Weak Insertion of an α−Continuous Function

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Abstract: A sufficient condition in terms of lower cut sets are given for the weak α−insertion property and the weak insertion of an α−continuous function between two comparable real-valued functions. Also several insertion theorems are obtained as corollaries of this result.

Keywords: Weak Insertion, Strong Binary Relation, Preopen Set, Semi-Open Set, α−Open Set, Lower Cut Set

1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [1].

A subset A of a topological space (X, τ) is called preopen or locally dense or nearly open if A ⊆ Int(Cl(A)). A set A is called preclosed if its complement is preopen or equivalently if Cl(Int(A)) ⊆ A.

The term, preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [2], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [1].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [3]. A subset A of a topological space (X, τ) is called semi-open [3] if A ⊆ Cl(Int(A)).

A set A is called semi-closed if its complement is semi-open or equivalently if Int(Cl(A)) ⊆ A.

Recall that a subset A of a topological space (X, τ) is called α−open if A is the difference of an open and a nowhere dense subset of X.

A set A is called α−closed if its complement is α−open or equivalently if A is union of a closed and a nowhere dense set.

We have a set is α−open if and only if it is semi-open and preopen.

Recall that a real-valued function f defined on a topological space X is called A−continuous [4] if the preimage of every open subset of R belongs to A, where A is a collection of subset of X.

Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to [5, 6].

Hence, a real-valued function f defined on a topological space X is called precontinuous (resp. semi-continuous or α−continuous) if the preimage of every open subset of R is preopen (resp. semi-open or α−open) subset of X.

Precontinuity was called by V. Ptk nearly continuity [7]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [8].

Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [9].

Results of M. Kat’etov [10, 11] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to F. Brooks [12], are used in order to give a sufficient condition for the insertion of an α−continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write g ≤ f in case g(x) ≤ f(x) for all x in X.

The following definitions are modifications of conditions considered in [13].

A property P defined relative to a real-valued function on a topological space is an α−property provided that any constant function has property P and provided that the sum of a function with property P and any α−continuous function also has property P.

If P1 and P2 are α−property, the following terminology is used:

A space X has the weak α−insertion property for (P1,P2) if and only if for any functions g and f on X such that g ≤ f, g has property P1 and f has property P2, then there exists an α−continuous function h such that g ≤ h ≤ f.
2. The Main Result

Before giving a sufficient condition for insertability of an \(\alpha\)-continuous function, the necessary definitions and terminology are stated.

Let \((X, \tau)\) be a topological space, the family of all \(\alpha\)-open, \(\alpha\)-closed, semi-open, semi-closed, preopen and preclosed will be denoted by \(\alpha O(X, \tau), \alpha C(X, \tau), \sigma O(X, \tau), \sigma C(X, \tau), pO(X, \tau)\) and \(pC(X, \tau)\), respectively.

Definition 2.1. Let \(A\) be a subset of a topological space \((X, \tau)\). Respectively, we define the \(\alpha\)-closure, \(\alpha\)-interior, \(\sigma\)-closure, \(\sigma\)-interior, \(p\)-closure and \(p\)-interior of a set \(A\), denoted by \(\alpha Cl(A), \alpha Int(A), \sigma Cl(A), \sigma Int(A), p Cl(A)\) and \(p Int(A)\) as follows:

\[
\begin{align*}
\alpha Cl(A) &= \cap \{F : F \supseteq A, F \in \alpha C(X, \tau)\}, \\
\alpha Int(A) &= \cup \{O : O \subseteq A, O \in \alpha O(X, \tau)\}, \\
\sigma Cl(A) &= \cap \{F : F \supseteq A, F \in \sigma C(X, \tau)\}, \\
\sigma Int(A) &= \cup \{O : O \subseteq A, O \in \sigma O(X, \tau)\}, \\
p Cl(A) &= \cap \{F : F \supseteq A, F \in p C(X, \tau)\} \text{ and } \\
p Int(A) &= \cup \{O : O \subseteq A, O \in p O(X, \tau)\}.
\end{align*}
\]

Respectively, we have \(\alpha Cl(A), \sigma Cl(A), p Cl(A)\) are \(\alpha\)-closed, semi-closed, preclosed and \(\alpha Int(A), \sigma Int(A), p Int(A)\) are \(\alpha\)-open, semi-open, preopen. The following first two definitions are modifications of conditions considered in [10, 11].

Definition 2.2. If \(\rho\) is a binary relation in a set \(S\) then \(\rho^\circ\) is defined as follows: \(x \rho^\circ y\) if and only if \(y \rho v\) implies \(x \rho v\) and \(y \rho^\circ x\) implies \(y \rho v\) for any \(u\) and \(v\) in \(S\).

Definition 2.3. A binary relation \(\rho\) in the power set \(P(X)\) of a topological space \(X\) is called a strong binary relation in \(P(X)\) in case \(\rho\) satisfies each of the following conditions:

i). If \(A_i \rho B_j\) for any \(i \in \{1,\ldots,n\}\) and for any \(j \in \{1,\ldots,n\}\), then there exists a set \(C\) in \(P(X)\) such that \(A_i \rho^\circ C\) and \(C \rho B_j\) for any \(i \in \{1,\ldots,n\}\) and any \(j \in \{1,\ldots,n\}\).

ii). If \(A \subseteq B\), then \(A \rho^\circ B\).

iii). If \(A \rho B\), then \(\alpha Cl(A) \subseteq B\) and \(A \subseteq \alpha Int(B)\).

The concept of a lower indefinite cut set for a real-valued function was defined by F. Brooks [12] as follows:

Definition 2.4. If \(f\) is a real-valued function defined on a space \(X\) and \(i\):

\[
\{x \in X : f(x) < t\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \leq t\} \text{ for a real number } l, \ (7)
\]

then \(A(f, l)\) is called a lower indefinite cut set in the domain of \(f\) at the level \(l\).

We now give the following main result:

Theorem 2.1. Let \(g\) and \(f\) be real-valued functions on a topological space \(X\) with \(g \leq f\). If there exists a strong binary relation \(\rho\) on the power set \(X\) and if there exist lower indefinite cut sets \(A(f, t)\) and \(A(g, t)\) in the domain of \(f\) and \(g\) at the level \(t\) for each rational number \(t\) such that if \(t_1 < t_2\) then \(A(f, t_1) \rho A(g, t_2)\), then there exists an \(\alpha\)-continuous function \(h\) defined on \(X\) such that \(g \leq h \leq f\).

Proof. Let \(g\) and \(f\) be real-valued functions defined on \(X\) such that \(g \leq f\). By hypothesis there exists a strong binary relation \(\rho\) on the power set of \(X\) and there exist lower indefinite cut sets \(A(f, t)\) and \(A(g, t)\) in the domain of \(f\) and \(g\) at the level \(t\) for each rational number \(t\) such that if \(t_1 < t_2\) then \(A(f, t_1) \rho A(g, t_2)\).

Define functions \(F\) and \(G\) mapping the rational numbers \(Q\) into the power set of \(X\) by \(F(t) = A(f, t)\) and \(G(t) = A(g, t)\). If \(t_1\) and \(t_2\) are any elements of \(Q\) with \(t_1 < t_2\), then \(F(t_1) \rho G(t_2)\). By Lemmas 1 and 2 of [11] it follows that there exists a function \(H\) mapping \(Q\) into the power set of \(X\) such that if \(t_1\) and \(t_2\) are any rational numbers with \(t_1 < t_2\), then \(F(t_1) \rho H(t_2)\), \(H(t_1) \rho H(t_2)\) and \(H(t_1) \rho G(t_2)\).

For any \(x\) in \(X\), let \(h(x) = \inf\{t \in Q : f(x) < t\}\). (8)

We first verify that \(g \leq h \leq f\). If \(x\) is in \(H(t)\) then \(x\) is in \(G(t)\) for any \(t < t\); since \(x\) is in \(G(t) = A(g, t)\) implies that \(g(x) \leq t\), it follows that \(g(x) \leq t\). Hence \(g \leq f\). If \(x\) is not in \(H(t)\), then \(x\) is not in \(G(t)\) for any \(t < t\); since \(x\) is not in \(F(t) = A(f, t)\) implies that \(f(x) > t\), it follows that \(f(x) > t\). Hence \(h \leq f\).

Also, for any rational numbers \(t_1\) and \(t_2\) with \(t_1 < t_2\), we have \(h(t_1, t_2) = \alpha Int(H(t_2)) \subseteq \alpha Cl(H(t_1))\). Hence \((t_1, t_2)\) is an \(\alpha\)-open subset of \(X\), i.e., \(h\) is an \(\alpha\)-continuous function on \(X\).

The above proof used the technique of proof of Theorem 1 of [10].

3. Applications

The abbreviations \(pc\) and \(sc\) are used for precontinuous and semicontinuous, respectively.

Corollary 3.1. If for each pair of disjoint preclosed (resp. semi-closed) sets \(F_1, F_2\) of \(X\), there exist \(\alpha\)-open sets \(G_1, G_2\) of \(X\) such that \(F_1 \subseteq G_1, F_2 \subseteq G_2\) and \(G_1 \cap G_2 = \emptyset\) then \(X\) has the weak \(\alpha\)-insertion property for \((pc, pc)\) (resp. \((sc, sc)\)).

Proof. Let \(g\) and \(f\) be real-valued functions defined on \(X\), such that \(f\) and \(g\) are \(pc\) (resp. \(sc\)), and \(g \leq f\). If a binary relation \(\rho\) is defined by \(A \rho B\) in case \(p Cl(A) \subseteq p Int(B)\) (resp. \(s Cl(A) \subseteq s Int(B)\)), then by hypothesis \(\rho\) is a strong binary relation in the power set of \(X\). If \(t_1\) and \(t_2\) are any elements of \(Q\) with \(t_1 < t_2\), then

\[
A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2), \ (9)
\]

since \(\{x \in X : f(x) \leq t_1\}\) is a preclosed (resp. semi-closed) set and since \(\{x \in X : g(x) < t_2\}\) is a preopen (resp. semi-open) set, it follows that \(p Cl(A(f, t_1)) \subseteq p Int(A(g, t_2))\) (resp. \(s Cl(A(f, t_1)) \subseteq s Int(A(g, t_2))\)). Hence \(t_1 < t_2\) implies that \(A(f, t_1) \rho A(g, t_2)\). The proof follows from Theorem 2.1.

Corollary 3.2. If for each pair of disjoint preclosed (resp. semi-closed) sets \(F_1, F_2\), there exist \(\alpha\)-open sets \(G_1, G_2\) such that \(F_1 \subseteq G_1, F_2 \subseteq G_2\) and \(G_1 \cap G_2 = \emptyset\) then every precontinuous (resp. semi-continuous) function is \(\alpha\)-continuous.

Proof. Let \(f\) be a real-valued precontinuous (resp. semi-continuous) function defined on \(X\). Set \(g = f\), then by Corollary 3.1, there exists an \(\alpha\)-continuous function \(h\) such...
that \( g = h = f \).

Corollary 3.3. If for each pair of disjoint subsets \( F_1, F_2 \) of \( X \), such that \( F_1 \) is preclosed and \( F_2 \) is semi-closed, there exist \( \alpha \)-open subsets \( G_1 \) and \( G_2 \) of \( X \) such that \( F_1 \subseteq G_1, F_2 \subseteq G_2 \) and \( G_1 \cap G_2 = \emptyset \) then \( X \) have the weak \( \alpha \)-insertion property for (pc, sc) and (sc, pc).

Proof. Let \( g \) and \( f \) be real-valued functions defined on the \( X \), such that \( g \) is pc (resp. sc) and \( f \) is sc (resp. pc), with \( g \leq f \). If a binary relation \( \rho \) is defined by \( \text{sCl}(A) \subseteq \text{pInt}(B) \) (resp. \( \text{pCl}(A) \subseteq \text{sInt}(B) \)), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( Q \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]

since \( \{ x \in X : f(x) \leq t_1 \} \) is a semi-closed (resp. preclosed) set and since \( \{ x \in X : g(x) < t_2 \} \) is a preopen (resp. semi-open) set, it follows that \( \text{sCl}(A(f, t_1)) \subseteq \text{pInt}(A(g, t_2)) \) (resp. \( \text{pCl}(A(f, t_1)) \subseteq \text{sInt}(A(g, t_2)) \)). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 2.1. •

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**References**


