}=-\frac{ady_{n-2}z_{n-1}z_n}{(f+dz_{n-2}z_{n-1}z_n)^2},\\ &F_{z_{n-1}}=-\frac{ady_{n-2}z_{n-2}z_n}{(f+dz_{n-2}z_{n-1}z_n)^2}, F_{z_n}=-\frac{ady_{n-2}z_{n-2}z_{n-1}}{(f+dz_{n-2}z_{n-1}z_n)^2},\\ &G_{z_{n-2}}=\frac{b}{e+cy_{n-2}y_{n-1}y_n}, G_{y_{n-2}}=-\frac{bcz_{n-2}y_{n-1}y_n}{(e+cy_{n-2}y_{n-1}y_n)^2},\\ &G_{y_{n-1}}=-\frac{bcz_{n-2}y_{n-2}y_n}{(e+cy_{n-2}y_{n-1}y_n)^2}, G_{y_n}=-\frac{bcz_{n-2}y_{n-2}y_{n-1}}{(e+cy_{n-2}y_{n-1}y_n)^2}.\end{aligned} \tag{20}$$

Moreover, the linearized system of Eq. (18) about the equilibrium point $\overline{X}_1$ is given by

$$\phi_{n+1}=D_1\phi_n, \tag{21}$$

Where

$$\varphi_n=\begin{bmatrix} y_n\\ y_{n-1}\\ y_{n-2}\\ z_n\\ z_{n-1}\\ z_{n-2}\end{bmatrix}, D_1=\begin{bmatrix} 0&0&\frac{a}{f}&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&\frac{b}{e}\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\end{bmatrix}.$$

The characteristic equation of the matrix D_1 is shown as follows

$$f(\lambda)=\frac{(a-f\lambda^3)(b-e\lambda^3)}{ef}=0,$$

Then roots of this characteristic equation are given by

$$\lambda_{1,2,3}=\sqrt[3]{\frac{a}{f}},\lambda_{4,5,6}=\sqrt[3]{\frac{b}{e}}.$$

Since all eigenvalues of the Jacobian matrix D_1 about the equilibrium point $\overline{X}_1$ lie in open unit disk $|\lambda|_k<1$, for all $k=1,2,\cdots,6$. Hence from Lemma 2.2 the equilibrium point $\overline{X}_1$ is locally asymptotically stable.

(ii) Now, it is easy to see that if $a>f$ or $b>e$, then there exists at least one root λ of the characteristic equation $f(\lambda)=0$ such that $|\lambda|>1$. Therefore, if $a>f$ or $b>e$, then the equilibrium point $\overline{X}_1$ of the systems (18) is unstable. The proof is completed.

Theorem 3.4 If $a>f,b>e$, the equilibrium point $\overline{X}_2=((\frac{b-e}{c})^{1/3},(\frac{a-f}{d})^{1/3})$ is unstable.

Proof. Let

$$A_1=(\frac{b-e}{c})^{1/3},B_1=(\frac{a-f}{d})^{2/3},A_2=(\frac{b-e}{c})^{2/3},B_2=(\frac{a-f}{d})^{1/3},$$

From (20), the linearized system of Eq. (18) about the equilibrium point $\overline{X}_2$ is provided by

$$\varphi_{n+1}=D_2\varphi_n, \tag{22}$$

Where

$$\varphi_n=\begin{bmatrix} y_n\\ y_{n-1}\\ y_{n-2}\\ z_n\\ z_{n-1}\\ z_{n-2}\end{bmatrix}, D_2=\begin{bmatrix} 0 & 0 & 1 & -\frac{d}{a}A_1B_1 & -\frac{d}{a}A_1B_1 & -\frac{d}{a}A_1B_1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ -\frac{c}{b}A_2B_2 & -\frac{c}{b}A_2B_2 & -\frac{c}{b}A_2B_2 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}.$$

The characteristic equation of Eq. (22) is given by

$$f(\lambda)=\frac{(\lambda^2+\lambda+1)^2(ab\lambda^2-2ab\lambda+ab-A_1A_2B_1B_2cd)}{ab}=0, \tag{23}$$

then roots of this characteristic equation are shown as follows

$$\lambda_{1,2}=-\frac{1+\sqrt{3}i}{2},\lambda_{3,4}=-\frac{-1+\sqrt{3}i}{2},$$
$$\lambda_5=1+\sqrt{\frac{A_1A_2B_1B_2cd}{ab}}=1+\sqrt{\frac{(b-e)(a-f)}{ab}},$$
$$\lambda_6=1-\sqrt{\frac{A_1A_2B_1B_2cd}{ab}}=1-\sqrt{\frac{(b-e)(a-f)}{ab}}.$$

Now, it is sufficient to prove that one of these roots has absolute value greater than one. It is obvious that if $a>f,b>e$, then there exists at least one root λ of the characteristic equation (23) such that $|\lambda|>1$. Therefore, If $a>f,b>e$ then the equilibrium point $\overline{X}_2$ of the system (18) is unstable. The proof is completed.

Lemma 3.3 [12] Let I_x,I_y be some intervals of real numbers and assume that $f:I_x^{k+1}\times I_y^{l+1}\to I_x$ and $g:I_x^{k+1}\times I_y^{l+1}\to I_y$ be continuously differential functions satisfying mixed monotone property. If there exists

$$\begin{cases} m_0\le\min\{x_{-k},\cdots,x_0,y_{-l},\cdots,y_0\}\le\max\{x_{-k},\cdots,x_0,y_{-l},\cdots,y_0\}\le M_0,\\ n_0\le\min\{x_{-k},\cdots,x_0,y_{-l},\cdots,y_0\}\le\max\{x_{-k},\cdots,x_0,y_{-l},\cdots,y_0\}\le N_0,\end{cases}$$

Such that

$$m_0\le f([m_0]_p,[M_0]_q,[n_0]_s,[N_0]_t)\le f([M_0]_p,[m_0]_q,[N_0]_s,[n_0]_t)\le M_0,$$
$$n_0\le g([m_0]_{p_1},[M_0]_{q_1},[n_0]_{s_1},[N_0]_{t_1})\le g([M_0]_{p_1},[m_0]_{q_1},[N_0]_{s_1},[n_0]_{t_1})\le N_0,$$

Then there exist $(m,M)\in[m_0,M_0]^2$ and $(n,N)\in[n_0,N_0]^2$ satisfying

$$\begin{cases} M=f([M]_p,[m]_q,[N]_s,[n]_t), m=f([m]_p,[M]_q,[n]_s,[N]_t),\\ N=g([M]_{p_1},[m]_{q_1},[N]_{s_1},[n]_{t_1}), n=g([m]_{p_1},[M]_{q_1},[n]_{s_1},[N]_{t_1}).\end{cases}$$

Moreover, if $m=M, n=N,$ then the equations (7) has a unique equilibrium point $(\bar{x},\bar{y})\in[m_0,M_0]\times[n_0,N_0]$ and every solution of (7) converges to $(\bar{x},\bar{y})$.

Theorem 3.5 If $a=b<e=f, c=d,$ then the equilibrium point $\bar{X}_1=(0,0)$ of the system (18) is global attractor.

Proof. In view of $a=b<e=f, c=d,$ the system (19) is changed to

$$\begin{aligned} y_{n+1}&=\frac{ay_{n-2}}{f+dz_{n-2}z_{n-1}z_n},\\ z_{n+1}&=\frac{az_{n-2}}{f+dy_{n-2}y_{n-1}y_n}, n=0,1,\cdots. \end{aligned} \tag{24}$$

Let $(f,g):(0,\infty)^3\times(0,\infty)^3\to(0,\infty)\times(0,\infty)$ be a function defined by

$$\begin{aligned} f(y_n,y_{n-1},y_{n-2},z_n,z_{n-1},z_{n-2})&=\frac{ay_{n-2}}{f+dz_{n-2}z_{n-1}z_n},\\ g(y_n,y_{n-1},y_{n-2},z_n,z_{n-1},z_{n-2})&=\frac{az_{n-2}}{f+dy_{n-2}y_{n-1}y_n}. \end{aligned} \tag{25}$$

Set

$$f=\frac{au}{f+dvws}, g=\frac{au^*}{f+dv^*w^*s^*}, \tag{26}$$

Then

$$\begin{aligned} &f_u=\frac{a}{f+dvws}>0, f_v=-\frac{aduws}{(f+dvws)^2}<0,\\ &f_w=-\frac{aduvs}{(f+dvws)^2}<0, f_s=-\frac{aduvw}{(f+dvws)^2}<0,\\ &g_{u^*}=\frac{a}{f+dv^*w^*s^*}>0, g_{v^*}=-\frac{adu^*w^*s^*}{(f+dv^*w^*s^*)^2}<0,\\ &g_{w^*}=-\frac{adu^*v^*s^*}{(f+dv^*w^*s^*)^2}<0, g_{s^*}=-\frac{adu^*v^*w^*}{(f+dv^*w^*s^*)^2}<0, \end{aligned} \tag{27}$$

Which implies that f and g have a mixed monotone property.

Let $M_0=N_0=\max\{y_{-2},y_{-1},y_0,z_{-2},z_{-1},z_0\}$, $\sqrt[3]{(a-f)/d}<n_0=m_0<0$, then

$$m_0\le\frac{am_0}{f+dN_0^3}\le\frac{aM_0}{f+dn_0^3}\le M_0, n_0\le\frac{an_0}{f+dM_0^3}\le\frac{aN_0}{f+dm_0^3}\le N_0.$$

Obviously, since $m_i=n_i, M_i=N_i, i=0,1,\cdots.$ then from the system (18) and Lemma 3.3, there exist $m,M\in[m_0,M_0], n=m, N=M$ such that

$$m=\frac{am}{f+dN^3}, n=\frac{an}{f+dM^3}, M=\frac{aM}{f+dn^3}, N=\frac{aN}{f+dm^3},$$

Then $M=m, N=n.$

Hence, it is proved that the equilibrium point $(0,0)$ of the system (18) is global attractor from Lemma 3.3. The proof is completed.

Moreover, from Definition 2.5 it follows that $\bar{X}_1$ is asymptotically stable.

Finally, the stability of the trivial solution of the fuzzy difference equation (6) will be discussed. Firstly, the following definition is introduced.

Definition 3.1[17] The trivial solution $x=\hat{0}$ of Eq. (6) is said to be stable, if given $\varepsilon>0,$ there exists a $\delta(\varepsilon)>0$ with $D(x_i,\hat{0})<\delta, i=-2,-1,0,$ implies $D(x_n,\hat{0})<\varepsilon$ for any $n>0,$ such that for any $x_i\in D_\delta, i=-2,-1,0,$ the solution $x_n\in D_\varepsilon, n>0;$ attractive if there is a $\delta>0$ such that $D(x_i,\hat{0})<\delta, i=-2,-1,0,$ one has

$$\lim_{n\to\infty} D(x_n,\hat{0})=0;$$

(iii) asymptotically stable if (i) and (ii) hold simultaneously.

Theorem 3.6 If the parameters A,B,C are positive real numbers and $A<B,$ then the trivial solution $x=\hat{0}$ of Eq. (6) is asymptotically stable with respect to D when the initial conditions are positive fuzzy numbers with $[x_i]_\alpha\subset(0,+\infty), i=-2,-1,0, \alpha\in(0,1].$

Proof. The result follows from Theorem 3.3 and Theorem 3.5.

4. Numerical Simulation

In this section a numerical example is given in order to support our theoretical results. The example represents the asymptotically behavior of solutions for the fuzzy difference system (6).

Example 4.1 Consider the following fuzzy difference equation

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_{n-2}x_{n-1}x_n},n=0,1,2,\ldots, \tag{28}$$

where A,B,C are positive trivial fuzzy numbers. By Theorem 3.6, let

$$[A]_\alpha=[A,A]=0.01,$$
$$[B]_\alpha=[B,B]=4,[C]_\alpha=[C,C]=2,\alpha\in(0,1].$$

In addition, from Theorem 3.6, we denoted the initial conditions x_{-2},x_{-1},x_0 with $[x_i]_\alpha\subset(0,+\infty),i=-2,-1,0,$ $\alpha\in(0,1]$ Such that

$$x_0(x)=\begin{cases}\frac{1}{3}x-\frac{5}{3},5\le x\le 8,\\ -\frac{1}{5}x+\frac{13}{5},8\le x\le 13,\end{cases}\quad x_{-1}(x)=\begin{cases}\frac{1}{4}x-\frac{1}{4},1\le x\le 5,\\ -x+6,5\le x\le 6,\end{cases}\quad x_{-2}(x)=\begin{cases}\frac{1}{5}x-\frac{2}{5},2\le x\le 7,\\ -\frac{1}{3}x+\frac{10}{3},7\le x\le 10.\end{cases} \tag{29}$$

In view of (29),

$$[x_0]_\alpha=[5+3\alpha,13-5\alpha],[x_{-1}]_\alpha=[4+4\alpha,6-\alpha],[x_{-2}]_\alpha=[2+5\alpha,10-3\alpha].$$

Moreover, from Eq. (28), a coupled system of difference equation with parameter α is obtained

$$\begin{aligned}L_{n+1,\alpha}&=\frac{0.01L_{n-2,\alpha}}{4+2R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}},\\ R_{n+1,\alpha}&=\frac{0.01R_{n-2,\alpha}}{4+2L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}},\alpha\in(0,1],n=0,1,\cdots.\end{aligned} \quad)$$

It is easy to prove the conditions of Theorem 3.6 is satisfied. So from Theorem 3.6 the trivial solution $x=\hat{0}$ of Eq. (6) is asymptotically stable with respect to D as $n\to\infty$ (see Figureures 1-6).

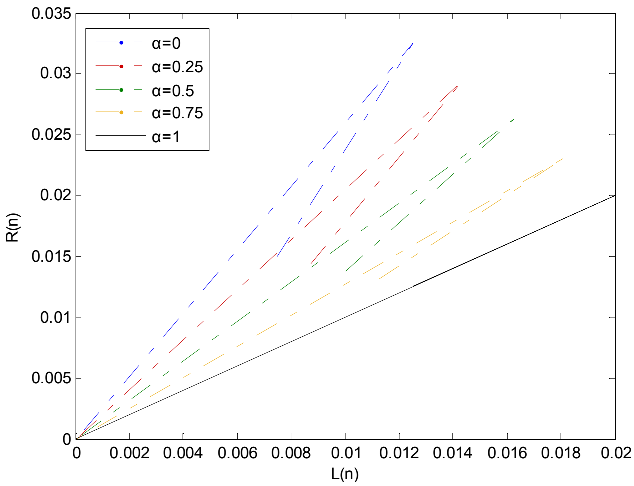

Figure 1. *The dynamics of system (30).*

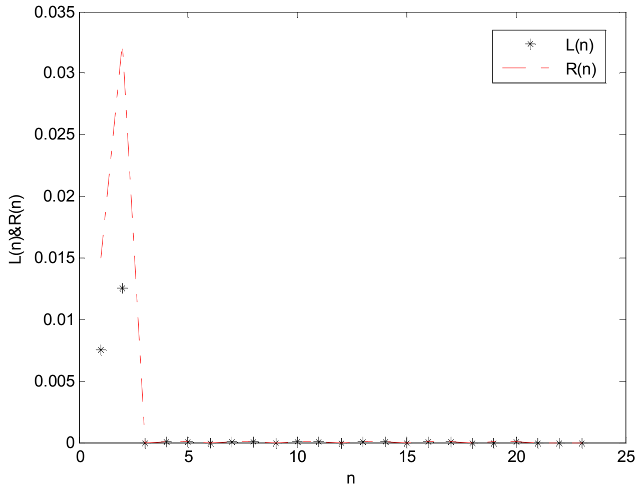

Figure 2. *The solution of system (30) when $\alpha=0$.*

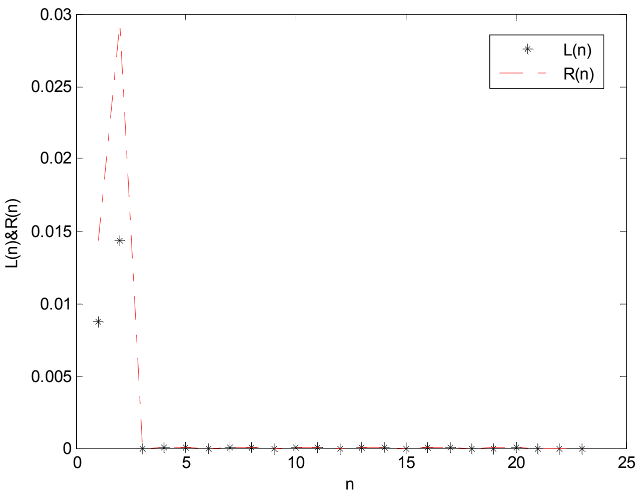

Figure 3. *The solution of system (30) when $\alpha=0.25$.*

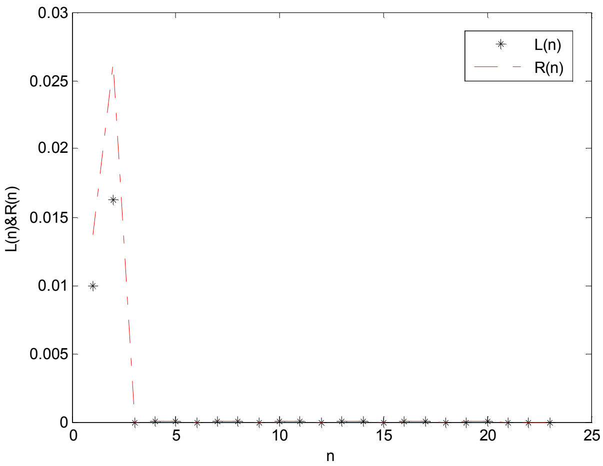

Figure 4. *The solution of system (30) when* $\alpha = 0.5$.

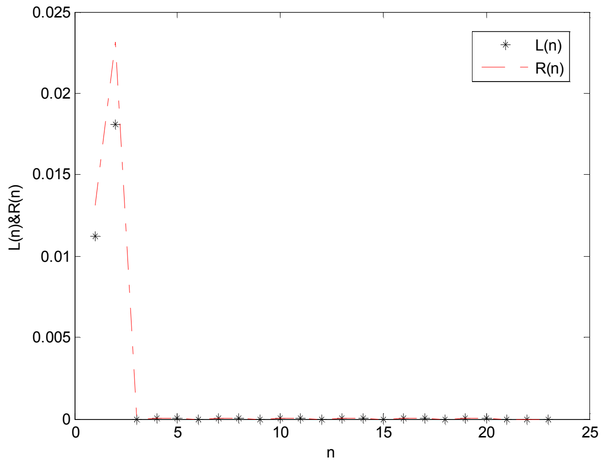

Figure 5. *The solution of system (30) when* $\alpha = 0.75$.

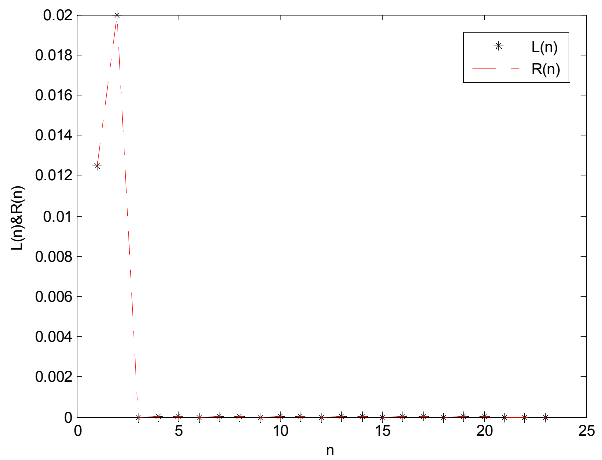

Figure 6. *The solution of system (30) when* $\alpha = 1$.

5. Conclusion

The main purpose of this paper is to deal with the dynamics behavior for a class of nonlinear third-order fuzzy difference equations. Firstly, the existence and uniqueness of positive fuzzy solutions is proved. Secondly, It is obtained that the nonzero equilibrium points of the corresponding ordinary difference equations is unstable by using linearization method. Finally, It is find that the zero trivial solution of the fuzzy equation is stable when the parameters A, B, C are positive real numbers, $A < B$ and the initial conditions are any positive fuzzy numbers. In particular, an example is given to show the effectiveness of the obtained results. In addition, the sufficient conditions obtained in this paper are very simple, which provide flexibility for the application and analysis of nonlinear fuzzy difference equation. For further work, it is our next research target to study the higher order fuzzy difference equations using new iteration method, inequality skills and comparison theorem.

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