Stability and Oscillatory Behavior of the Solutions on a Class of Coupled Van der Pol-Duffing Equations with Delays

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Abstract: In the present paper, a class of coupled van der Pol-Duffing oscillators with a nonlinear friction of higher polynomial order model which involves time delays is investigated. The coefficients of the highest order of the polynomial determine the boundedness of the solutions. With special attention to the boundedness of the solutions and the instability of the unique equilibrium point of linearized system, some sufficient conditions to guarantee the existence of oscillatory solutions for the model are obtained based on the generalized Chafee's criterion. Convergence of the trivial solution is determined by the negative real part of eigenvalues of the linearized system. Examples are provided to demonstrate the reduced conservativeness for the parameters of the proposed results. The results obtained shown that the passive decay rate in the model affects the oscillatory frequency and amplitude. When a permanent oscillation occurred, time delays affect mainly oscillatory frequency and amplitude slightly.

Keywords: Coupled Van der Pol-Duffing Equation, Delay, Stability, Oscillation

1. Introduction

It is known that the van der Pol (VDP) oscillator could model the typical self-excited or self-sustained oscillation. Various coupled van der Pol or van der Pol-Duffing equations have been applied in physics and engineering. Many good results have appeared [1-12]. For the following three-dimensional autonomous van der Pol-Duffing type oscillator system:

\begin{equation}
\left\{ \begin{array}{l}
x'(t) = y(t), \\
y'(t) = -x(t) + \beta x^2(t) + \varepsilon(1-x^2(t))y(t) - kz(t), \\
z'(t) = y(t) - z(t).
\end{array} \right.
\end{equation}

By analyzing the stability of the equilibrium points, the existence of Hopf bifurcation is established [1]. Barron has considered the stability of a ring of coupled van der Pol oscillators with non-uniform distribution of the coupling parameter as follows:

\begin{equation}
x''_i(t) + a(x^2_i(t) - 1)x_i(t) + x_i(t) = b_i(x_{i-1}(t) - 2x_i(t) + x_{i+1}(t))
\end{equation}

where \(1 \leq i \leq n, b_i\) are the coupling parameter corresponding to the \(i\)th oscillator. For a modified hybrid van der Pol-Duffing-Rayleigh oscillator for modeling the lateral walking force on a rigid floor:

\begin{equation}
x''(t) + \mu(1 - x^2(t) + ax^4(t) - bx^6(t))x'(t) + x(t) = \theta_0 \sin \omega t
\end{equation}

Kumar et al. have studied the stability of the equation (3) by the perturbation and energy balance method [3]. Rompala et al. have considered a system of three van der Pol oscillators [5]. For a ring of four mutually coupled biological systems described by coupled van der Pol oscillators, the stability boundaries and the main dynamical states have been considered on the stability maps by Kadji et al. [6]. A driven van der Pol-like oscillator with a nonlinear friction of higher polynomial order model as follows:

\begin{equation}
x''(t) + \mu(1 - x^2(t) + ax^4(t) - bx^6(t))x'(t) + x(t) = \theta_0 \sin \omega t
\end{equation}

The effects of noise correlation on the coherence of a forced van der Pol type birhythmic system has been
phenomena such as manufacturing process, nuclear reactors, rocket motors, mechanical controlling systems, population dynamics, and so on. Naturally the time delay coupled van der Pol equations also have been extensively studied by many researchers [13-24]. For example, Li et al. have studied the coupled van der Pol oscillators with two kinds of delays [13]:

\[
\begin{align*}
\dot{y}_1'(t) + w^2 y_1(t) - \varepsilon (1 - y_1^2(t)) y_1'(t) &= \varepsilon \alpha (y_2(t - \tau) + y_1'(t - \tau)), \\
\dot{y}_2'(t) + w^2 y_2(t) - \varepsilon (1 - y_2^2(t)) y_2'(t) &= \varepsilon \alpha (y_1(t - \tau) + y_2'(t - \tau)).
\end{align*}
\]

(5)

Zhang and Gu used the theory of normal form and central manifold theorem to discuss the following time delay system [14]:

\[
\begin{align*}
x_1'(t) + \varepsilon (x_1^3(t) - 1) x_1'(t) + x_1(t) &= \alpha (y_1(t - \tau) - x_1(t)), \\
y_1'(t) + \varepsilon (y_1^3(t) - 1) y_1'(t) + y_1(t) &= \alpha (x_1(t - \tau) - y_1(t)).
\end{align*}
\]

(6)

Motivated by the above models, in this paper we consider the following a ring of time delays Duffing-van der Pol-like oscillator with a nonlinear friction of higher polynomial order system:

\[
\begin{align*}
u_i'(t) + \varepsilon_i [\bar{u}_i u_i(t) - \bar{k}_i u_i(t) + u_i^3(t) - \bar{a}_i u_i(t) + \bar{c}_i u_i(t)] = &\bar{b}_i [u_i(t - \tau_{ni}) - 2 u_i(t - \tau_{i}) + u_i(t - \tau_{ni})], \\
u_i'(t) + \varepsilon_i [\bar{u}_i u_i(t) - \bar{k}_i u_i(t) + u_i^3(t) - \bar{a}_i u_i(t) + \bar{c}_i u_i(t)] = &\bar{b}_i [u_i(t - \tau_{ni}) - 2 u_i(t - \tau_{ni}) + u_i(t - \tau_{ni})], \\
\ldots &
\end{align*}
\]

(7)

where \(0 < l_i, \bar{k}_i \) and \(0 < \bar{e}_i < 1; \bar{c}_i, \bar{b}_i, \bar{a}_i, \bar{b}_i \in \mathbb{R} \) for each \(i = 1, 2, \ldots, n \). \(0 \leq t_i \) are time delays. Our aim is to investigate the dynamical behavior of \(n \) coupled oscillators by means of the generalized Chafee's criterion [25, 26].

2. Preliminaries

For convenience, setting \(\xi_i = 2 \xi_i, \bar{l}_i = l_{2i}, \bar{k}_i = k_{2i}, \bar{a}_i = a_{2i}, \bar{c}_i = c_{2i-1}, \bar{b}_i = b_{2i-1}, \bar{b}_i = b_{2i-1}, \bar{\tau}_i = \tau_{2i-1} (1 \leq i \leq n) \). Then the coupled system (7) can be written as the following equivalent system:

\[
\begin{align*}
x_1'(t) &= x_2(t), \\
x_2'(t) &= -\bar{c}_1 x_1(t) - \bar{\beta}_1 x_1^3(t) + \bar{b}_1 [x_2(t - \tau_{ni}) - 2 x_1(t - \tau_{i}) + x_1(t - \tau_{ni})] + \varepsilon_2 x_2 x_1'(t) - \varepsilon_2 x_2^3 x_1'(t) x_2(t) + \varepsilon_2 k x_2^3 x_2(t) x_1(t) - \varepsilon_2 [x_2(t) x_1(t) - x_2(t) x_1(t)], \\
x_1'(t) &= -\bar{c}_3 x_3(t) - \bar{\beta}_3 x_3^3(t) + \bar{b}_3 [x_3(t - \tau_{ni}) - 2 x_3(t - \tau_{ni}) + x_3(t - \tau_{ni})] + \varepsilon_4 x_4 x_3'(t) - \varepsilon_4 x_4^3 x_3'(t) x_4(t) + \varepsilon_4 k x_4^3 x_4(t) x_3(t) - \varepsilon_4 [x_4(t) x_3(t) - x_4(t) x_3(t)], \\
\ldots &
\end{align*}
\]

(8)

The matrix form of system (8) is the following: \((b_{ij})_{2n \times 2n} \) are 2n by 2n matrices as follows:

\[
X'(t) = AX(t) + BX(t - \tau) + g(X)
\]

(9)

where \(X(t) = [x_1(t), x_2(t), \ldots, x_{2n}(t)]'\),

\[
X(t - \tau) = [x_1(t - \tau), 0, x_3(t - \tau), \ldots, x_{2n-1}(t - 2 \tau_{ni}), 0]'\text{,}
\]

\[
g(X) = [0, -\bar{c}_1 x_1^3(t) - \bar{\beta}_1 x_1^3(t) x_2(t) + \bar{b}_1 [x_2(t - \tau_{ni}) - 2 x_1(t - \tau_{ni}) + x_1(t - \tau_{ni})] - \varepsilon_2 x_2 x_1'(t) x_2(t) + \varepsilon_2 k x_2^3 x_2(t) x_1(t) - \varepsilon_2 x_2 x_1'(t) x_2(t) + \varepsilon_2 k x_2^3 x_2(t) x_1(t) - \varepsilon_2 x_2 x_1'(t) x_2(t) +\ldots - c_{2n-1} a_{2n2n}]
\]

Both \(A = (a_{ij})_{2n \times 2n} \) and \(B =

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-\bar{c}_1 & a_{22} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & -\bar{c}_3 & a_{44} & \cdots & 0 & 0 \\
0 & 0 & 0 & \ldots & a_{22} & 0 & \cdots \\
\end{pmatrix}
\]
where \(a_{22} = \varepsilon_2 a_2, a_{44} = \varepsilon_4 a_4, a_{2n2} = \varepsilon_{2n} a_{2n}.\)

\[
B = (b_{ij})_{2n \times 2n} = \\
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
-2b_1 & 0 & b_1 & 0 & \cdots & b_1 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
b_{2n-1} & 0 & 0 & \cdots & 0 & -2b_{2n-1} \\
\end{pmatrix}
\]

Obviously, the origin \(x_k = 0\) \((k = 1, 2, \ldots, 2n)\) is an equilibrium of system (8). The linearization of system (9) at origin is

\[
X'(t) = AX(t) + BX(t - \tau)
\]

\[
x'_i = 0,
\]

\[
-c_1x'_i - \beta_1(x_i)^2 + b_1[x_{2i-1} - 2x_i + x'_i] + \varepsilon_2 a_2 x'_i - \varepsilon_2(x_i)^2 x'_2 + \varepsilon_2 k_2(x_i)^4 x'_2 - \varepsilon_2 l_2(x_i)^6 x'_2 = 0,
\]

\[
x'_i = 0,
\]

\[
-c_3 x'_i - \beta_3(x_i)^2 + b_3[x_i^2 - 2x_i + x'_i] + \varepsilon_4 a_4 x'_i - \varepsilon_4(x_i)^2 x'_4 + \varepsilon_4 k_4(x_i)^4 x'_4 - \varepsilon_4 l_4(x_i)^6 x'_4 = 0,
\]

\[
x'_{2n-2} = 0,
\]

\[
-c_{2n-3} x'_{2n-3} - \beta_{2n-3}(x_{2n-3})^3 + b_{2n-3}[x_{2n-3} - 2x_{2n-3} + x'_i] + \varepsilon_2 a_{2n-2} x''_{2n-2} - \varepsilon_2 x''_{2n-2} = 0,
\]

\[
x'_i = 0,
\]

\[
-c_{2n-1} x'_{2n-1} - \beta_{2n-1}(x_{2n-1})^3 + b_{2n-1}[x_{2n-1} - 2x_{2n-1} + x'_i] + \varepsilon_2 a_{2n-1} x''_{2n-1} - \varepsilon_2 x''_{2n-1} = 0.
\]

Since \(x'_{2i} = 0\) \((1 \leq i \leq n)\), from (11) we get

\[
-c_1x'_i - \beta_1(x_i)^2 + b_1[x_{2i-1} - 2x_i + x'_i] = 0,
\]

\[
c_3 x'_i - \beta_3(x_i)^2 + b_3[x_i^2 - 2x_i + x'_i] = 0,
\]

\[
-c_{2n-3} x'_{2n-3} - \beta_{2n-3}(x_{2n-3})^3 + b_{2n-3}[x_{2n-3} - 2x_{2n-3} + x'_i] = 0,
\]

\[
-c_{2n-1} x'_{2n-1} - \beta_{2n-1}(x_{2n-1})^3 + b_{2n-1}[x_{2n-1} - 2x_{2n-1} + x'_i] = 0.
\]

System (12) can be written as a matrix form as the have

\[
D X' = 0
\]

where \(D = (d_{ij})_{n \times n} = \begin{pmatrix} d_{11} & b_1 & 0 & \cdots & b_{1} \\
0 & d_{21} & b_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_{n1} \\
b_{2n-1} & 0 & 0 & \cdots & 0 \end{pmatrix},
\]

where \(d_{ii} = -2b_{2i-1} - c_{2i-1} - \beta_{2i-1}(x'_{2i-1})^2(1 \leq i \leq n).\)

According to standard results in linear algebra, if \(D\) is a nonsingular matrix, system (13) has only one solution, namely, the trivial solution. When \(x'_{2i-1} = 0\) \((1 \leq i \leq n),\) matrix \(D\) changes to \(C.\) The proof is completed.  

Lemma 2 All solutions of system (8) are uniformly bounded.

Proof Construct a Lyapunov function \(V(t) = \sum_{i=1}^{2n} \frac{1}{2} x_i^2(t).\)

Calculating the derivative of \(V(t)\) through system (8) we have

\[
V'(t)(8) = \sum_{i=1}^{2n} x_i(t) x'_i(t)
\]

\[
= x_i(t) x'_i(t) + x_i(t) (-c_1 x_i(t) - \beta_1 x_i^2(t))
\]

\[
+ b_1 [x_{2i-1} - 2x_i + x'_i] + \varepsilon_2 a_2 x'_i - \varepsilon_2 x''_{2i-1} - 2x_i(t) - \tau_i
\]

\[
+ \varepsilon_4 a_4 x'_i - \varepsilon_4 x''_{2i-1} + \varepsilon_4 x''_{2i-1} + \varepsilon_4 l_4 x''_i
\]

\[
+ b_{2n-1} x_{2i-1} - 2x_{2i-1} + x'_i + \varepsilon_2 a_{2n-2} x''_{2i-1} - \varepsilon_2 x''_{2i-1} - 2x_{2i-1} + x'_i
\]

\[
+ \varepsilon_2 a_{2n-1} x''_{2i-1} - \varepsilon_2 x''_{2i-1} - 2x_{2i-1} + x'_i
\]

\[
= (1 - c_1) x_i(t) x'_i(t) - \beta_1 x_i^2(t) x'_i(t) + b_1 x_i(t) [x_{2i-1} - 2x_i(t) - \tau_i] + \varepsilon_2 a_2 x'_i - \varepsilon_2 x''_{2i-1} - 2x_i(t) - \tau_i
\]

Let \(C = (c_{ij})_{n \times n} = \begin{pmatrix} c_{11} & 0 & b_1 & 0 & \cdots & b_1 \\
0 & c_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
b_{2n-1} & 0 & 0 & \cdots & 0 \\
\end{pmatrix},\)

where \(c_{11} = -2b_1 - c_1, c_{2n-3} = -2b_{2n-1} - c_{2n-1}.\)

Lemma 1 Assume that the matrix \(C\) is a nonsingular matrix, then system (8) (or (9)) has a unique equilibrium.
matrix $R$ equilibrium point, for selected parameter values of (9) is stable. The proof is completed.

As equation, we know that the trivial solution of system (8) (or (9)) is unstable, then the unique equilibrium point, namely, the trivial solution of system (8) is stable.

Proof Since $Re(\gamma_i) (i = 1, 2, \ldots, 2n) < -r < 0$, hence there exists a positive constant $K \geq 1$ such that $\|e^{(A + B)x}\| \leq Ke^{-rt}$. In (15) for $t \geq \tau^*$ we have

$$X'(t) = (A + B)X(t) - B \int_{t-\tau^*}^{t} X'(s)ds = (A + B)X(t) - B \int_{t-\tau^*}^{t} (AX(s) + B(s - tau^*)) ds$$

From (16), for $t \geq \tau^*$ we get

$$X'(t) = e^{(A + B)(t - \tau^*)}X(\tau^*) - B \int_{\tau^*}^{t} ds e^{(A + B)(t - s)} (AX(s) + B(u - tau^*)) du$$

Therefore,

$$\|X(t)\| \leq LKe^{-r(t - \tau^*)} + K\|B\| \int_{\tau^*}^{t} ds \int_{s-\tau^*}^{s} e^{-r(t-s)} (\|A\|\|X(s)\| + \|B\|\|X(s)\|) du$$

where $L = \sup_{t \in [\tau^*, \tau^* + \tau]} \|X(t)\|$. We shall prove that there exists a positive constant $c (< r)$ such that $\|X(t)\| \leq LKe^{-c(t - \tau^*)}, t \geq \tau^*$. Indeed, select $c (< r)$ such that

$$K\|B\|(\|A\| + \|B\|e^{ct})(e^{ct} - 1) = c(r - c)$$

then from (18) we have

This means that $X(t) \to 0$ as $t \to \infty$ in system (15). Since $t_{2i-1} \leq \tau^* (i = 1, 2, \ldots, n)$, and $g(X)$ is higher infinitesimal as $X(t) \to 0$, based on the property of delayed differential equation, we know that the trivial solution of system (8) (or (9)) is unstable. Therefore, we only need to consider the stability of the trivial solution of system (10). The characteristic equation corresponding to system (10) is

$$\det(\lambda I_{ij} - a_{ij} - b_{ij}e^{-\lambda t}) = 0$$

noting that each characteristic value of matrix $B$ is zero. So we have

$$\det(\lambda I_{ij} - a_{ij} - b_{ij}e^{-\lambda t}) = \prod_{i=1}^{2n} \lambda_i - \rho_i = 0$$

By the assumption, there exits at least one $k$ such that Re

3. Main Result

First we discuss the stability of the trivial solution of system (8) (or (9)). Noting that $g(X)$ is a higher order infinitesimal in a neighborhood of $\|X\| = 0$. Therefore, the stability of trivial solution of system (8) guarantees the stability of trivial solution of system (8). We consider the following auxiliary system:

$$X'(t) = AX(t) + BX(t - \tau^*)$$

where $\tau^* = \max\{\tau_1, \tau_2, \ldots, \tau_{2n-1}\}$, $X(t - \tau^*) = [x_1(t - \tau^*), 0, x_3(t - \tau^*), 0, \ldots, x_{2n-1}(t - \tau^*), 0]^{T}$.

Theorem 1 Assume that system (8) has a unique equilibrium point, for selected parameter values of $a_{2i}, b_{2i-1}, c_{2i-1},$ and $e_{2i} (1 \leq i \leq n)$. Let the eigenvalues of matrix $R = A + B$ be $\gamma_i (1 \leq i \leq 2n)$. Let the eigenvalues of matrix $B$ be $\rho_i (1 \leq i \leq 2n)$ such that $\rho_k > 0$, then the unique equilibrium point of system (8) is unstable, implying that system (8) generates an oscillatory solution.
(λ_k) = Re (ρ_k) > 0, this means that the trivial solution of system (10) is unstable, implying that the trivial solution of system (8) (or 9) is unstable. Since all solutions of system (8) (or 9) are bounded, and system (8) has a unique unstable equilibrium. Based on the generalized Chafee's criterion, this instability of the unique equilibrium will force system (8) to generate an oscillatory solution.

Theorem 3 Assume that system (8) has a unique equilibrium point, for selected parameter values of a_{2i}, b_{2i-1}, c_{2i-1}, and e_{2i} (1 ≤ i ≤ n). If there exists one e_{2k}a_{2k}, k ∈ {1, 2, ..., n} such that
\[ c_{2k-1} - e_{2k}a_{2k} < 0 \]
(22)
then the unique equilibrium point of system (10) is unstable, implying that system (8) generates an oscillatory solution.

Proof As theorem 2, we only need to consider the instability of the trivial solution of system (10). For some k ∈ {1, 2, ..., n}, consider an auxiliary equation
\[ y_{2k}'(t) = -c_{2k-1}y_{2k}(t) + b_{2k}[y_{2k}(t - \tau_{2k-3}) - 2y_{2k}(t - \tau_{2k-1}) + y_{2k}(t - \tau_{2k+1}) + e_{2k}a_{2k}y_{2k}(t)] \]
(23)
The characteristic equation of (23) is the following
\[ \lambda + c_{2k-1} - e_{2k}a_{2k} - b_{2k}e^{-\lambda \tau_{2k-3}} + 2b_{2k}e^{-\lambda \tau_{2k-1}} - 2b_{2k}e^{-\lambda \tau_{2k+1}} = 0 \]
(24)
We show that the characteristic equation (24) of the auxiliary equation has a real positive root say λ'(> 0). Define a function
\[ h(\lambda) = \lambda + c_{2k-1} - e_{2k}a_{2k} - b_{2k}e^{-\lambda \tau_{2k-3}} + 2b_{2k}e^{-\lambda \tau_{2k-1}} - b_{2k}e^{-\lambda \tau_{2k+1}} \]
(25)
Obviously, h(λ) is a continuous function of λ. Under the restrictive condition (22) we have h(0) = c_{2k-1} - e_{2k}a_{2k} - b_{2k} + 2b_{2k} - b_{2k} = c_{2k-1} - e_{2k}a_{2k} < 0. Noting that e^{-\lambda \tau_{2k-3}} → 0 as λ → ∞, e^{-\lambda \tau_{2k-1}} → 0 as λ → ∞, and e^{-\lambda \tau_{2k+1}} → 0 as λ → ∞. Therefore, there exists a suitably large λ say λ_1(> 0) such that h(λ_1) = λ_1 + c_{2k-1} - e_{2k}a_{2k} - b_{2k}e^{-\lambda_1 \tau_{2k-3}} + 2b_{2k}e^{-\lambda_1 \tau_{2k-1}} - b_{2k}e^{-\lambda_1 \tau_{2k+1}} > 0.

Example 1 Consider the case of n = 3 in the following:

The parameter values are selected as c_1 = 5.45, c_2 = 5.65, c_5 = 5.85; b_1 = 0.075, b_2 = 0.085, b_3 = 0.095; a_2 = -1.05, a_3 = -0.55, a_5 = -0.75; β_1 = 0.35, β_2 = 0.25, β_5 = 0.45; l_2 = 0.45, l_3 = 0.25, l_5 = 0.38; ε_2 = 0.0005, ε_4 = 0.0062, ε_5 = 0.0004, and k_3 = 0.35, k_4 = 0.42, k_6 = 0.68, respectively, the eigenvalues of matrix R_k = A_{2k} + B_{k} are -0.0002 ± 2.4001i, and -0.0002 ± 0.7792i. Obviously, the conditions of Theorem 1 are satisfied. The solutions of system (26) are convergent (see Figure 1). When the parameter values are selected as c_1 = 5.12, c_2 = 5.15, c_5 = 5.18; b_1 = 0.00175, b_3 = 0.00185, b_5 = 0.00165; a_2 = 0.05, a_3 = 0.05, a_5 = 0.75; β_1 = 0.25, β_2 = 0.15, β_3 = 0.45; l_2 = 0.15, l_4 = 0.24, l_5 = 0.18; ε_2 = 0.0005, ε_4 = 0.0002, ε_5 = 0.0004, and k_3 = 0.25, k_4 = 0.42, k_6 = 0.16, respectively. The eigenvalues of matrix A_k are 0.0004 ± 2.2803i, 0.0001 ± 2.2694i, and 0.0002 ± 2.2760i, the conditions of Theorem 2 are satisfied. System (26) generates an oscillatory solutions (see Figure 2). When the parameter values are selected as c_1 = 25.12, c_2 = 25.15, c_5 = 25.19, the other parameters are the same as in Figure 2, we see that the oscillation of the solutions is maintained. However, the oscillatory amplitude and frequency both are changed (see Figure 3), implying that the values of c_1, c_2, and c_5 affect the oscillatory amplitude and frequency very much of the solutions. When the parameter values are selected as c_1 = 0.002, c_2 = 0.0015, c_5 = 0.0018; b_1 = 0.00175, b_3 = 0.00155, b_5 = 0.00165; a_2 = 10.95, a_3 = 10.55, a_5 = 10.75; β_1 = 0.25, β_2 = 0.15, β_3 = 0.45; l_2 = 0.15, l_4 = 0.24, l_5 = 0.18; ε_2 = 0.0005, ε_4 = 0.0002, ε_5 = 0.0004, and k_3 = 0.25, k_4 = 0.32, k_6 = 0.61, respectively, we have c_1 - e_{2k}a_{2k} = -0.0035 < 0, c_3 - e_{2k}a_{2k} = -0.0006 < 0, and c_5 - e_{2k}a_{2k} = -0.0025 < 0. The conditions of Theorem 3 are satisfied. System (26) generates an oscillatory solutions (see Figure 4).

Example 2 Consider the case of n = 4 in the following:
The parameter values are selected as $c_4 = 0.0025$, $c_3 = 0.0024$, $c_3 = 0.0022$, $c_2 = 0.0028$; $b_4 = 0.00135$, $b_3 = 0.00125$, $b_2 = 0.00115$, $b_1 = 0.00125$; $a_2 = 12.65$, $a_4 = 12.75$, $a_2 = 12.45$; $\beta_1 = 0.28$, $\beta_3 = 0.16$, $\beta_5 = 0.22$, $\beta_7 = 0.25$; $l_2 = 0.24$, $l_4 = 0.18$, $l_6 = 0.02$, $\epsilon_2 = 0.00024$, $\epsilon_4 = 0.00035$, $\epsilon_6 = 0.00032$, $\epsilon_9 = 0.00036$, and $k_2 = 0.35$, $k_4 = 0.32$, $k_6 = 0.26$, $k_9 = 0.36$, respectively. We have $c_1 - \epsilon_2 a_2 = -0.0005 < 0$, $c_3 - \epsilon_4 a_4 = -0.002 < 0$, $c_5 - \epsilon_6 a_6 = -0.0019 < 0$, and $c_7 - \epsilon_9 a_9 = -0.0019 < 0$. The conditions of Theorem 3 are satisfied. System (27) generates an oscillatory solutions (see Figures 5, 6 and Figure 7).

$$x'_1(t) = x_1(t),$$
$$x'_2(t) = -c_2 x_2(t) - \beta_2 x_2^2(t) + b_2 [x_2(t - t_2) - 2x_2(t - t_2)] + \epsilon_4 a_4 x_4(t) - \epsilon_4 x_4^2(t) x_4(t) + \epsilon_4 k_4 x_4^2(t) x_4(t) - \epsilon_4 d_4 x_4^2(t) x_4(t),$$
$$x'_3(t) = x_3(t),$$
$$x'_4(t) = -c_4 x_4(t) - \beta_4 x_4^2(t) + b_4 [x_4(t - t_2) - 2x_4(t - t_2)] + \epsilon_4 a_4 x_4(t) - \epsilon_4 x_4^2(t) x_4(t) + \epsilon_4 k_4 x_4^2(t) x_4(t) - \epsilon_4 d_4 x_4^2(t) x_4(t),$$
$$x'_5(t) = x_5(t),$$
$$x'_6(t) = -c_6 x_6(t) - \beta_6 x_6^2(t) + b_6 [x_6(t - t_2) - 2x_6(t - t_2)] + \epsilon_6 a_6 x_6(t) - \epsilon_6 x_6^2(t) x_6(t) + \epsilon_6 k_6 x_6^2(t) x_6(t) - \epsilon_6 d_6 x_6^2(t) x_6(t),$$
$$x'_7(t) = x_7(t),$$
$$x'_8(t) = -c_8 x_8(t) - \beta_8 x_8^2(t) + b_8 [x_8(t - t_2) - 2x_8(t - t_2)] + \epsilon_8 a_8 x_8(t) - \epsilon_8 x_8^2(t) x_8(t) + \epsilon_8 k_8 x_8^2(t) x_8(t) - \epsilon_8 d_8 x_8^2(t) x_8(t),$$
$$x'_9(t) = x_9(t).$$

(27)
generate a permanent oscillation, the delays affect oscillatory frequency and amplitude slightly.

References


