



Conjugacy Class Lengths of Finite Groups with Prime Graph a Tree

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Abstract: For a finite group G , we write $c\rho(G)$ to denote the prime divisor set of the various conjugacy class lengths of G and $c\sigma(G)$ the maximum number of distinct prime divisors of a single conjugacy class length of G . It is a famous open problem that $|c\rho(G)|$ can be bounded by $c\sigma(G)$. Let G be an almost simple group such that the graph $\Gamma(G)$ built on element orders is a tree. By using Lucido's classification theorem, we prove $|c\rho(G)| = c\sigma(G)$ except possibly when G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$, where p is an odd prime and α is a field automorphism of odd prime order f . In the exceptional case, $|c\rho(G)| \leq c\sigma(G) + 2$. Combining with our known result, we also prove that for a finite group G with $\Gamma(G)$ a forest, the inequality $|c\rho(G)| \leq 2c\sigma(G)$ is true.

Keywords: Prime Graph, Conjugacy Class Length, Almost Simple Group

1. Introduction

Throughout the paper, we only consider finite groups. For a finite group G , write $c\rho(G)$ to denote the prime divisor set of the various conjugacy class lengths of G and $c\sigma(G)$ the maximum number of different prime divisors of a single conjugacy class length of G . And $\pi(n)$ stands for prime divisor set of the positive integer n and write $\pi(G)$ for $\pi(|G|)$, here $|G|$ denotes the order of G . Considering some similarities between the influence of character degrees and conjugacy class lengths on groups, in 1989, Huppert once asked [1] whether the inequality $|c\rho(G)| \leq 2c\sigma(G)$ holds for every solvable group. It was shown in [2, 4, 6] that this is true when $\Gamma(G)$ is at most 3. Specifically, the case G is proven in [4], the case $\pi(G)$ for the solvable G is in [6], and the cases pq for the nonsolvable $\pi_i, i = 1, 2, \dots$, and $\Gamma(G)$ are finished in [2], respectively. For solvable groups, it is

proved $2 \in \pi_1$ in [7], and an improved version G in [20]. Generally, Casolo proved in [1, Corollary 2] that the inequality is true except possibly when $\Gamma(G)$ is $\Gamma(G)$ nilpotent with abelian Sylow G subgroups for at least two prime divisors of $\Gamma(G)$. And yet Casolo and Dolfi in [3, Example 2] show the inequality is invalid by constructing an infinite family of group examples $|c\rho(G)| = c\sigma(G)$. Specifically, the quotient G approaches $PSL_2(p^f)\langle\alpha\rangle$ (from below) as p approaches infinity. The prime number α divides each f . When take $|c\rho(G)| \leq c\sigma(G) + 2$, we may obtain an infinity family of counterexamples such that each G is metabelian and super solvable and the constant is 3 in that inequality. Note that the subscripts $\Gamma(G)$ in these counterexamples are sufficient large. By observation of these counterexamples, in 1998, Zhang further conjectured [20] the weak version of that inequality should hold by saying that "Now, it seems true that $|c\rho(G)| \leq 2c\sigma(G)$ for any finite solvable group and probably also for any finite group." We

attach a prime graph $\Gamma(G)$ to a finite group G : its vertices are $\pi(G)$, and any two vertices are adjacent by an edge just when G has an element of order pq . We use $\pi_i, i = 1, 2, \dots$, to denote the connected components of the graph $\Gamma(G)$, in particular, $2 \in \pi_1$ if G is of even order. Furthermore, the graph $\Gamma(G)$ is called a tree when it is a connected graph without any loop; and $\Gamma(G)$ is called a forest when each connected component is a tree. In this note, we prove the following results.

Theorem A. *Suppose that G is an almost simple group such that $\Gamma(G)$ is a tree. Then $|c\rho(G)| = c\sigma(G)$ except possibly when G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$, where p is an odd prim and α is a field automorphism of odd prime order f . In the exceptional case, $|c\rho(G)| \leq c\sigma(G) + 2$.*

Theorem B. *Suppose that G is a finite group such that $\Gamma(G)$ is a forest. Then $|c\rho(G)| \leq 2c\sigma(G)$.*

In the proof of Theorem A, we apply the classification result due to Lucido (Theorem 2.2). In the process, GAP [6] plays a crucial role. In essence, we use finite simpl egroup classification theorem.

Unless otherwise specified, we adopt the standard notation and terminology as presented in [9].

2. Preliminaries

The following fact is useful and basic on conjugacy class lengths, which will be frequently applied without reference.

Lemma 2.1 ((Lemma 33.2 of [9])). Let N be a normal subgroup of G and $\bar{G} = G/N$. Then

1. $|x^N|$ divides $|x^G|$ for any $x \in N$, and so $c\sigma(N) \leq c\sigma(G)$.
2. $|\bar{y}^{\bar{G}}|$ divides $|y^G|$ for any $y \in G$, and thus $c\sigma(\bar{G}) \leq c\sigma(G)$.

Theorem 2.2. If G is an almost simple group with $\Gamma(G)$ a tree, then G is one of the following types of groups.

1. $Aut(A_6)$ for the alternating group A_6 of degree 6.
2. $PSL_2(p^f)\langle\alpha\rangle$ such that $p > 3$ is a prime, f is an odd prime and α is a field automorphism of order f .
3. $PGL_4(3) \leq G \leq Aut(PSL_4(3))$.
4. $Aut(B_2(3))$.
5. $PSU_4(3)\langle\delta\rangle \leq G \leq Aut(PSU_4(3))$, where δ is a diagonal automorphism of order 2.
6. $PGL_3(4) \leq G \leq PGL_3(4)\langle\alpha\rangle$ with α is a graph-field automorphism of order 2.
7. $PSL_3(q)\langle\alpha\rangle$, with $q = 9, 25, 49$ and α a field

automorphism of order 2.

$$8. PGU_3(8) \leq G \leq Aut(PSU_3(8)).$$

$$S_{\mathbb{Z}}(2^{2^f})\langle\alpha\rangle \text{ with } \alpha \text{ a field automorphism of order } f.$$

Here f is an odd prime.

Proof. This is Lemma 3 of [11].

Lemma 2.3. Assume that G is a finite group with disconnected $\Gamma(G)$. Then the inequality $|c\rho(G)| \leq 2c\sigma(G)$ is true.

Proof. This is Theorem A of [8].

Lemma 2.4. Let G be a finite group with $|\pi(G)| \leq 8$.

Then $|c\rho(G)| \leq 2c\sigma(G)$.

Proof. By [2, 3, 5], the inequality is valid when $c\sigma(G) \leq 3$. Otherwise, we see $c\sigma(G) \geq 4$ and so $|c\rho(G)| \leq |\pi(G)| \leq 8 \leq 2c\sigma(G)$, as wanted.

Theorem 2.5. Let G be a finite group such that $\Gamma(G)$ is a tree, then $|\pi(G)| \leq 8$.

Proof. This is Theorem 6 of [11].

Lemma 2.6. Let G be an almost simple group over the nonabelian simple group S , then $\pi(G) = c\rho(G)$.

Proof. It is known $c\rho(G) = \pi(G/Z(G))$ by Theorem 33.4 of [9]. If G has a maximal central Hall subgroup $H > 1$, then we can write $G = K \times H$. As S is a simple group, the intersection $S \cap H$ is trivial. It follows that $|S|$ divides $|K|$, and so $S \leq K$. We obtain H acts trivially on S , however, this is a contradiction since $G \leq Aut(S)$. Therefore H is trivial an $\pi(G) = \pi(G/Z(G))$.

3. Main Results

Theorem 3.1. Suppose that G is an almost simple group such that $\Gamma(G)$ is a tree. Then $|c\rho(G)| = c\sigma(G)$ except possibly when G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$, where p is an odd prime and α is a field automorphism of odd prime order f . In the exceptional case, $|c\rho(G)| \leq c\sigma(G) + 2$.

Proof. We apply Theorem 2.2 above to prove the claims.

If G is isomorphic to $Aut(A_6)$, then we know via GAP [6] that $Z(G)$ is trivial and $c\sigma(G) = 3 = |\pi(G)|$, thus we obtain $|c\rho(G)| = c\sigma(G)$, as desired.

Assume that G is isomorphic to $PSL_2(p^f)\langle\alpha\rangle$. Here $p > 3$ is an odd prime and α is a field automorphism of order f which is also an odd prime. By Lemma 4.1 of [10], we know $Z(G)$ is trivial and so $c\rho(G) = \pi(G)$. It is known $|PSL_2(p^f)| = (p^f - 1)p^f(p^f + 1)/2$ when p is an odd prime. As in [4], denote by d an element of order p in

$SL_2(p^f)$ and by z the element of order 2 in the centre of $SL_2(p^f)$. Then it is known that the class size of $d\langle z \rangle$ in $PSL_2(p^f)$ is $(p^f - 1)(p^f + 1) / 2$ (which is the conjugacy class size corresponding to $c\sigma(PSL_2(p^f))$), and so we conclude that $|c\rho(PSL_2(p^f))| = c\sigma(PSL_2(p^f)) + 1$ for odd p . The field automorphism α indeed leave the class of $d\langle z \rangle$ invariant by [10, Lemma 4.1]. We further get

$$\begin{aligned} |c\rho(G)| &= |\pi(G)| \leq |\pi(PSL_2(p^f))| + 1 \\ &= c\sigma(PSL_2(p^f)) + 2 \leq c\sigma(G) + 2 \end{aligned}$$

for odd p . Note that f is an odd prime.

Assume now that $PGL_4(3) \leq G \leq Aut(PSL_4(3))$. Then we see

$$c\sigma(PGL_4(3)) \leq c\sigma(G) \leq c\sigma(Aut(PSL_4(3)))$$

and

$$c\rho(PGL_4(3)) \subseteq c\rho(G) \subseteq c\rho(Aut(PSL_4(3))),$$

By using GAP [6], we know that $PGL_4(3)$ has 43 conjugacy class lengths,

$c\rho(PGL_4(3)) = \{2, 3, 5, 13\}$ and $c\sigma(PGL_4(3)) = 4$; and $Aut(PSL_4(3))$ has 56 conjugacy class lengths, $c\rho(Aut(PSL_4(3))) = \{2, 3, 5, 13\}$ and $c\sigma(Aut(PSL_4(3))) = 4$.

Hence we obtain $|c\rho(G)| = c\sigma(G)$, as required.

If G is isomorphic to $Aut(B_2(3))$, then the application of GAP yields that G has twenty conjugacy class lengths, $c\rho(G) = \{2, 3, 5\}$ and $c\sigma(G) = 3$, and so $|c\rho(G)| = c\sigma(G)$, as wanted.

Next, consider the case that $PSU_4(3)\langle\delta\rangle \leq G \leq Aut(PSU_4(3))$, where δ is a diagonal automorphism of order 2. Using GAP, we reach that $PSU_4(3)$ has 20 conjugacy class lengths, $c\rho(PSU_4(3)) = \{2, 3, 5, 7\}$ and $c\sigma(PSU_4(3)) = 4$. Its automorphism group $Aut(PSU_4(3))$ has 61 conjugacy class lengths, $c\rho(Aut(PSU_4(3))) = \{2, 3, 5, 7\}$ and $c\sigma(Aut(PSU_4(3))) = 4$. As G is an almost simple group, we achieve that $c\rho(PSU_4(3)) \subseteq c\rho(G) \subseteq c\rho(Aut(PSU_4(3)))$ and $c\sigma(PSU_4(3)) \leq c\sigma(G) \leq c\sigma(Aut(PSU_4(3)))$, thus $|c\rho(G)| = 4 = c\sigma(G)$, as desired.

Now, suppose that $PGL_3(4) \leq G \leq PGL_3(4)\langle\alpha\rangle$ with α is a graph-field automorphism of order 2. As α is of

order 2, we attain G is either $PGL_3(4)$ or else

$PGL_3(4)\langle\alpha\rangle$. When $G = PGL_3(4)$, by GAP, we know $\pi(PGL_3(4)) = \{2, 3, 5, 7\} = c\rho(PGL_3(4))$ and $c\sigma(G) = 4$, and so $|c\rho(G)| = c\sigma(G)$. When $G = PGL_3(4)\langle\alpha\rangle$, we also get that $|c\rho(G)| = |\pi(G)| = c\sigma(G)$ because α is of order 2 which is in $\pi(PGL_3(4))$.

The next case is $PSL_3(q)\langle\alpha\rangle$, with $q = 9, 25, 49$ and α a field automorphism of order 2. For $PSL_3(9)$, we get via GAP that $\pi(PSL_3(9)) = \{2, 3, 5, 7, 13\} = c\rho(PSL_3(9))$ and $c\sigma(PSL_3(9)) = 5$. Also since

$c\rho(PSL_3(9)\langle\alpha\rangle) \subseteq \pi(PSL_3(9)\langle\alpha\rangle) = \pi(PSL_3(9))$ this yields $|c\rho(PSL_3(9)\langle\alpha\rangle)| = c\sigma(PSL_3(9)\langle\alpha\rangle)$. For $PSL_3(25)$, by GAP, we attain that $\pi(PSL_3(25)) = \{2, 3, 5, 7, 13, 31\} = c\rho(PSL_3(25))$ and $c\sigma(PSL_3(25)) = 6$. Also $c\rho(PSL_3(25)\langle\alpha\rangle) \subseteq \pi(PSL_3(25)\langle\alpha\rangle) = \pi(PSL_3(25))$, this implies $|c\rho(PSL_3(25)\langle\alpha\rangle)| = c\sigma(PSL_3(25)\langle\alpha\rangle)$.

For $PSL_3(49)$, we get by GAP that $\pi(PSL_3(49)) = \{2, 3, 5, 7, 19, 43\} = c\rho(PSL_3(49))$ and $c\sigma(PSL_3(49)) = 6$. Also $c\rho(PSL_3(49)\langle\alpha\rangle) \subseteq \pi(PSL_3(49)\langle\alpha\rangle) = \pi(PSL_3(49))$, this shows $|c\rho(PSL_3(49)\langle\alpha\rangle)| = c\sigma(PSL_3(49)\langle\alpha\rangle)$, as desired.

Assume that $PGU_3(8) \leq G \leq Aut(PSU_3(8))$. By using GAP, it follows that $\pi(PSU_3(8)) = \pi(Aut(PSU_3(8))) = c\rho(PSU_3(8)) = \{2, 3, 7, 19\}$ and $c\sigma(PSU_3(8)) = 4$. Note that G is an almost simple group with $PSU_3(8) \leq G \leq Aut(PSU_3(8))$, thus $|c\rho(G)| = c\sigma(G)$.

Assume finally that G is isomorphic to $S_z(2^{2f^2})\langle\alpha\rangle$, where α is a field automorphism of odd prime order f . Let Q be a Sylow 2-subgroup of $S = S_z(2^{2f^2})$.

By Proposition 1 of [12], we see $C_s(g) \leq Q$ for any nontrivial element $g \in Q$.

Using Lemmas 1 and 2 of [12], we may pick the noncentral element $g = (\alpha, \beta, x) \in S(q; x)$, which is indeed a specific form of Q (by [12, Theorem 7]). Here neither of α or β can equal 0, 1. Then the centralizer $C_s(g)$ is properly contained in Q , and so $\pi(S) = \pi |g^S|$. If f belongs to $\pi(S)$, then $|c\rho(G)| = c\sigma(G)$. Otherwise, $\langle\alpha\rangle$ is a Sylow f -subgroup. By [7, Theorem 9], we have $\pi(G) = \pi |g^G|$ and so $|c\rho(G)| = c\sigma(G)$. The whole proof is complete.

Some remarks on $c\sigma(PSL_2(p^f))$ are made here. By [4],

we see that $c\sigma(PSL_2(p^f)) = |\pi(p^{2f} - 1)|$ for the odd $p^f > 5$ and $c\sigma(PSL_2(p^f)) = |\pi(2^{2f} - 1)|$ for $2^f > 4$.

Lemma 2 of [11] yields $|\pi(p^{2f} - 1)| \geq 3$ unless $p^f = 7, 8, 9, 17$. Some direct calculations by GAP show that $c\sigma(PSL_2(7))$, $c\sigma(PSL_2(9))$ and $c\sigma(PSL_2(17))$ are all equal to 3, but $c\sigma(PSL_2(5))$ and $c\sigma(PSL_2(8))$ are 2. Observe that $PSL_2(4) \cong PSL_2(5) \cong A_5$, $c\rho(PSL_2(5)) = \{2, 3, 5\}$ and $c\rho(PSL_2(8)) = \{2, 3, 7\}$, moreover $c\rho(\text{Aut}(PSL_2(8))) = \{2, 3, 7\}$.

Theorem 3.2. Suppose that G is a finite group such that $\Gamma(G)$ is a forest. Then

$$|c\rho(G)| \leq 2c\sigma(G).$$

Proof. If $\Gamma(G)$ is disconnected, then Lemma 2.3 yields the result. Otherwise, $\Gamma(G)$ is a tree, and so $|\pi(G)| \leq 8$ (by Lemma 2.5). Applying Lemma 2.4, we get the result.

In this note, we show Huppert's problem has affirmative answer for the finite group whose prime graph a forest, and even has better result when the group is an almost simple group with prime graph a tree.

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