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The New Proof of Ptolemy's Theorem & Nine Point Circle Theorem

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Abstract: The main purpose of the paper is to present a new proof of the two celebrated theorems: one is "Ptolemy's Theorem" which explains the relation between the sides and diagonals of a cyclic quadrilateral and another is "Nine Point Circle Theorem" which states that in any arbitrary triangle the three midpoints of the sides, the three feet of altitudes, the three midpoints of line segments formed by joining the vertices and Orthocenter, total nine points are concyclic. Our new proof is based on a metric relation of circumcenter.

Keywords: Ptolemy's Theorem, Circumcenter, Cyclic Quadrilateral, Nine Point Circle Theorem, Pedals Triangle, Medial Triangle

1. Introduction

The Ptolemy's Theorem states that "The multiple of the lengths of the diagonals of a Cyclic Quadrilateral is equal to the addition of separate multiples of the opposite side lengths of the Cyclic Quadrilateral" (refer [6]).

In this short paper, we deal with an elementary proof for the Ptolemy's theorem as well as Nine Point Circle Theorem ([24], [25]). Many other different simple approaches for proving the two theorems and their further generalizations are well known in the literature of Euclidean geometry. (some of them can be found in [1], [5], [6], [12], [13], [14], [17], [18], [21], [22], [23] and [26]). In this article we present a new proof for these two theorems based on a metric relation of circumcenter. Our proof actually follows from a lemma related to a circumcircle which gives a necessary condition for the four points to be concylic and in the conclusion of the article we will try to prove some remarks and related inequalities based on these lemmas.

2. Some Back Ground Material

The standard notation is used throughout:

For any arbitrary triangle ABC, we denote a, b, c for the
The following formulas are well known

sides of the lengths of BC, CA, AB, its semi perimeter by $s=\frac{1}{2}(a+b+c)$ and its area by Δ. Its classical centers are the circumcenter (S), the incenter (I), the centroid (G), the orthocenter (O) and nine-point center (N) respectively. Let D, E, F be the feet of the medians and K, L, M are the feet of the altitudes of the triangle ABC which lies on the sides BC, CA, AB and T, U, V are the mid points of AO, BO and CO. The median and the altitude through A (and their lengths) are denoted by m_a and h_a respectively. The classical radii are the circumradius R (=SA=SB=SC), the inradius r and the exradii r_1, r_2, r_3.

The Medial Triangle:

The triangle formed by the feet of the medians is called as Medial triangle. Its sides are parallel to the sides of given triangle ABC and by Thales Theorem the sides and angles of medial triangle are $\frac{a}{2}$, $\frac{b}{2}$, $\frac{c}{2}$, A, B and C respectively. Its area is $\frac{\Delta}{4}$, circumradius $\frac{R}{2}$, inradius $\frac{r}{2}$.

(a) $\Delta=\frac{abc}{4R}=rs=r_1(s-a)=r_2(s-b)=r_3(s-c)=\sqrt{s(s-a)(s-b)(s-c)}$

(b) $(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc$

(c) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C=\frac{abc}{2R^3}=\frac{2\Delta}{R^2}$

(d) For any triangle ABC, $\sum_{a,b,c} a^3\cos(B-C)=3abc$

(e) For any triangle ABC, $a^2\sin 2A+b^2\sin 2B+c^2\sin 2C=\frac{2\Delta}{R^2}\left(a^2+b^2+c^2-6R^2\right)$

(f) $\cos A\cos B\cos C=\frac{a^2+b^2+c^2-8R^2}{8R^2}$

3. Basic Lemma's

Lemma-1

Let M be any point in the plane of the triangle ABC, if S its circumcenter then

$$SM^2=\frac{R^2}{2\Delta}\left(\sin 2A.AM^2+\sin 2B.BM^2+\sin 2C.CM^2-2\Delta\right) \quad (1)$$

$$SM^2=\frac{1}{16\Delta^2}\left[a^2\left(b^2+c^2-a^2\right)AM^2+b^2\left(a^2+c^2-b^2\right)BM^2+c^2\left(b^2+a^2-c^2\right)CM^2-16R^2\Delta\right] \quad (2)$$

Proof:

The proof of (1) is available in [3].

Now for (2), we proceed as follows

We have

$\sin 2A=\frac{4a^2\left(b^2+c^2-a^2\right)\Delta}{2a^2b^2c^2}=\frac{a^2\left(b^2+c^2-a^2\right)}{8R^2\Delta}$

By replacing sin2A, sin2B, sin2C in **(1)** and by little computation, we get the conclusion **(2)**.

Lemma-2

If X is any point on the circumcircle of the triangle ABC then

$$\sin 2A.\,AX^2+\sin 2B.\,BX^2+\sin 2C.\,CX^2=4\Delta \quad (3)$$

$$a^2\left(b^2+c^2-a^2\right)AX^2+b^2\left(a^2+c^2-b^2\right)BX^2+c^2\left(b^2+a^2-c^2\right)CX^2=2a^2b^2c^2 \quad (4)$$

Proof:

The proof of **(3)** is available in **[3]**.

By replacing sin2A, sin2B, sin2C in **(1)** and with a little algebra, we get the conclusion **(4)**.

Lemma-3

If D, E, F are the feet of medians of ΔABC drawn from the vertices A, B, C on the sides BC, CA, AB and M is any point in the plane of the triangle then

$$4DM^2=2CM^2+2BM^2-a^2 \quad (5)$$

$$4EM^2=2CM^2+2AM^2-b^2 \quad (6)$$

$$4FM^2=2AM^2+2BM^2-c^2 \quad (7)$$

Proof:

Clearly for the triangle BMC, DM is a cevian.

So by using Stewart's theorem,

We have $DM^2=\frac{BD}{BC}CM^2+\frac{CD}{BC}BM^2-BD.CD$

By replacing BD=CD=$\frac{a}{2}$, we get the conclusion **(5)**.

In the similar manner we can prove the conclusions **(6)** and **(7)**.

Lemma-4

If ABCD is a cyclic quadrilateral and P is the point of intersection of diagonals AC and BD such that AB=a, BC=b, CD=c, DA=d and AC=p, BD=q then

AP.PC=BP.PD (chords property) (8)

$$\frac{p}{q}=\frac{ad+bc}{ab+cd} \quad (9)$$

Proof:

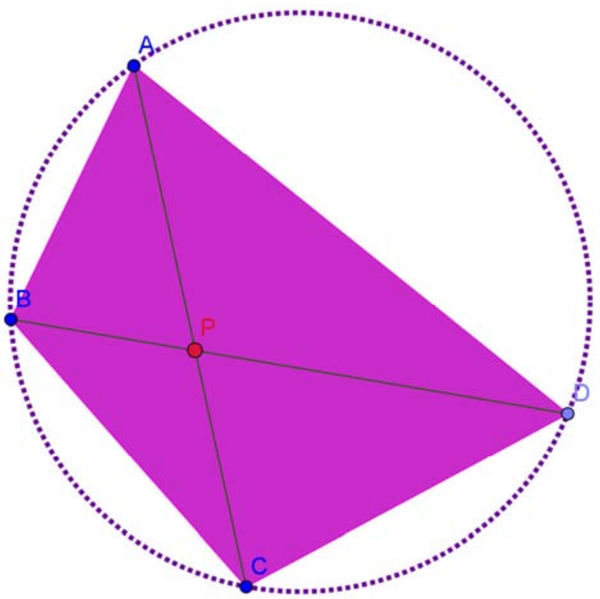

Figure 1. Cyclic quadrilateral.

Clearly from the figure1, Using the property "*In any circle Angles in the same segment are equal"* and by little angle chasing in the triangles ΔAPB, ΔDPC, ΔAPD and ΔBPC.

We can prove that pair of the triangles ΔAPB, ΔDPC and ΔAPD, ΔBPC are similar

It implies that AP.PC=BP.PD

Now for **(8)**, we proceed as follows

Since P is the point of intersection of diagonals and BP, CP are the cevians of the triangles ΔABC and ΔADC,

So we have $\frac{AP}{CP}=\frac{[\Delta ABP]}{[\Delta BPC]}=\frac{[\Delta APD]}{[\Delta CPD]}$

([ΔXYZ] represents area of the triangle XYZ)

It implies that $\frac{AP}{CP}=\frac{[\Delta ABP]+[\Delta APD]}{[\Delta BPC]+[\Delta CPD]}=\frac{[\Delta ABD]}{[\Delta BCD]}$

$$=\frac{\frac{1}{2}ad\sin A}{\frac{1}{2}bc\sin(180-A)}=\frac{ad}{bc}$$

In the similar manner we can prove that $\frac{BP}{DP}=\frac{ab}{cd}$

Now let AP=adx, CP=bcx and BP=aby, DP=cdy for some non zero real x, y

But from **(8)**, AP.PC = BP.PD

It implies that x=y

So $\frac{AC}{BD}=\frac{AP+CP}{BP+DP}=\frac{(ad+bc)x}{(ab+cd)y}$

It implies the conclusion **(9)**.

Lemma-5

If ABCD is a cyclic quadrilateral such that AB=a, BC=b, CD=c, DA=d and AC=p, BD=q where $p\neq q$ then

$$\left[(ac+bd)^2-p^2q^2\right]\left[p^2-q^2\right]=\left[(ab+cd)^2p^2-(ad+bc)^2q^2\right] \quad (10)$$

$$p^2q^2\left(a^2+b^2+c^2+d^2-p^2-q^2\right)-\left(a^2-b^2+c^2-d^2\right)\left(a^2c^2-b^2d^2\right)-p^2\left(a^2-d^2\right)\left(b^2-c^2\right)+q^2\left(a^2-b^2\right)\left(c^2-d^2\right)=0 \quad (11)$$

Proof:

Since the Quadrilateral ABCD is a cyclic Quadrilateral, the 4 points A, B, C and D are concyclic. Hence one point among A, B, C and D will lie on the circumcircle of the triangle formed by the remaining three points.

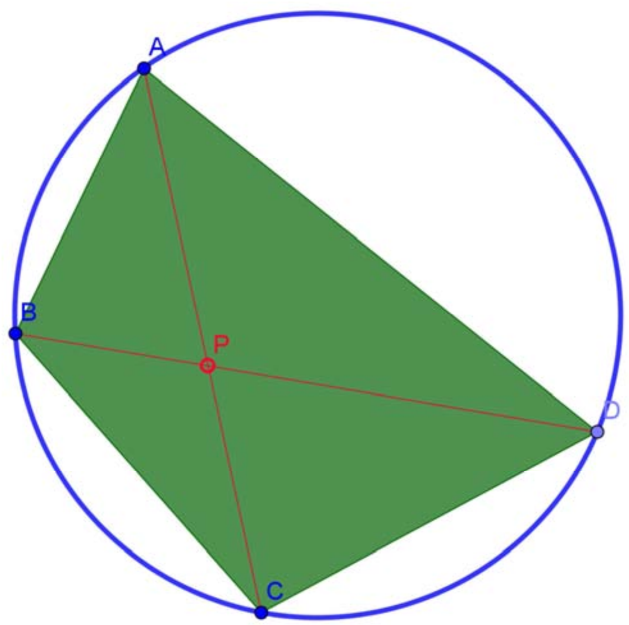

Figure 2. Cyclic quadrilateral.

For triangle ABC, the point D lies on the circumcircle of triangle ABC.
So using lemma-2, by fixing X as D in the conclusion **(4)**,
We get

$$b^2\left(a^2+p^2-b^2\right)AD^2+p^2\left(a^2+b^2-p^2\right)BD^2+a^2\left(b^2+p^2-a^2\right)CD^2=2a^2b^2p^2 \tag{12}$$

Now for triangle ADC, the point B lies on the circumcircle of triangle ADC.
And using lemma-2, by fixing X as B in the conclusion **(4)**,
We get

$$c^2\left(d^2+p^2-c^2\right)AB^2+p^2\left(c^2+d^2-p^2\right)DB^2+d^2\left(c^2+p^2-d^2\right)CB^2=2c^2d^2p^2 \tag{13}$$

In the similar manner for triangles Δ ADB, Δ ABD we get

$$q^2\left(d^2+a^2-q^2\right)AC^2+d^2\left(a^2+q^2-d^2\right)BC^2+a^2\left(d^2+q^2-a^2\right)DC^2=2a^2d^2q^2 \tag{14}$$

$$c^2\left(b^2+q^2-c^2\right)BA^2+q^2\left(b^2+c^2-q^2\right)CA^2+b^2\left(c^2+q^2-b^2\right)DA^2=2b^2c^2q^2 \tag{15}$$

Now by replacing AB=a, BC=b, CD=c, DA=d, AC=p and BD=q in **(12)**, **(13)**, **(14)** and **(15)** we get

$$b^2d^2\left(a^2+p^2-b^2\right)+p^2q^2\left(a^2+b^2-p^2\right)+a^2c^2\left(b^2+p^2-a^2\right)=2a^2b^2p^2 \tag{16}$$

$$c^2a^2\left(d^2+p^2-c^2\right)+p^2q^2\left(c^2+d^2-p^2\right)+b^2d^2\left(c^2+p^2-d^2\right)=2c^2d^2p^2 \tag{17}$$

$$p^2q^2\left(d^2+a^2-q^2\right)+b^2d^2\left(a^2+q^2-d^2\right)+a^2c^2\left(d^2+q^2-a^2\right)=2a^2d^2q^2 \tag{18}$$

$$a^2c^2\left(b^2+q^2-c^2\right)+p^2q^2\left(b^2+c^2-q^2\right)+b^2d^2\left(c^2+q^2-b^2\right)=2b^2c^2q^2 \tag{19}$$

Now by the computation of [**(16)**+**(17)**]-[**(18)**+**(19)**], we get

$$\left[a^2c^2+b^2d^2-p^2q^2\right]\left[p^2-q^2\right]=\left(a^2b^2+c^2d^2\right)p^2-\left(a^2d^2+b^2c^2\right)q^2$$

It implies $\left[(ac+bd)^2-p^2q^2\right]\left[p^2-q^2\right]=\left[(ab+cd)^2p^2-(ad+bc)^2q^2\right]$
Which completes the proof of **(10)**.
Now for the conclusion **(11)**,
We compute [**(16)**+**(17)**+**(18)**+**(19)**].
Remark:
The conclusion **(11)** usually called as ***Euler's four point relation*** (In this article we proved this relation for a cyclic quadrilateral, actually it is true for any convex quadrilateral). **[12]**
Lemma-6
If X is any point on the circumcircle of the medial triangle DEF whose angles are A, B, C and area is $\frac{\Delta}{4}$ then

$$\boldsymbol{\sin 2A.\,DX^2+\sin 2B.\,EX^2+\sin 2C.\,FX^2=\Delta} \tag{20}$$

$$\boldsymbol{(\sin 2B+\sin 2C)AX^2+(\sin 2A+\sin 2C)BX^2+(\sin 2A+\sin 2B)CX^2=\Delta\left(a^2+b^2+c^2-5R^2\right)} \tag{21}$$

Proof:
Using the lemma-2**(3)**,
We have, If X is any point on the circumcircle of triangle ABC then
$\sin 2A.AX^2+\sin 2B.BX^2+\sin 2C.CX^2=4\Delta$
Now for medial triangle DEF,
If X is any point on the Circumcircle then

$$\sin 2A.DX^2+\sin 2B.EX^2+\sin 2C\,FX^2=4\frac{\Delta}{4}$$

It implies the conclusion **(20)**.
Now for the conclusion **(21)**,
We proceed as follows
By using lemma-3**(5)**, **(6)**, **(7)** Replace DX, EX, FX in **(20)**
We get

$$\sum 2(\sin 2B+\sin 2C)AX^2 - \sum a^2\sin 2A = 4\Delta \qquad (22)$$

Using **(g)**, we have

$$a^2\sin 2A+b^2\sin 2B+c^2\sin 2C=\frac{2\Delta}{R^2}\left(a^2+b^2+c^2-6R^2\right)$$

By replacing $\sum a^2\sin 2A$ using **(g)** in **(22)** and by little algebra gives the conclusion **(21)**.

Lemma-7

If triangle DEF is a medial triangle and the points K, L and M are the foot of perpendiculars drawn from the Vertices A, B and C of triangle ABC to the sides BC, CA and AB respectively then the 6 points D, E, F, K, L and M are concyclic.

Proof:

By using lemma-6**(20)**,

If X is any point on the circumcircle of medial triangle DEF then

$$\sin 2A.\,DX^2+\sin 2B.\,EX^2+\sin 2C.\,FX^2=\Delta \qquad (23)$$

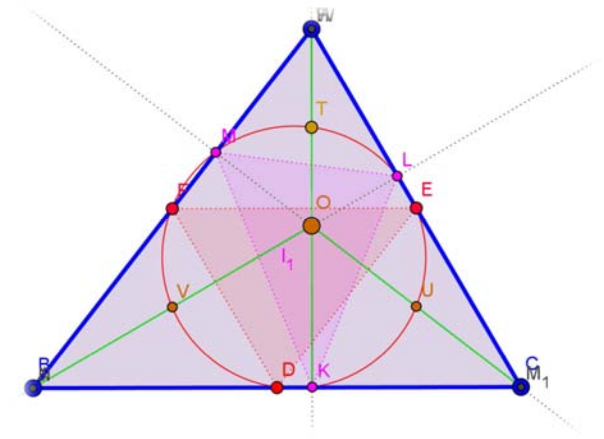

Figure 3. *D, E, F, K, L and M are concylic.*

Clearly to prove D, E, F, K, L and M are concyclic, it is enough to prove that the X present in **(23)** is satisfied by K, L and M simultaneously.

Consider, $\sin 2A.DK^2+\sin 2B.EK^2+\sin 2C.FK^2$

It is clear that from the triangles ΔBFK and ΔCEK, $EK=CE=AE=\frac{b}{2}$, $FK=AF=BF=\frac{c}{2}$ (Mid Point of hypotenuse of a right triangle acts as circumcenter, so it is equidistant from the three vertices)

$$DK=|BK-BD|=\left|c\cos B-\frac{a}{2}\right| \quad \sin 2A.\,DK^2+\sin 2B.\,EK^2+\sin 2C.\,FK^2=\sin 2A\left(c\cos B-\frac{a}{2}\right)^2+\sin 2B\frac{b^2}{4}+\sin 2C\frac{c^2}{4}$$

$$=\frac{1}{4}\left[\sum\left(a^2\sin 2A\right)\right]+c\cos B\sin 2A(c\cos B-a)$$

$$=\frac{1}{4}\left[\sum\left(a^2\sin 2A\right)\right]-4\Delta\cos A\cos B\cos C \qquad (24)$$

Now using **(e)** and **(f)** we can prove **(24)** is equal to Δ.

$$\Rightarrow \sin 2A.\,DK^2+\sin 2B.\,EK^2+\sin 2C.\,FK^2=\Delta$$

Hence using **(23)** we can conclude that the point K lies on the circumcircle of the triangle DEF, that is D, E, F and K are concyclic.

In the similar manner using **(23)** we can prove L and M also lie on the circumcircle of the triangle DEF.

That is the 6 points D, E, F, K, L and M are concyclic. (see figure 3).

Lemma- 8

If the triangle DEF is a medial triangle and the points T, U and V are the mid points of AO, BO and CO respectively where O is the orthocenter of the triangle ABC then the 6 points D, E, F, T, U and V are concyclic.

Proof:

By using lemma-6 **(20)**,

If X is any point on the circumcircle of medial triangle DEF then

$$\sin 2A.\,DX^2+\sin 2B.\,EX^2+\sin 2C.\,FX^2=\Delta$$

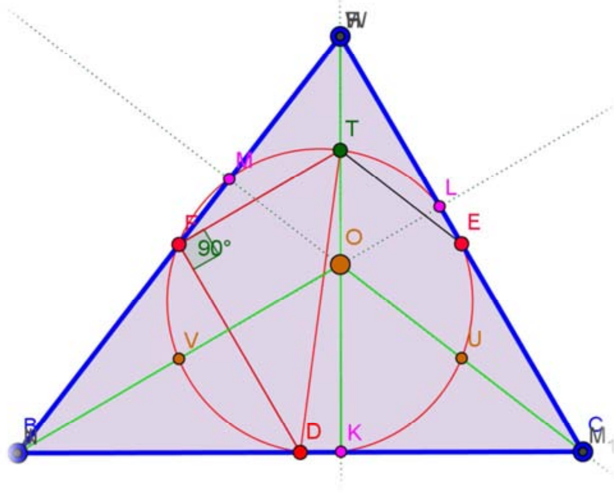

Figure 4. *D, E, F, T, U and V are concyclic.*

Clearly, to prove D, E, F, T, U and V are concyclic, it is enough to prove that the X present in (23) is satisfied by T, U and V simultaneously.

Consider,
$\sin 2A.\,DT^2 + \sin 2B.\,ET^2 + \sin 2C.\,FT^2$

Clearly In triangles ΔABO and ΔACO, the points F, E and T are the midpoints of sides AB, AC and AO.

So by the Mid Points Theorem, we can prove that the pairs of the lines (DF, AC), (FT, BO) and (ET, CO) are parallel.

It implies angle AFT = angle ABO=90^0-A = angle AET = angle ACO.

And angle BFD = angle BAC = angle A.

It implies angleDFT=180° – angle BFD – angleTFA= 90°

And again by Mid Points Theorem we have

$$FT = \frac{BO}{2} = \frac{2R\cos B}{2} = R\cos B \text{ and } ET = \frac{CO}{2} = \frac{2R\cos C}{2} = R\cos C\ ,$$

$$DF = \frac{AC}{2} = \frac{b}{2}$$

Now from triangle DFT, Since angle DFT = 90°,

So by Pythagorus Theorem, we have

$$DT^2 = DF^2 + FT^2 = \frac{b^2}{4} + R^2\cos^2 B = R^2\sin^2 B + R^2\cos^2 B = R^2$$

$$\text{So}\quad \sin 2A.\,DT^2 + \sin 2B.\,ET^2 + \sin 2C.\,FT^2 = \sin 2A.\,R^2 + \sin 2B.\,R^2\cos^2 C + \sin 2C.\,R^2\cos^2 B$$

$$= R^2\left(\sum \sin 2A\right) - 2R^2 \sin B \sin C\left(\sin B\cos C + \sin C\cos B\right) \qquad (25)$$

Now using (g) we can prove (25) is equal to Δ.

$$\Rightarrow\ \sin 2A.\,DT^2 + \sin 2B.\,ET^2 + \sin 2C.\,FT^2 = \Delta$$

Hence using (23) we can conclude that the point T lies on the circumcircle of the triangle DEF, that is D, E, F and T are concyclic.

In the similar manner using (23) we can prove U and V also lie on the circumcircle of the triangle DEF.

That is the 6 points D, E, F, T, U and V are concyclic (see figure-4).

Hence proved.

Now we are in a position to deal with the proof of most celebrated theorems in a very prominent way.

4. Main Theorems

A). PTOELMY'S THEOREM

Let ABCD is any arbitrary cyclic quadrilateral such that AC and BD are its diagonals then $pq = ac + bd$

Proof:

Using lemma-4**(9)**,

We have $\frac{p}{q} = \frac{ad+bc}{ab+cd}$,

It implies

$$\left[(ab+cd)^2 p^2 - (ad+bc)^2 q^2\right] = 0 \qquad (26)$$

Now using lemma-5**(10)**
we have

$$\left[(ac+bd)^2 - p^2q^2\right]\left[p^2 - q^2\right] = \left[(ab+cd)^2 p^2 - (ad+bc)^2 q^2\right]$$

By combining **(10)** and **(26)** we get $pq = ac + bd$

It completes the proof of the Ptolemy's Theorem.

B). NINE POINT CIRCLE THEOREM

Let D, E, F are the foot of the medians and K, L, M are the foot of altitudes of the triangle ABC and T, U, V are the midpoints of AO, BO and CO where O is orthocenter then D, E, F, K, L, M, T, U and V are concyclic.

Proof:

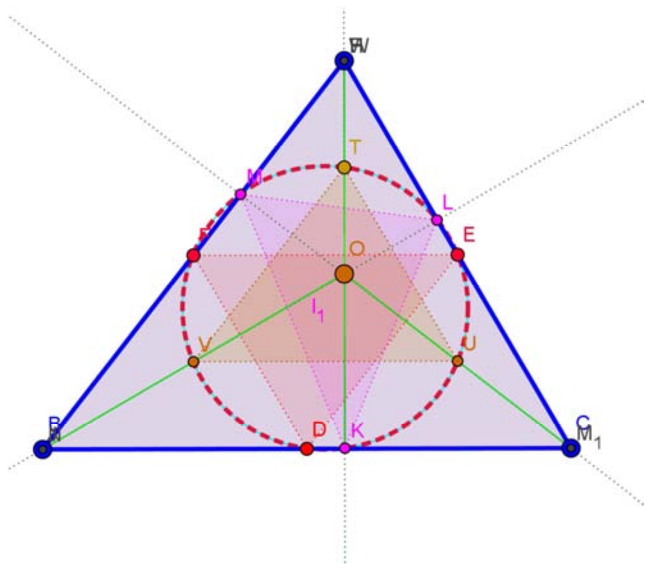

Figure 5. *Nine Point Circle.*

By combining lemma-7 and lemma-8 we can conclude that the 9 points D, E, F, K, L, M, T, U and V are concyclic (see figure -5).

It proves the Nine Point Circle Theorem.

5. Remarks

$$16\Delta^2 = 2a^2b^2 + 2a^2b^2 + 2a^2b^2 - a^4 - b^4 - c^4 \qquad (27)$$

Proof:

We have by lemma-1**(2)**

$$SM^2 = \frac{1}{16\Delta^2}\left[a^2\left(b^2+c^2-a^2\right)AM^2 + b^2\left(a^2+c^2-b^2\right)BM^2 + c^2\left(b^2+a^2-c^2\right)CM^2 - 16R^2\Delta\right] \qquad (28)$$

Since the conclusion **(2)** is true for any M, let us fix M as S (circumcenter),
By replacing AS = BS = CS = R and $SM^2 = SS^2 = 0$,
We get

$$0 = \frac{1}{16\Delta^2}\left[a^2\left(b^2+c^2-a^2\right)R^2 + b^2\left(a^2+c^2-b^2\right)R^2 + c^2\left(b^2+a^2-c^2\right)R^2 - 16R^2\Delta\right]$$

Further simplification gives the conclusion **(27)**.

1. If triangle ABC is an Equilateral and X be any point on the circumcircle of the triangle ABC then

$$\boldsymbol{AX^2 + BX^2 + CX^2 = \frac{8\Delta}{\sqrt{3}} = 6R^2} \tag{29}$$

Proof:

We have from lemma-2 **(3)**,

If X is any point on the circumcircle of triangle ABC then
$sin2A.AX^2 + sin2B.BX^2 + sin2C.CX^2 = 4\Delta$

If the triangle ABC is an equilateral triangle then angle A = angle B = angle C = 60°

So $sin2A = sin2B = sin2C = sin120^0 = \frac{\sqrt{3}}{2}$

And we have area of an equilateral triangle $\Delta = \frac{\sqrt{3}}{4}a^2 = \frac{3\sqrt{3}}{4}R^2$, where a and R is the length of the side and circumradius of the triangle respectively.

By replacing these in **(3)**, we get conclusion **(29)**.

2. If X is a point on the circumcircle of triangle ABC which acts as a feet of the internal angular bisector of angle A then

$$\boldsymbol{\frac{AX}{IX} = \frac{cos\left(\frac{B-C}{2}\right)}{cos\left(\frac{B+C}{2}\right)}} \tag{30}$$

$$\boldsymbol{\frac{AX^2 - 2R^2}{IX^2 - 2R^2} = \frac{cos\left(B-C\right)}{cos\left(B+C\right)}} \tag{31}$$

$$\boldsymbol{AX = 2Rcos\left(\frac{B-C}{2}\right), IX = 2Rcos\left(\frac{B+C}{2}\right)} \tag{32}$$

Proof:

Since X is the feet of internal angular bisector of angle A,

So by angle chasing we can prove that BX = CX = IX

Hence by Ptolemy's Theorem for the cyclic quadrilateral ABXC,

We have $AX.BC = AC.BX + AB.CX$

Further simplification gives

$$\frac{AX}{IX} = \frac{AC+AB}{BC} = \frac{\sin B + \sin C}{\sin A} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)}$$

Which proves **(30)**

Now for **(31)** we proceed as follows

We know by lemma-2**(3)**,

If X is any point on the circumcircle of triangle ABC then

$sin2A.AX^2 + sin2B.BX^2 + sin2C.CX^2 = 4\Delta$

Clearly BX=CX=IX

So $\sin 2A.AX^2 + \left(\sin 2B + \sin 2C\right)IX^2 = 4\Delta$

Further simplification (by replacing sin2A and 4Δ) using the transformations of the angles gives **(31)**

Now by combining **(30)**, **(31)** we get **(32).**

3. If M is any point in the plane of the triangle ABC which minimizes the sum

sin2A.AM² + sin2B.BM² + sin2C.CM² as 2Δ then M must be circumcenter (S). (33)

Proof:

We have by lemma-2,

$$SM^2 = \frac{R^2}{2\Delta}\left(sin2A.AM^2 + sin2B.BM^2 + sin2C.CM^2 - 2\Delta\right)$$

Now Since the square of any real is nonnegative,

We have

$$SM^2 \geq 0$$

And it is clear that the equality holds when M coincides with S (circumcenter).

It gives $sin2A.AM^2 + sin2B.BM^2 + sin2C.CM^2 \geq 2\Delta$

Which proves the required conclusion **(33)**.

$$\boldsymbol{sin2A + sin2B + sin2C = \frac{2\Delta}{R^2}} \tag{34}$$

Proof:

Using transformations of angles we can prove **(34)** which can be available in any academic trigonometry book.

But here we will prove this identity using lemma-1**(1)**

We have by lemma-1**(1)**,

If S is the circumcenter of a triangle and M be any point in the plane of triangle then

$$SM^2 = \frac{R^2}{2\Delta}\left(sin2A.AM^2 + sin2B.BM^2 + sin2C.CM^2 - 2\Delta\right)$$

It is true for any M, let us fix M as S(circumcenter), then

$$SS^2 = \frac{R^2}{2\Delta}\left(sin2A.AS^2 + sin2B.BS^2 + sin2C.CS^2 - 2\Delta\right)$$

By replacing AS = BS = CS = R and SS=0

We get conclusion **(34)**.

In the similar manner we can derive some more remarks using the lemma's discussed earlier in section 3.

For few more generalizations and historical details about Ptolemy's theorem and Nine Point Circle Theorem refer [2], [4], [7], [8], [9], [10], [11], [13], [14], [15], [16], [19], [20] and [26].

6. Conclusion

This article has shown New and more elegant approach to prove that the give three points are concyclic, based on this approach we proved two famous theorems related to concyclic points. The proofs presented here only require basic knowledge of trigonometry and no advanced knowledge of synthetic projective geometry, the proofs are certainly an impressive tour –de-force of algebraic-trig manipulation and application. The approach which we dealt in this paper is also useful to study and investigate the further properties of concyclic points. We can extend this approach to prove the necessary condition for the points to be concyclic.

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Biography

Dasari Naga Vijay Krishna is the son of Sri D Adhiseshu and Sri D Sulochana, He is renowned faculty in mathematics who has served in all the reputed Instutions in India with more than a decade's of experience in the training of the students for various competitive exams. Currently he is working as a senior lecturer in the Department of Mathematics, Narayana Educational Instutions. He received his B.Sc Honors Degree in Mathematics from Magadha University. He also holds Master Degree in Mathematics from Kalinga University. His primary research is Euclidean Geometry. His several research articles were published in reputed journals.