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Hermite-Hadamard Type Integral Inequalities for Log-η-Convex Functions

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Abstract: In this paper by using the concept of log-η-convexity of functions some interesting inequalities are investigated. In fact new Hermite-Hadamard type integral inequalities involving log-η-convex function are established. The obtained results have as particular cases those previously obtained for log-convex.

Keywords: Log-η-Convex Functions, Integral Inequalities, Hermite-Hadamard Type Inequalities

1. Introduction and Preliminaries

The elegance in shape and interesting properties of convex functions make it attractive to study this class of function in mathematical analysis specially in applied mathematical analysis. In the last 60 years many efforts have gone on generalization of notion of convexity. In our opinion the following classification in generalization of convex functions holds:

(1) Works that change the form of defining convex functions to a generalized form such as quasi-convex [5], pseudo-convex [16], strongly convex [19], logarithmically convex [18], approximately convex [11], delta-convex[20], h-convex [22], midconvex functions [13], etc.

(2) Works that extend the domain set of convex functions such as E-convex functions [23], α-convex functions, all works on convex functions from R^n to R [3], invex functions [10], etc.

(3) Works that extend the range set of convex functions such as works on functions with range in vector spaces [12], all kind of multivalued convex functions [2, 14], etc.

On the other hand logarithmically convex (log-convex) functions are interesting class of functions to study in many fields of mathematics. They have been found to play an important role in the theory of special functions and mathematical statistics. To see recent works about log-convex functions see [4, 17, 18]).

Motivated by above works, we use the concept of log-η-convex function to establish some new Hermite-Hadamard type integral inequalities involving log-η-convex function. In fact obtained results have as particular cases those previously obtained for log-convex. We start with two definitions and one example.

Let I be an interval in real line $\mathbb{R}$. Consider $\eta: A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

Definition 1. [4] A function $f: I \to \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)), \qquad (1)$$

for all $x, y \in I$ and $t \in [0,1]$.

In fact above definition geometrically says that if a function is η-convex on I, then its graph between any $x, y \in I$ is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one.

Definition 2. Consider $f: I \longrightarrow (0, +\infty)$ and $\eta: \ln f(I) \times \ln f(I) \longrightarrow \mathbb{R}$. If

$$f(tx + (1-t)y) \leq f(y) \exp\left(t\eta(\ln f(x), \ln f(y))\right) \quad (2)$$

for every $x, y \in I$ and $t \in [0,1]$, then f is called *log-η-convex* function.

In the above definition if we set $\eta(x, y) = x - y$, then we recapture the classic definition of a log-convex function. It is clear that $f: I \longrightarrow (0, +\infty)$ is *log-η-convex* iff $(\ln f)$ is η-

convex and when f is η-convex then (exp f) is log-η-convex.

The following are two simple examples of log-η-convex functions.

Example 1.

a. Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0; \\ e^{x}, & x < 0. \end{cases}$$

and define a bifunction η as $\eta(x,y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that f is a log-η-convex function.

b. Define the function $f: \mathbb{R}^+ = [0, +\infty) \to \mathbb{R}^+$ by

$$f(x) = \begin{cases} e^{x}, & 0 \leq x \leq 1; \\ e, & x > 1. \end{cases}$$

and define the bifunction $\eta: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\eta(x,y) = \begin{cases} x + y, & x \leq y; \\ 2(x+y), & x > y. \end{cases}$$

Then f is log-η-convex.

The following result is of importance [3]:

Theorem 1. Suppose that $f: I \to \mathbb{R}$ is a η-convex function and η is bounded from above on $f(I) \times f(I)$. Then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Hence, f is absolutely continuous on $[a,b]$ and continuous on I°.

Note. As a consequence of Theorem 1, if $f: [a,b] \to \mathbb{R}$ is a log-η-convex function where η is bounded from above on $f([a,b]) \times f([a,b])$, then $\ln(f)$ is integrable and so $f = \exp(\ln(f))$ is integrable. For other results see [2, 4].

Some Hermite-Hadamard type inequalities related to η-convex functions are proved in [3, 4, 7]. Some log-η-convex version of this type inequalities are investigated in the following.

$$f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq$$
$$\frac{1}{2}[f(a) + f(b)] + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))] \leq$$
$$\frac{1}{2}[f(a) + f(b)] + \frac{M_\eta}{2}.$$

provided that $f: [a,b] \to \mathbb{R}$ is a η-convex function, η is bounded from above on $\ln f([a,b]) \times \ln f([a,b])$ and M_η is upper bound of η.

Now if $f: [a,b] \longrightarrow (0, +\infty)$ is log-η-convex, since $(\ln f)$ is η-convex we have

$$\ln f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \leq \frac{1}{b-a}\int_a^b \ln f(x)dx \leq$$
$$\frac{1}{2}[\ln f(a) + \ln f(b)] + \frac{1}{4}[\eta(\ln f(a), \ln f(b)) + \eta(\ln f(b), \ln f(a))] \leq$$
$$\frac{1}{2}[\ln f(a) + \ln f(b)] + \frac{M_\eta}{2}.$$

Consequently

$$\exp\left(\ln f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2}\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)dx\right) \leq$$
$$\exp\left(\frac{1}{2}[\ln f(a) + \ln f(b)] + \frac{1}{4}[\eta(\ln f(a), \ln f(b)) + \eta(\ln f(b), \ln f(a))]\right) \leq$$
$$\exp\left(\frac{1}{2}[\ln f(a) + \ln f(b)] + \frac{M_\eta}{2}\right).$$

So

$$f\left(\frac{a+b}{2}\right)\exp\left(-\frac{M_\eta}{2}\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)dx\right) \leq$$
$$\sqrt{f(a)f(b)}\exp\left(\frac{1}{4}[\eta(\ln f(a), \ln f(b)) + \eta(\ln f(b), \ln f(a))]\right) \leq$$
$$\sqrt{f(a)f(b)}\exp\left(\frac{M_\eta}{2}\right).$$

Also if we consider

(a) (Arithmetic mean) $A(a,b) = \frac{a+b}{2}$, for any $a, b \in \mathbb{R}$,

(b) (Geometric mean) $G(a,b) = \sqrt{ab}$, for any $a, b \in \mathbb{R}^+$,

then we have

$$f(A(a,b))\exp\left(-\frac{M_\eta}{2}\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)dx\right) \leq G(f(a), f(b))\exp\left(\frac{M_\eta}{2}\right). \tag{3}$$

For more results about this inequalities see [2, 5].

The following theorem is a consequence of Theorem 1 of [1], which we use these results frequently in this paper.

Theorem 2. If f_1 and f_2 are positive increasing functions on $[0,1]$. Then

$$\int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \leq \int_0^1 f_1(x)f_2(x)dx.$$

Also if f_1 and f_2 are positive decreasing functions on $[0,1]$ and K is an upper bound for f_1 and f_2, then $K - f_1$ and $K - f_2$ are positive increasing functions and we have

$$\int_0^1 (K - f_1(x))dx \int_0^1 (K - f_2(x))dx \leq \int_0^1 (K - f_1(x))(K - f_2(x))dx,$$

which gives again

$$\int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \leq \int_0^1 f_1(x)f_2(x)dx.$$

2. Main Results

In this section by using log-η-convexity property of a function some inequalities which generalize those previously obtained for log-convex functions are given.

Theorem 3. Let $f: I \longrightarrow (0, +\infty)$ be a log-η-convex function with η bounded from above on $\ln f([a,b]) \times \ln f([a,b])$ and M_η be the upper bound of the function η. Consider $a, b \in I$ with $a < b$. Then

$$(f(A(a,b)))^2 \exp(-M_\eta) \leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq \quad (4)$$

$$A((f(a))^2 \frac{1}{2\eta(\ln f(b), \ln f(a))}, (f(b))^2 \frac{1}{2\eta(\ln f(a), \ln f(b))}) \exp(2M_\eta - 1)$$

Proof. For any $x, y \in I$ and $t \in [0,1]$,

$$(*) \begin{cases} f(tx + (1-t)y) \leq f(y) \exp(t\eta(\ln f(x), \ln f(y))), \\ f(tx + (1-t)y) \leq f(x) \exp((1-t)\eta(\ln f(y), \ln f(x))), \end{cases}$$

and

$$(**) \begin{cases} f(ty + (1-t)x) \leq f(x) \exp(t\eta(\ln f(y), \ln f(x))), \\ f(ty + (1-t)x) \leq f(y) \exp((1-t)(\eta(\ln f(x), \ln f(y))). \end{cases}$$

Then for $t = 1/2$

$$f(\frac{x+y}{2}) \leq f(y) \exp(\frac{1}{2}\eta(\ln f(x), \ln f(y))),$$

and

$$f(\frac{x+y}{2}) \leq f(x) \exp(\frac{1}{2}\eta(\ln f(y), \ln f(x))).$$

So

$$f(\frac{x+y}{2}) \leq \sqrt{f(x)f(y)} \exp(\frac{1}{4}[\eta(\ln f(x), \ln f(y)) + \eta(\ln f(y), \ln f(x))])$$

$$\leq \sqrt{f(x)f(y)} \exp(\frac{1}{4}[M_\eta + M_\eta]),$$

or

$$f(\frac{x+y}{2}) \exp(-\frac{M_\eta}{2}) \leq \sqrt{f(x)f(y)}.$$

Now choosing $x = ta + (1-t)b$ and $y = (1-t)a + tb$ for all $t \in [0,1]$ we get

$$f(\frac{a+b}{2}) \exp(-\frac{M_\eta}{2}) \leq G(f(ta + (1-t)b), f((1-t)a + tb)). \quad (5)$$

Now the left side of (4) is a consequence of (5) with integration over $t \in [0,1]$.

For the right side of (4), using the elementary inequality $G(x,y) \leq K(x,y) := \sqrt{\frac{x^2+y^2}{2}}$ $(x, y \geq 0)$ and relations

$$\begin{cases} f(ta + (1-t)b) \leq f(b) \exp\{t\eta(\ln f(a), \ln f(b))\} \\ f((1-t)a + tb) \leq f(b) \exp\{(1-t)\eta(\ln f(a), \ln f(b))\}, \end{cases}$$

we get

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx = \int_0^1 f(ta + (1-t)b)f((1-t)a + tb)dt \leq$$

$$\frac{1}{2}[\int_0^1 \{f(ta + (1-t)b)\}^2 dt + \int_0^1 \{f((1-t)a + tb)\}^2 dt] \leq$$

$$\frac{1}{2}\int_0^1 \{(f(b)) \exp(t\eta(\ln f(a), \ln f(b)))\}^2 dt + \frac{1}{2}\int_0^1 \{(f(b)) \exp((1-t)\eta(\ln f(a), \ln f(b)))\}^2 dt =$$

$$\frac{1}{2}(f(b))^2 \int_0^1 \exp(2t\eta(\ln f(a), \ln f(b)))dt + \frac{1}{2}(f(b))^2 \int_0^1 \exp(2(1-t)\eta(\ln f(a), \ln f(b)))dt =$$

$$(f(b))^2 \int_0^1 \exp(2t\eta(\ln f(a), \ln f(b)))dt \leq (f(b))^2 \frac{1}{2\eta(\ln f(a), \ln f(b))}(\exp(2M_\eta) - 1).$$

With the same argument we can obtain that

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq (f(a))^2 \frac{1}{2\eta(\ln f(b), \ln f(a))}(\exp(2M_\eta) - 1).$$

So

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq \min\{(f(b))^2 \frac{1}{2\eta(\ln f(a), \ln f(b))}(\exp(2M_\eta) - 1), (f(a))^2 \frac{1}{2\eta(\ln f(b), \ln f(a))}(\exp(2M_\eta) - 1)\} \leq$$

$$\frac{1}{2}((f(a))^2 \frac{1}{2\eta(\ln f(b), \ln f(a))} + (f(b))^2 \frac{1}{2\eta(\ln f(a), \ln f(b))})(\exp(2M_\eta) - 1),$$

where for the last inequality we used the property that $\min\{c, d\} \leq \frac{c+d}{2}$.

When a log-η-convex function is positive and increasing, we can use Theorem 2 to obtain the following inequalities as well.

Theorem 4. Let $f: I \longrightarrow (0, +\infty)$ be an increasing log-η-

convex function with η bounded from above on $\ln f(I) \times \ln f(I)$. Also consider $a, b \in I$ $(a < b)$, $\eta(\ln f(b), \ln f(a)) > 0$ and $M_\eta \neq 0$. Then

$$\frac{8f(a)}{M_\eta(b-a)}(\exp(\eta(\ln f(b), \ln f(a))) - 1)\int_a^b f(x)dx \leq$$

$$\frac{1}{b-a}\int_a^b f^4(x)dx + (\frac{exp(4M_\eta) - 1}{4\,exp(\eta(ln\ f(b), ln\ f(a))})f^4(a) + 8.$$

Proof. For any $t \in [0,1]$ we have

$$f(tb + (1-t)a) \leq f(a)\exp(t\eta(\ln f(b), \ln f(a))),$$

and for every $x, y \in \mathbb{R}$ we have [6],

$$8xy \leq x^4 + y^4 + 8.$$

So we can write

$$\begin{aligned} 8f(tb+(1-t)a)f(a)\exp(t\eta(\ln f(b), \ln f(a))) &\leq f^4(tb+(1-t)a) \\ &+ f^4(a)\exp(4t\eta(\ln f(b), \ln f(a))) + 8 \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 8f(tb+(1-t)a)f(a)\exp(t\eta(\ln f(b), \ln f(a)))dt &\leq \\ &\int_0^1 f^4(tb+(1-t)a) \\ &+ \int_0^1 f^4(a)\exp(4t\eta(\ln f(b), \ln f(a))) \\ &+ 8. \end{aligned}$$

Also

$$\int_0^1 f(tb+(1-t)a)dt\int_0^1 f(a)\exp(t\eta(\ln f(b), \ln f(a)))dt \leq$$

$$\int_0^1 f(tb+(1-t)a)f(a)\exp(t\eta(\ln f(b), \ln f(a)))dt.$$

Therefor we have

$$\begin{aligned} 8\int_0^1 f(tb+(1-t)a)dt\int_0^1 f(a)\exp(t\eta(\ln f(b), \ln f(a)))dt &\leq \\ &\int_0^1 f^4(tb+(1-t)a) \\ &+ \int_0^1 f^4(a)\exp(4t\eta(\ln f(b), \ln f(a))) \\ &+ 8. \end{aligned}$$

On the other hand

$$\int_0^1 f(tb+(1-t)a)dt = \frac{1}{b-a}\int_a^b f(x)dx,$$

$$\int_0^1 f(a)\exp(t\eta(\ln f(b), \ln f(a)))dt$$
$$= \frac{f(a)}{\eta(ln\ f(b), ln\ f(a))}(\exp(\eta(\ln f(b), \ln f(a))) - 1),$$

$$\int_0^1 f^4(a)\exp(4t\eta(\ln f(b), \ln f(a)))dt$$
$$= \frac{f^4(a)}{4\eta(ln\ f(b), ln\ f(a))}(\exp(4\eta(\ln f(b), \ln f(a))) - 1).$$

Hence

$$\begin{aligned} \frac{8f(a)}{\eta(ln\ f(b), ln\ f(a))(b-a)}&(\exp(\eta(\ln f(b), \ln f(a))) - 1)\int_a^b f(x)dx \leq \\ &\frac{1}{b-a}\int_a^b f^4(x)dx \\ &+ \frac{f^4(a)}{4\eta(ln\ f(b), ln\ f(a))}(\exp(4\eta(\ln f(b), \ln f(a))) - 1) + 8, \end{aligned}$$

which gives

$$\frac{8f(a)}{M_\eta(b-a)}(\exp(\eta(\ln f(b), \ln f(a))) - 1)\int_a^b f(x)dx \leq \quad (6)$$

$$\frac{1}{b-a}\int_a^b f^4(x)dx + (\frac{exp(4M_\eta) - 1}{4\,exp(\eta(ln\ f(b), ln\ f(a))})f^4(a) + 8.$$

The following result is obtained for the multiplication of two positive increasing log-η-convex functions under some special conditions.

Theorem 5. Let $f, g: I \longrightarrow (0, +\infty)$ be increasing log-η-convex functions with η bounded from above on $\ln f(I) \times \ln f(I)$. Also consider $a, b \in I$ with $a < b$, $\eta(\ln f(a), \ln f(b)) < 0$, $\eta(\ln g(a), \ln g(b)) < 0$ and $M_\eta \neq 0$. Then the following inequality holds:

$$\frac{exp(M) - 1}{M_\eta(b-a)}[f(b)\int_a^b g(x)dx + g(b)\int_a^b f(x)dx] \leq$$

$$\frac{f(b)g(b)}{P}[\exp(P) - 1] + \frac{1}{b-a}\int_a^b f(x)g(x)dx,$$

where
$M = \min\{\eta(\ln f(a), \ln f(b)), \eta(\ln g(a), \ln g(b))\}$ and
$P = \eta(\ln f(a), \ln f(b)) + \eta(\ln g(a), \ln g(b))$.

Proof. Since f, g are log-η-convex functions, we have

$$f(ta + (1-t)b) \leq f(b)\exp(t\eta(\ln f(a), \ln f(b)))$$
$$g(ta + (1-t)b) \leq g(b)\exp(t\eta(\ln g(a), \ln g(b)))$$

for all $t \in [0,1]$. So

$$[f(ta+(1-t)b)-f(b)\exp(t\eta(\ln f(a),\ln f(b)))][g(ta+(1-t)b)-g(b)\exp(t\eta(\ln g(a),\ln g(b)))]\geq 0$$

Therefor one can write:

$$f(ta+(1-t)b)g(b)\exp(t\eta(\ln g(a),\ln g(b)))+ \quad (7)$$

$$g(ta+(1-t)b)f(b)\exp(t\eta(\ln f(a),\ln f(b)))\leq$$

$$f(b)g(b)\exp(t\eta(\ln f(a),\ln f(b)))\exp(t\eta(\ln g(a),\ln g(b)))+$$

$$f(ta+(1-t)b)g(ta+(1-t)b).$$

$$f(b)g(b)\exp(tP)+f(ta+(1-t)b)g(ta+(1-t)b).$$

Integration from (7) over t on $[0,1]$ gives

$$\int_0^1 f(ta+(1-t)b)g(b)\exp(t\eta(\ln g(a),\ln g(b)))dt+\int_0^1 g(ta+(1-t)b)f(b)\exp(t\eta(\ln f(a),\ln f(b)))dt\leq$$

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt+\int_0^1 f(b)g(b)\exp(tP)dt.$$

On the other hand

$$\int_0^1 f(ta+(1-t)b)g(b)\exp(t\eta(\ln g(a),\ln g(b)))dt+\int_0^1 g(ta+(1-t)b)f(b)\exp(t\eta(\ln f(a),\ln f(b)))dt\geq$$

$$\int_0^1 f(ta+(1-t)b)dt\int_0^1 g(b)\exp(t\eta(\ln g(a),\ln g(b)))dt+\int_0^1 g(ta+(1-t)b)dt\int_0^1 f(b)\exp(t\eta(\ln f(a),\ln f(b)))dt=$$

$$\frac{g(b)}{\eta(ln\ g(a),ln\ g(b))(b-a)}(\exp(\eta(\ln g(a),\ln g(b)))-1)\int_a^b f(x)dx+$$

$$\frac{f(b)}{\eta(ln\ f(a),ln\ f(b))(b-a)}(\exp(\eta(\ln f(a),\ln f(b)))-1)\int_a^b g(x)dx\geq$$

$$\frac{g(b)}{M_\eta(b-a)}(\exp(\eta(\ln g(a),\ln g(b)))-1)\int_a^b f(x)dx+$$

$$\frac{f(b)}{M_\eta(b-a)}(\exp(\eta(\ln f(a),\ln f(b)))-1)\int_a^b g(x)dx\geq$$

$$\frac{g(b)}{M_\eta(b-a)}(\exp(M)-1)\int_a^b f(x)dx+$$

$$\frac{f(b)}{M_\eta(b-a)}(\exp(M)-1)\int_a^b g(x)dx,$$

and

$$\int_0^1 f(b)g(b)\exp(tP)dt+\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt=$$

$$\frac{f(b)g(b)}{P}[\exp(P)-1]+\frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

There for

$$\frac{exp\ (M)-1}{M_\eta(b-a)}[f(b)\int_a^b g(x)dx+g(b)\int_a^b f(x)dx]\leq$$

$$\frac{f(b)g(b)}{P}[\exp(P)-1]+\frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

The dual form of Theorem 5, is stated as the following.

Theorem 6. Let $f,g: I\longrightarrow(0,+\infty)$ be increasing log-η-convex functions with η bounded from above on $\ln f(I)\times\ln f(I)$. Also consider $a,b\in I$ with $a<b$, $\eta(\ln f(b),\ln f(a))>0$, $\eta(\ln g(b),\ln g(a))>0$ and $M_\eta\neq 0$. Then the following inequality holds:

$$\frac{exp\ (M)-1}{M_\eta(b-a)}[f(a)\int_a^b g(x)dx+g(a)\int_a^b f(x)dx]\leq$$

$$\frac{f(a)g(a)}{P}[\exp(P)-1]+\frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

where $M=\min\{\eta(\ln f(b),\ln f(a)),\eta(\ln g(b),\ln g(a))\}$ and $P=\eta(\ln f(b),\ln f(a))+\eta(\ln g(b),\ln g(a))$.

Proof. Change the role of a and b in proof of Theorem 5.

Using an elementary inequality between real numbers leads to an inequality related to square of a positive increasing log-η-convex function.

Theorem 7. Let $f: I\longrightarrow(0,+\infty)$ be an increasing log-η-convex function with η bounded from above on $\ln f(I)\times\ln f(I)$. Also consider $a,b\in I$ with $a<b$ and $M_\eta<0$. Then

$$\left(\frac{exp\ (M_\eta)-1}{M_\eta}\right)\left[\frac{f(a)+f(b)}{b-a}\right]\int_a^b f(x)dx\leq$$

$$[f^2(a)+f^2(b)]\left(\frac{exp\ (2M_\eta)-1}{2M_\eta}\right)+\frac{1}{b-a}\int_a^b f^2(x)dx.$$

Proof. Since f is log-η-convex function on I, we have

$$f(ta+(1-t)b)\leq f(b)\exp(t\eta(\ln f(a),\ln f(b)))$$

$$f(ta+(1-t)b)\leq f(a)\exp((1-t)\eta(\ln f(b),\ln f(a)))$$

for all $t\in[0,1]$. Using the elementary inequality

$$xy+yz+xz\leq x^2+y^2+z^2\ (x,y,z\in\mathbb{R})$$

and the fact that η is bounded from above we have

$$f^2(ta+(1-t)b)+f^2(a)\exp(2(1-t)M_\eta)+f^2(b)\exp(2M_\eta)\geq \tag{8}$$

$$f(ta+(1-t)b)f(b)\exp(tM_\eta)+f(ta+(1-t)b)f(a)\exp((1-t)M_\eta)+f(a)f(b)\exp(M_\eta).$$

Then by integration over $t\in[0,1]$ in (8),

$$\int_0^1 f^2(ta+(1-t)b)dt+\int_0^1 f^2(a)\exp(2(1-t)M_\eta)dt+\int_0^1 f^2(b)\exp(2M_\eta)\geq \tag{9}$$

$$\int_0^1 f(ta+(1-t)b)f(b)\exp(tM_\eta)dt+\int_0^1 f(ta+(1-t)b)f(a)\exp((1-t)M_\eta)dt+\int_0^1 f(a)f(b)\exp(M_\eta)dt.$$

It is easy to check the following from (9):

$$\frac{1}{b-a}\int_a^b f^2(x)dx+f^2(a)(\frac{1-exp(2M_\eta)}{-2M_\eta})+f^2(b)(\frac{exp(2M_\eta)}{2M_\eta})\geq$$

$$\frac{1}{(b-a)}\int_a^b f(x)dx.f(b).\int_0^1\exp(tM_\eta)dt+\frac{1}{(b-a)}\int_a^b f(x)dx.f(a).\int_0^1\exp((1-t)M_\eta)dt=$$

$$f(b)(\frac{exp(M_\eta)-1}{M_\eta})\frac{1}{(b-a)}\int_a^b f(x)dx+f(a)(\frac{1-exp(M_\eta)}{-M_\eta})\frac{1}{(b-a)}\int_a^b f(x)dx=$$

$$(\frac{exp(M_\eta)-1}{M_\eta})[\frac{f(a)+f(b)}{b-a}]\int_a^b f(x)dx.$$

So

$$(\frac{exp(M_\eta)-1}{M_\eta})[\frac{f(a)+f(b)}{b-a}]\int_a^b f(x)dx\leq$$

$$[f^2(a)+f^2(b)](\frac{exp(2M_\eta)-1}{2M_\eta})+\frac{1}{b-a}\int_a^b f^2(x)dx.$$

3. Conclusion

Logarithmically convex (log-convex) functions have some nice results in mathematical inequalities and are of interest in many areas of mathematics. They play a valuable and important role in the theory of special functions and mathematical statistics. On the other hand it should be noticed that in new problems related to convexity, generalized notions about convexity are required to obtain applicable results. One of these generalizations may be notion of log-η-convex functions which results in many interesting integral inequalities such as generalized form of Hermite-Hadamard type integral inequalities.

References

[1] R. Ahlswede and D. E. Daykin, *Integrals inequalities for increasing functions*, Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press, 86(3)(1979), 391–394.

[2] A. Aleman, *On some generalizations of convex sets and convex functions*, Anal. Numer. Theor. Approx. 14 (1985), 1-6.

[3] C. R. Bector and C. Singh, *B-Vex functions*, J. Optim. Theory. Appl. 71(2) (1991), 237-253.

[4] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, (2000).

[5] B. Definetti, *Sulla stratificazioni convesse*, Ann. Math. Pura. Appl. 30 (1949), 173-183.

[6] M. Eshaghi Gordji, S. S. Dragomir and M. Rostamian Delavar, *An inequality related to η-convex functions (II)*, Int. J. Nonlinear Anal. Appl. 6(2) (2015), 26-32.

[7] M. Eshaghi Gordji, M. Rostamian Delavar and M. De La Sen, *On φ-convex functions*, J. Math. Inequal. 10(1) (2016), 173-183.

[8] M. Eshaghi Gordji, M. Rostamian Delavar and S. S. Dragomir, *Some inequalities related to η-convex functions*, Preprint, RGMIA *Res. Rep. Coll.* 18(2015), Art. 08. [Online http://rgmia.org/papers/v18/v18a08.pdf].

[9] M. Eshaghi, F. Sajadian and M. Rostamian Delavar, *Inequalities for log-η-convex functions*, to apear in Int. J. Nonlinear Anal. Appl.

[10] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. 80 (1981), 545-550.

[11] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. 3 (1952), 821-828.

[12] I. Hsu and R. G. Kuller, *Convexity of vector-valued functions*, Proc. Amer. Math. Soc. 46 (1974), 363-366.

[13] J. L. W. V. Jensen, *On konvexe funktioner og uligheder mellem middlvaerdier*, Nyt. Tidsskr. Math. B. 16 (1905), 49-69.

[14] D. Kuroiwa, *Convexity for set-valued maps*, Appl. Math. Lett. 9 (1996), 97-101.

[15] R. B. Manfrino, R. V. Delgado and J.A.G. Ortega, *Inequalities a Mathematical Olympiad Approach*, Birkhӓuser, (2009).

[16] O. L. Mangasarian, *Pseudo-convex functions*, SIAM Journal on Control, 3 (1965), 281-290.

[17] D. S. Mitrinovic´, J. E. Pecˇaric´, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, (1993).

[18] J. E. Pecaric, F. Proschan and Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, Boston, (1992).

[19] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. 7 (1966), 72-75.

[20] T. Rajba, *On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order*, Math. Inequal. Appl. 18(1) (2015), 267-293.

[21] M. Rostamian Delavar and S. S. Dragomir, *On η-convexity*, to appear in Math. Inequal. Appl.

[22] S. Varosanec, *On h-convexity*, J. Math. Anal. Appl. 326(1) (2007), 303-311.

[23] X. M. Yang, *E-convex sets, E-convex functions and E-convex programming*, J. Optim. Theory. Appl. 109 (2001), 699-704.