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Optimal Prediction of Expected Value of Assets Under Fractal Scaling Exponent Using Seemingly Black-Scholes Parabolic Equation

Bright O. Osu[1], Joy Ijeoma Adindu-Dick[2]

[1]Department of Mathematics, College of Physical and Applied Sciencs, Michael Okpara University of Agriculture, Umudike, Nigeria
[2]Department of Mathematics, Faculty of Physical and Biological Sciences, Imo State University, Owerri, Imo State, Nigeria

Email address:
megaobrai@hotmail.com (B. O. Osu), ji16adindudick@yahoo.com (J. I. Adindu-Dick)

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Abstract: Assessing the stock price indices is the foundation of forecasting the market risk. In this paper, we derived a seemingly Black-Scholes parabolic equation. We then solved this equation under given conditions for the optimal prediction of the expected value of assets.

Keywords: Fractal Scaling Exponent, Black-Scholes Equation, Assets Price Return, Optimal Value, Parabolic Equation

1. Introduction

The problem associated with random behavior of stock exchange has been addressed extensively by many authors (see for example, Black and Scholes, 1973; and Black, *et al*., 1991). The concept of "fractal world" was proposed by Mandelbrot in 1980's and was based on scale-invariant statistics with power law correlation (Mandelbrot, 1982). Fang *et al*., (1994) examined the relevance of fractal dynamics in major currency futures market. Fractal dynamics are forms of dynamics characterized by irregular cyclical fluctuations and long term dependence. They estimated directly the fractal structure in currency futures prices based on a time series model of fractional processes. Based on the self-similarity property of fractal, Tokinaga and Moriyasu, (1997) forecasted the time series by the fractal dimension which was obtained via the wavelet transform. Xiong, (2002) also applied the wavelet to measure the fractal dimension of Chinese stock market. Muzy, *et al*., (2000) estimated the statistical self-similarity exponents from the data and made a quadratic fit for some low order moments. Several studies have examined the cyclic long-term dependence property of financial prices, including stock prices (Greene and Fielitz, (1977); Aydogan and Booth, (1988)). These studies used the classical rescaled range (R/S) analysis, first proposed by Hurst (1951) and later refined by Mandelbrot and Wallis, (1969) and Wallis and Matalas, (1970), among others. Using R/S analysis, Greene and Fielitz, (1977) studied 200 daily stock returns of securities listed on the New York stock exchange and they found significant long range dependence. A problem with the classical R/S analysis is that the distribution of its regression-based test statistics is not well defined. As a result, Lo (1991) proposed the use of a modified R/S procedure with improved robustness. The modified R/S procedure has been applied to study dynamic behavior of stock prices (Lo, 1991; and Cheung, *et al*., 1994). Teverovsky *et al*., (1999) and Willinger *et al.,* (1999) identified a number of problems associated with Lo's method. In particular, they showed that Lo's method has a strong preference for accepting the null hypothesis of no long range dependence. This happens even with long-range dependent synthetic data. To account for the long-range dependence observed in financial data, Cutland *et al*., (1995) proposed to replace Brownian motion with fractional Brownian motion as the building block of stochastic models for asset prices. An account of the historical development of these ideas can be traced from Cutland *et al*., (1995), Mandelbrot, (1997) and Shiryaev, (1999). In this paper, we will derive a seemingly Black-Scholes parabolic equation. This equation is being solved under given conditions for the optimal prediction of the expected value of assets.

2. The Model

Consider a portfolio comprising h unit of assets in long position and one unit of the option in short position. At time, T the value of the portfolio is

$$hP - V, \tag{1}$$

measured by the fractal index $C^{\phi}(E) - V^{\phi}(E) \neq 0$.

After an elapse of time, Δt, the value of the portfolio will change by the rate $h(\Delta P + D_1\Delta t) - \Delta V$ in view of the dividend received on h units held. By Ito's lemma this equals

$$h(uP\Delta t + \sigma P\Delta z + D_1\Delta t)$$
$$-\left\{\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial P}uP + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right)\Delta t + \frac{\partial V}{\partial P}\sigma P\Delta z\right\}$$

OR

$$\left\{(huP + hD_1) - \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial P}uP + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right)\right\}\Delta t$$
$$+ (h\sigma P - \frac{\partial V}{\partial P}\sigma P)\Delta z$$

If we take

$$h = \frac{\partial V}{\partial P} \tag{2}$$

the uncertainty term disappears, thus the portfolio in this case is temporarily riskless. It should therefore grow in value by the riskless rate in force i.e.

$$\left\{(huP + hD_1) - \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial P}uP + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right)\right\}\Delta t$$
$$= (hP - V)r\Delta t.$$

Thus

$$D_1\frac{\partial V}{\partial P} - \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right) = (\frac{\partial V}{\partial P}P - V)r$$

So

$$\frac{\partial V}{\partial t} + (rP - D_1)\frac{\partial V}{\partial P} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = rV. \tag{3}$$

Proposition 1: Let $D_1 = 0$ (where D_1 is the market price of risk), then the solution of equation (3), which coincides with the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = 0 \tag{4}$$

with

$$V(P, t) = 0, \tag{5a}$$

$$\frac{\partial V(P,t)}{\partial P} = 0 \ \forall t, \tag{5b}$$

and P^2 is assumed constant, is given by

$$\mathrm{V}(P, \mathrm{t}) = \mathrm{V}_0 \exp\left\{\frac{-2\alpha \mathrm{t} \mathrm{P}^{-2}}{\sigma^2} + \lambda \mathrm{P}\right\} e^{rt}.\text{(using equation (5a))} \tag{6}$$

with

$$\lambda + \frac{4\alpha P^{-3}t}{\sigma^2} = 0 \text{ (using equation (5b)).} \tag{7}$$

Where V is the investment output, r the discount rate, and σ^2 the variance of the stock market price.

Proof: Let $D_1 = 0$ (where D_1 is the market price of risk), then equation (3) becomes

$$\frac{\partial V}{\partial t} + rP\frac{\partial V}{\partial P} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = rV. \tag{8}$$

In order to remove the effect of the discount rate (r) from equation (8), we let $r = 0$ and set

$$V = e^{-rt}\bar{V} \tag{9a}$$

and

$$P = e^{-rt}\bar{P}. \tag{9b}$$

Hence equation (8) becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = 0. \tag{10}$$

By the method of separation of variables, we assume a solution of the form $V = f(P)g(t)$.

Hence

$$\frac{\partial^2 V}{\partial P^2} = f''g, \tag{11a}$$

and

$$\frac{\partial V}{\partial t} = fg'. \tag{11b}$$

Substituting equations (11a) and (11b) in equation (10) gives

$$\frac{\sigma^2 P^2}{2}\frac{f''}{f} = \frac{-g'}{g} = \alpha \tag{12}$$

or

$$g' = \frac{-2\alpha g}{\sigma^2 P^2} \tag{13a}$$

and

$$f'' = \alpha f. \tag{13b}$$

These are ordinary differential equations for $f \ and \ g$ with $lng = \frac{-2\alpha}{\sigma^2 P^2}t$ (using equation (13a)) and having solution

$$g(t) = V_0 e^{-\frac{2\alpha P^{-2}}{\sigma^2}t}. \tag{14}$$

Also

$$f'' - \alpha f = 0 \text{ (using equation (13b))}$$

So that $\lambda^2 = \alpha$ and $\lambda = \pm\sqrt{\alpha}$ with solution

$$f(P) = e^{\lambda P}. \quad (15)$$

Hence, we obtain a special solution of the form

$$V(P,t) = V_0 exp\left\{\frac{-2\alpha P^{-2}}{\sigma^2}t + \lambda P\right\}. \quad (16)$$

But

$$\frac{\partial V}{\partial P} = 0$$

$$= V_0\lambda e^{\frac{-2\alpha P^{-2}}{\sigma^2}t}e^{\lambda P} + \frac{4V_0\alpha P^{-3}t}{\sigma^2}e^{\frac{-2\alpha P^{-2}t}{\sigma^2}}e^{\lambda P}$$

$$= \lambda + \frac{4\alpha P^{-3}t}{\sigma^2} \text{ (as in equation (7)).}$$

Solving for P in the above equation gives

$$P^{-3} = \frac{-\lambda\sigma^2}{4\alpha t},$$

and

$$P = \left(\frac{-4\alpha t}{\lambda\sigma^2}\right)^{\frac{1}{3}}.$$

Equating this result to equation (9b) gives

$$e^{-rt}\overline{P_t} = \left(\frac{-4\alpha t}{\lambda\sigma^2}\right)^{\frac{1}{3}}$$

and

$$\overline{P_t} = \left(\frac{-4\alpha t}{\lambda\sigma^2}\right)^{\frac{1}{3}}e^{rt}. \quad (17)$$

Equating equation (9a) to equation (16) gives

$$\mathrm{V}(P,\mathrm{t}) = \mathrm{V_0}\exp\left\{\frac{-2\alpha\mathrm{tP}^{-2}}{\sigma^2} + \lambda\mathrm{P}\right\}e^{rt} \quad (18)$$

Proposition 2: Let $D_1 = 0$ (where D_1 is the market price of risk), then the solution of equation (3) where P^2 is not a constant, coincides with the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = 0. \quad (19)$$

with

$$V(P,t) = 0 \quad (20a)$$

and

$$\frac{\partial V(P,t)}{\partial P} = 0 \; \forall t, \quad (20b)$$

is given by

$$V(P,t) = V_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t + rt}\left\{AP^{\lambda_1} + BP^{\lambda_2}\right\} \quad (21)$$

with

$$A\lambda_1 P^{\lambda_1 - 1} + B\lambda_2 P^{\lambda_2 - 1} = 0 \text{ (using equation (20b)).} \quad (22)$$

Where V is the investment output, r the discount rate, σ^2 the variance of the stock market price, A and B are arbitrary constants.

Proof: From equation (8), to equation (9b) we have equation (3) reduced to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = 0. \quad (23)$$

By the method of separation of variables, let the solution of equation (23) be $V = f(P)g(t)$. Hence, $\frac{\partial V}{\partial t} = fg'$ and $\frac{\partial^2 V}{\partial P^2} = f''g$. Equation (23) becomes $fg' + \frac{1}{2}f''g\sigma^2P^2 = 0$. Therefore

$$\frac{1}{2}f''g\sigma^2P^2 = -fg'. \quad (24)$$

By separation of variables equation (24) becomes

$$P^2\frac{f''}{f} = -\frac{2g'}{\sigma^2 g} = \alpha^2. \quad (25)$$

From equation (25) we have

$$P^2\frac{f''}{f} = \alpha^2$$

$$P^2 f'' - \alpha^2 f = 0$$

That is,

$$P^2\frac{d^2f}{dP^2} - \alpha^2 f = 0. \quad (26)$$

We then solve equation (26) using Euler's substitution method.

Let $P = e^t$, then

$$lnP = t, \text{ and } \frac{dt}{dP} = \frac{1}{P}. \quad (27)$$

Also

$$\frac{df}{dP} = \frac{df}{dt}\frac{dt}{dP} = \frac{1}{P}\frac{df}{dt}, \quad (28)$$

$$\frac{d^2f}{dP^2} = \frac{d}{dP}\left(\frac{1}{P}\frac{df}{dt}\right) = \frac{1}{P}\frac{d}{dP}\left(\frac{df}{dt}\right) + \frac{df}{dt}\frac{d}{dP}\left(\frac{1}{P}\right),$$

$$\text{and } \frac{d^2f}{dP^2} = \frac{1}{P^2}\left(\frac{d^2f}{dt^2} - \frac{df}{dt}\right). \quad (29)$$

Equation (26) becomes

$$\frac{d^2f}{dt^2} - \frac{df}{dt} - \alpha^2 f = 0. \quad (30)$$

Let $f = e^{\lambda t}$ be the solution of equation (30), hence $f' = \lambda e^{\lambda t}$; $f'' = \lambda^2 e^{\lambda t}$. Equation (30) becomes

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - \alpha^2 e^{\lambda t} = 0.$$

Our auxiliary equation becomes

$$\lambda^2 - \lambda - \alpha^2 = 0.$$

Therefore

$$\lambda_1 = \frac{1+\sqrt{1+4\alpha^2}}{2} \quad (31)$$

and

$$\lambda_2 = \frac{1-\sqrt{1+4\alpha^2}}{2}. \quad (32)$$

From equation (27) we have $lnP = t$, but $f = e^{\lambda t}$, hence $f = P^{\lambda}$.

Our general solution becomes

$$f(P) = AP^{\lambda_1} + BP^{\lambda_2}. \quad (33)$$

From equation (25) we have

$$-\frac{2g'}{\sigma^2 g} = \alpha^2. \quad (34)$$

The solution of equation (34) becomes

$$g(t) = V_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t}. \quad (35)$$

But $V(P,t) = f(P)g(t)$, from equations (33) and (35) we have

$$V(P,t) = V_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t}\{AP^{\lambda_1} + BP^{\lambda_2}\}. \quad (36)$$

But

$$\frac{\partial V}{\partial P} = 0$$

$$= V_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t}\{A\lambda_1 P^{\lambda_1 - 1} + B\lambda_2 P^{\lambda_2 - 1}\}$$

$$= A\lambda_1 P^{\lambda_1 - 1} + B\lambda_2 P^{\lambda_2 - 1} \text{ (as in equation (22)).}$$

Solving for P in the above equation gives

$$P^{(\lambda_1 - \lambda_2)} = \frac{-B\lambda_2}{A\lambda_1}$$

and

$$P = \left(\frac{-B\lambda_2}{A\lambda_1}\right)^{\frac{1}{\lambda_1 - \lambda_2}}.$$

Equating this result to equation (9b) gives

$$e^{-rt}\bar{P}_t = \left(\frac{-B\lambda_2}{A\lambda_1}\right)^{\frac{1}{\lambda_1 - \lambda_2}}$$

and

$$\bar{P}_t = \left(\frac{-B\lambda_2}{A\lambda_1}\right)^{\frac{1}{\lambda_1 - \lambda_2}} e^{rt}. \quad (37)$$

Equating equation (9a) to equation (36) gives

$$V(P,t) = V_0 e^{-\frac{1}{2}\sigma^2\alpha^2 t + rt}\{AP^{\lambda_1} + BP^{\lambda_2}\} \quad (38)$$

where A and B are arbitrary constants; λ_1 and λ_2 are as defined in equations (31) and (32).

Proposition 3: For $D_1 \neq 0$, the solution of equation (3) is given as:

$$V(P) = \left(\frac{aq_n^2}{2P}\right)^{\beta}\left\{Ae^{\lambda_1 \frac{aq_n^2}{2P}} + Be^{\lambda_2 \frac{aq_n^2}{2P}}\right\}, \quad (39)$$

where

$$\lambda_1 = -\frac{2}{z} \pm \sqrt{\frac{4}{z^2} + \frac{8r}{z^2\sigma^2}} \text{ and } \lambda_2 = \pm\frac{1}{z}\sqrt{4 + \frac{8r}{\sigma^2}}. \quad (40)$$

Proof

We take

$$Z = \frac{\alpha}{P}; V(P) = Z^{\beta}W(Z) \quad (41)$$

Thus

$$\frac{dz}{dP} = -\frac{\alpha}{P^2} = -\frac{1}{\alpha}Z^2$$

$$\frac{dV}{dP} = \frac{dV}{dZ}.\frac{dZ}{dP}$$

$$= -\frac{1}{\alpha}Z^2(\beta Z^{\beta-1}W + Z^{\beta}\frac{dW}{dZ})$$

$$= -\frac{1}{\alpha}(\beta Z^{\beta+1}W + Z^{\beta+2}\frac{dW}{dZ}).$$

Hence

$$\frac{d^2V}{dP^2} = \frac{d}{dP}(\frac{dV}{dZ}).\frac{dZ}{dP}$$

$$= -\frac{1}{\alpha}Z^2(\beta(\beta+1)Z^{\beta}W + \beta Z^{\beta+1}\frac{dW}{dZ}$$

$$+(\beta+2)Z^{\beta+1}\frac{dW}{dZ} + Z^{\beta+2}\frac{d^2W}{dZ^2}).$$

In this case V is not dependent on r. Substituting into the given differential equation we have

$$rZ^{\beta}W = \frac{\sigma^2}{2}\left(\beta(\beta+1)Z^{\beta}W + \beta Z^{\beta+1}\frac{dW}{dZ} + (\beta+2)Z^{\beta+1}\frac{dW}{dZ} + Z^{\beta+2}\frac{d^2W}{dZ^2}\right) + \left(\frac{r\alpha}{Z} - D_1\right)\left(\frac{-1}{\alpha}\right)(\beta Z^{\beta+1}W + Z^{\beta+2}\frac{dW}{dZ}). \quad (42)$$

Cancelling by Z^{β} and collecting like terms we have

$$0 = \frac{\sigma^2}{2}Z^2\frac{d^2W}{dZ^2} + \frac{dW}{dZ}\left(\sigma^2(\beta+1)Z - rZ + \frac{D_1}{\alpha}Z^2\right) + W\left(\frac{\sigma^2}{2}\beta(\beta+1) - r\beta + \beta\frac{D_1}{\alpha}Z\right) - rw$$

Or

$$0 = \frac{\sigma^2}{2}Z^2\frac{d^2W}{dZ^2} + \frac{dW}{dZ}Z\left(\sigma^2(\beta+1) - r + \frac{D_1}{\alpha}Z\right) + W\left(\frac{\sigma^2}{2}\beta(\beta+1) - r(\beta+1) + \beta\frac{D_1}{\alpha}Z\right).$$

Let

$$\beta = 0 \text{ and } = \frac{D_1}{\alpha}Z. \quad (43)$$

We obtain

$$Z^2\frac{d^2W}{dZ^2}+2Z\frac{dW}{dZ}-\frac{2Wr}{\sigma^2}=0. \tag{44}$$

Let λ_1 and λ_2 be the roots of the equation, then

$$\lambda_1+\lambda_2=-\frac{2}{Z}$$

$$\lambda_1\lambda_2=-\frac{2r}{Z^2\sigma^2}..$$

Now,

$$\frac{d^2W}{dZ^2}-(\lambda_1+\lambda_2)\frac{dW}{dP}-\lambda_1\lambda_2W=0$$

or

$$\frac{d}{dZ}\left(\frac{dW}{dZ}-\lambda_2W\right)=\lambda_1\left(\frac{dW}{dP}-\lambda_2W\right)$$

Then

$$\frac{dW}{dZ}=Y,Y=\left(\frac{dW}{dP}-\lambda_2W\right)$$

Which gives $Y=Ce^{\lambda_2 z}$ with solution

$$e^{-\lambda_1 z}W=\int C\,e^{(\lambda_1-\lambda_2)z}dz+B \tag{45}$$

(Where C and B are arbitrary constants).
Hence

$$W(z)=Ae^{\lambda_1 z}+Be^{\lambda_2 z} \tag{46}$$

$$V(P)=Z^\beta W(Z)\;=\left(\frac{\alpha}{P}\right)^\beta\left\{Ae^{\lambda_1\frac{\alpha}{P}}+Be^{\lambda_2\frac{\alpha}{P}}\right\}V(P)$$

$$=\left(\frac{aq_n^2}{2P}\right)^\beta\left\{Ae^{\lambda_1\frac{aq_n^2}{2P}}+Be^{\lambda_2\frac{aq_n^2}{2P}}\right\}. \tag{47}$$

3. Conclusion

The Models: equations (6), (21) and (39) suggest the optimal prediction of the expected value of assets under fractal scaling exponent $(C^\phi(E)-V^\phi(E))=\frac{aq_n^2}{2}$ which we obtained. We derived a seemingly Black Scholes parabolic equation and its solution under given conditions for the prediction of assets values given the fractal exponent. Considering equation (6), we observed that when $a=0,\alpha=0$, the equation reduces to $V(P,t)=V_0e^{`rt}$ .This means that the expected value is being determined by the interest rate r and time t. If $a=4,\alpha=2q_n^2$, equation (6) reduces to $V(P,t)=V_0exp\left\{\frac{-4q_n^2tP^{-2}}{\sigma^2}\pm\sqrt{2}q_n\right\}e^{rt}$, this also means that the growth rate depends on price, time, and interest rate.

Considering equation (21), we observed that when our singularity strength, $a=0$, our fractal exponent, $\alpha=0$, equation (21) becomes $V(P,t)=V_0e^{rt}\{AP^{\lambda_1}+BP^{\lambda_2}\}$. If our singularity strength, $a=4$, our fractal exponent, $\alpha=2q_n^2$, equation (21) reduces to $V(P,t)=V_0e^{-2\sigma^2q_n^4t+rt}\{AP^{\lambda_1}+BP^{\lambda_2}\}$. When $q=1$, we have $V(P,t)=V_0e^{-2\sigma^2t+rt}\{AP^{\lambda_1}+BP^{\lambda_2}\}$. This means that the expected value depends on stock price, interest rate, and time. If λ_1 and λ_2 are positive, the stock price increases, hence the investment output increases. On the other hand, if λ_1 and λ_2 are negative, the stock price decreases and this leads to decrease in investment output. Considering equation (39), we also observed that when $a=0$, the equation becomes $V(P)=0$, this signifies no signal. If $a=4$, equation (39) becomes $V(P)=\left(\frac{2q_n^2}{P}\right)^\beta\left\{Ae^{\lambda_1\frac{2q_n^2}{P}}+Be^{\lambda_2\frac{2q_n^2}{P}}\right\}$, this implies that there is signal. We now further look at it when $q=1$ to have $V(P)=\left(\frac{2}{P}\right)^\beta\left\{Ae^{\frac{2\lambda_1}{P}}+Be^{\frac{2\lambda_2}{P}}\right\}$. Hence, if $\lambda_1\;and\;\lambda_2$ are negative, the equation decays exponentially. On the other hand, if $\lambda_1 and\;\lambda_2$ are positive, the equation grows exponentially.

References

[1] Aydogan, K., & Booth, G. G. (1988). Are there long cycles in common stock returns? *Southern Economic Journal*, 55, 141-149.

[2] Black, F., & Karasinski, P. (1991). Bond and options pricing with short rate and lognormal. *Financial Analysis Journal*, 47 (4), 52-59.

[3] Black, F., & Scholes, M. (1973). The valuation of options and corporate liabilities. *Journal of Econometrics*, 81, 637-654.

[4] Cheung, Y. W., Lai, K. S., & Lai, M. (1994). Are there long cycles in foreign stock returns? *Journal of International Financial Markets, Institutions and Money*, 3 (1), 33-48.

[5] Cutland, N., Kopp, P., & Willinger, W. (1995). Stock price returns and the Joseph effect: A fractal version of the Black-Scholes model. *Progress in Probability*, 36, 327-351.

[6] Fang, H., Lai, K., & Lai, M. (1994). Fractal structure in currency futures price dynamics. *The Journal of Futures Markets*, 14, 169-181.

[7] Greene, M. T., & Fielitz B. D. (1997). Long term dependence in common stock returns. *Journal of Financial Economics,* 5, 339-349.

[8] Hurst, H. E., (1951). Long term storage capacity of reservoir. *Transactions of the American Society of Civil Engineers*, 116, 770-799.

[9] Lo, A. W., (1991). Long term memory in stock market prices. *Econometrica*, 59, 1279-1313.

[10] Mandelbrot, B. B., (1982). *The fractal geometry of nature*. New York: Freeman.

[11] Mandelbrot, B. B., (1997). *Fractals and scaling in finance:Discontinuity, Concentration, Risk.* New York: Springer-Verlag.

[12] Mandelbrot, B. B., & Wallis, J. R. (1969). Robustness of the rescaled range in the measurement of non-cyclic long-run statistical dependence. *Water Resources Research*, 5, 967-988.

[13] Muzy, J., Delour, J., & Bacry, E., (2000). Modelling fluctuations of financial time series: from cascade process to stochastic volatility Model. Euro. Phys. *Journal B*, 17, 537-548.

[14] Shiryaev, A. N., (1999). *Essentials of stochastic finance*. Singapore: World Scientific.

[15] Teverovsky, V., Taqqu, M., & Willinger, W., (1999). A critical look at Lo's modified R/S statistic. *Journal of statistical planning and inference*, 80, 211-227.

[16] Tokinaga, S., Moriyasu, H., Miyazaki, A, & Shimazu, N. (1997). Forecasting of time series with fractal geometry by using scale transformations and parameter estimations obtained by the wavelet transform. *Electronics and Communications in Japan,* 80 (3), 8-17.

[17] Wallis, J. R., & Matalas, N. C., (1970). Small sample properties of H and K-estimators of the Hurst coefficient. *Water Resources Research*, 6, 1583-1594.

[18] Willinger, W., Taqqu, M., & Teverovsky, V., (1999). Stock market prices and long-range dependence. *Finance and Stochastic*, 3, 1-13.

[19] Xiong, Z., (2002). Estimating the fractal dimension of financial time Series by wavelets systems. *Engineering-Theory and Practice,* 12, 48-53.