Some Induced Averaging Aggregation Operators Based on Pythagorean Fuzzy Numbers

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To cite this article:

Received: May 29, 2017; Accepted: August 10, 2017; Published: September 6, 2017

Abstract: In this paper we present two new types aggregation operators such as, induced Pythagorean fuzzy ordered weighted averaging aggregation operator and induced Pythagorean fuzzy hybrid averaging aggregation operator. We also discuss of important properties of these proposed operators and construct some examples to develop these operators.

Keywords: Pythagorean Fuzzy Sets, I-PFOWA Operator, I-PFHA Operator

1. Introduction

Atanassov [1] introduced the concept of IFS characterized by a membership function and a non-membership function. It is more suitable for dealing with fuzziness and uncertainty than the ordinary fuzzy set developed by Zadeh [2] characterized by one membership function. Gau and Buehrer [3] proposed the notion of vague set. Chen and Tan [4] and Hong and Choi [5] presented some techniques for handling multi-criteria fuzzy decision-making problems based on vague sets. Bustine and Burillo [6] showed that the vague set is equivalent to IFS. In 1986, many scholars [7, 8, 9, 10, 11, 12] have done works in the field of AIFS and its applications. Particularly, information aggregation is a very crucial research area in IFS theory that has been receiving more and more focus. Xu [13] developed some basic arithmetic aggregation operators, including IFWA operator, IFOWA operator and IFHA operator and applied them to group decision making. Xu and Yager [14] defined some basic geometric aggregation operators such as, IFWG operator, IFOWG operator and IFHG operator, and applied them to multiple attribute decision making (MADM). Wang and Liu [15] introduced the notion of IFEWG operator and geometric IFEOWG operator and applied them to group decision making. In [16] Wang and Liu also introduced the concept of IFEWA operator and IFEOWA operator. Zhao and Wei [17] introduced the notion of two new types of hybrid aggregation operators such as, IFEHA operator and IFEHG operator. But there are many cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. Therefore, Yager [18] introduced the concept of PFS. PFS is more powerful tool to solve uncertain problems. In 2013, Yager and Abbasov [19] introduced the notion of two new Pythagorean fuzzy aggregation operators such as PFWA operator and PFOWA operator. In [20, 21, 22, 23] K. Rahman et al. introduced the concept of PFHA operator, PFWG operator, PFOWG operator and PFHG operator.

Thus keeping the advantages of the above mention aggregation operators in this article we introduce the notion of two new types aggregation operators based on PFNs, such as, induced Pythagorean fuzzy ordered weighted averaging aggregation operator and induced Pythagorean fuzzy hybrid averaging aggregation operator. We also discuss of important properties of these proposed operators and construct some examples to develop these operators.

The remainder paper can be constructed as. In Section 2, we present some basic definitions which will be used in our later sections. In Section 3, we introduce the notion of induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator and induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator. We also discuss some of their basic properties including idempotency, boundedness, commutativity and monotonicity. We also give some examples to develop these proposed operators. 

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2. Preliminaries

**Definition 1:** [16, 17] Let $Z$ be a fixed set, then Pythagorean fuzzy set can be defined as:

$$P = \left\{ \left( z, \mu_p(z), \eta_p(z) \right) \mid z \in Z \right\},$$

(1)

where $\mu_p(z)$ and $\eta_p(z)$ are mappings from $Z$ to $[0,1]$, such that $0 \leq \mu_p(z) \leq 1$, $0 \leq \eta_p(z) \leq 1$, and $0 \leq \mu^2_p(z) + \eta^2_p(z) \leq 1$, for all $z \in Z$, and represent the degrees of membership and nonmembership of the component $z \in Z$ to set $P$. Let

$$\pi_p(z) = \sqrt{1 - \mu^2_p(z) - \eta^2_p(z)},$$

(2)

then it is called the Pythagorean fuzzy index of the component $z \in Z$ to set $P$, and represents the degree of indeterminacy of $z$ to $P$. Also $0 \leq \pi_p(z) \leq 1$ for every $z \in Z$.

**Definition 2:** [31] Let $\alpha = (t_a, 1-f_a)$ be a PFN, then the score function and the accuracy function of $\alpha$ can be defined as:

$$S(\alpha) = t_a^2 - f_a^2,$$

(3)

and

$$H(\alpha) = t_a^2 + f_a^2,$$

(4)

where $S(\alpha) \in [-1,1]$ and $H(\alpha) \in [0,1]$.

If $\alpha$ and $\beta$ be two Pythagorean fuzzy numbers, then we have

1. If $S(\alpha) < S(\beta)$, then $\alpha < \beta$
2. If $S(\alpha) = S(\beta)$, then
   a. If $H(\alpha) = H(\beta)$, then $\alpha = \beta$,
   b. If $H(\alpha) < H(\beta)$ then $\alpha < \beta$,
   c. If $H(\alpha) > H(\beta)$ then $\alpha > \beta$.

3. Induced Pythagorean Fuzzy Averaging Aggregation Operators

In this section, we introduce the notion of two induced Pythagorean fuzzy aggregation operators such as, induced Pythagorean fuzzy ordered weighted averaging aggregation operator and induced Pythagorean fuzzy hybrid averaging aggregation operator. We also discuss some of their basic properties and construct some examples to develop these proposed operators.

3.1. Induced Pythagorean Fuzzy Ordered Weighted Averaging Operator

**Definition 3:** An induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator can be defined as:

$$\text{I-PFOWA}_\omega(\{u_1, \alpha_1, u_2, \alpha_2, \ldots, u_n, \alpha_n\}) = \left[ \prod_{j=1}^{n} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{n} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right],$$

(5)

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighted vector of $\alpha_j$ ($j = 1, 2, \ldots, n$) with conditions $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

$\alpha_{\alpha(j)}$ is the $\alpha_j$ value of the PFOWA pair $\langle u_j, \alpha_j \rangle$ having the $j^{th}$ largest $u_j$ and $u_j$ in $\{u_j, \alpha_j\}$ is referred to as the order inducing variable and $\alpha_j$ as the Pythagorean fuzzy argument variable.

**Theorem 1:** Let $\{u_j, \alpha_j\} (j = 1, 2, \ldots, n)$ be a collection of 2-tuples, then their aggregated value by using the I-PFOWA operator is also PFV, and

$$\text{I-PFOWA}_\omega(\{u_1, \alpha_1, u_2, \alpha_2, \ldots, u_n, \alpha_n\}) = \left[ \prod_{j=1}^{n} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{n} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right],$$

(6)

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighted vector of $\alpha_j$ ($j = 1, 2, 3, \ldots, n$) with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof: We can prove this theorem by mathematical induction. First we show that equation (6) holds for $n = 2$. Since

$$\alpha_1 \alpha_{\alpha(1)} = \left[ \prod_{j=1}^{2} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{2} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right],$$

$$\alpha_2 \alpha_{\alpha(2)} = \left[ \prod_{j=1}^{2} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{2} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right].$$

Then

$$\alpha_1 \alpha_{\alpha(1)} \oplus \alpha_2 \alpha_{\alpha(2)} = \left[ \prod_{j=1}^{2} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{2} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right].$$

Thus equation (6) holds for $n = 2$. Now we show that equation (6) holds for $n = k$, i.e.,

$$\text{I-PFOWA}_\omega(\{u_k, \alpha_k, \ldots, u_n, \alpha_n\}) = \left[ \prod_{j=1}^{n} \left( 1 - t_{\alpha(j)}^\omega_1 \right)^{\omega_1}, \prod_{j=1}^{n} \left( 1 - f_{\alpha(j)}^\omega_2 \right)^{\omega_2} \right].$$

(7)

Let us suppose that equation (6) holds for $n = k$, then we show that (6) holds for $n = k + 1$. 


be two set of 2-tuples, then
\[
I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right)
= \omega_1 \alpha_1(1) + \omega_2 \alpha_2(1) + \cdots + \omega_k \alpha_k
\]
where \( (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*) \) is any permutation of \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

Proof: As we know that
\[
I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right)
= \omega_1 \alpha_1(1) + \omega_2 \alpha_2(1) + \cdots + \omega_k \alpha_k
\]
and
\[
I-PFOWA \left(\left\{ \alpha_1^*, \alpha_2^*, \ldots, \alpha_n^* \right\} \right)
= \omega_1 \alpha_1^*(1) + \omega_2 \alpha_2^*(1) + \cdots + \omega_k \alpha_k^*
\]
Since \( (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*) \) is any permutation of \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \), thus equation (8) always holds.

**Theorem 3:** (Idempotency): Let \( \left\{ \alpha_j \right\}_{j=1}^{n} \) be a collection of 2-tuples, where \( \alpha_j = \alpha \) for all \( j \), then
\[
I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right)
= \omega_1 \alpha_1(1) + \omega_2 \alpha_2(1) + \cdots + \omega_k \alpha_k
\]
Proof: As \( \alpha_j = \alpha \) for all \( j \), then we have
\[
1 - f_{\alpha_1} = 0.5 \Leftrightarrow f_{\alpha_2} = 0.5
\]
\[
1 - f_{\alpha_3} = 0.6 \Leftrightarrow f_{\alpha_3} = 0.4
\]
\[
1 - f_{\alpha_4} = 0.7 \Leftrightarrow f_{\alpha_4} = 0.3
\]
\[
1 - f_{\alpha_5} = 0.8 \Leftrightarrow f_{\alpha_5} = 0.2
\]
\[
1 - f_{\alpha_6} = 0.9 \Leftrightarrow f_{\alpha_6} = 0.1
\]
Thus
\[
I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right)
= 1 - \frac{1}{\sum_{j=1}^{n} (1 - f_{\alpha_1})} - \frac{1}{\sum_{j=1}^{n} f_{\alpha_2}}
= \frac{1}{(1 - f_{\alpha_1})} - \frac{1}{f_{\alpha_2}}
\]
The proof is complete.

**Theorem 4:** (Boundedness): Let \( \left\{ \alpha_j \right\}_{j=1}^{n} \) be a collection of 2-tuples, and \( \omega = (\omega_1, \omega_2, \ldots, \omega_k) \) be the weighted vector of \( \alpha_j \), with \( \omega_j \in [0,1] \) and \( \sum \omega_j = 1 \), then
\[
\alpha_{max} \leq I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right) \leq \alpha_{max}
\]
Where
\[
I-PFOWA \left(\left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \right)
= \omega_1 \alpha_1(1) + \omega_2 \alpha_2(1) + \cdots + \omega_k \alpha_k
\]
Proof: Since
\[ \alpha_{\min} = \left( \min_j \left( t_{a_{r_{i_j}}} \right), 1 - \max_j \left( f_{a_{r_{i_j}}} \right) \right) \]  
(11)
\[ \alpha_{\max} = \left( \max_j \left( t_{a_{r_{i_j}}} \right), 1 - \min_j \left( f_{a_{r_{i_j}}} \right) \right) \]  
(12)

From equation (13), we have
\[ \alpha_{\min} \leq t_{a_{r_{i_j}}} \leq \max_j \left( t_{a_{r_{i_j}}} \right) \]  
(13)
\[ \min_j \left( f_{a_{r_{i_j}}} \right) \leq f_{a_{r_{i_j}}} \leq \max_j \left( f_{a_{r_{i_j}}} \right) \]  
(14)

Now from equation (14), we have
\[ \prod_{j=1}^{n} \left( 1 - \min_j \left( t_{a_{r_{i_j}}} \right) \right)^{\omega_j} \leq \prod_{j=1}^{n} \left( 1 - \max_j \left( t_{a_{r_{i_j}}} \right) \right)^{\omega_j} \]  
⇔ \[ \prod_{j=1}^{n} \left( 1 - \min_j \left( f_{a_{r_{i_j}}} \right) \right)^{\omega_j} \leq \prod_{j=1}^{n} \left( 1 - \max_j \left( f_{a_{r_{i_j}}} \right) \right)^{\omega_j} \]  
(16)

Let
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) = \alpha = \left( t_{a_r}, 1 - f_{a_r} \right) \]  
(17)

then
\[ S(\alpha) = t_{a_{r+}} - f_{a_{r-}} \leq \min_j \left( t_{a_{r_j}} \right)^2 - \min_j f^2 = S(\alpha_{\max}) \]  
(18)

Again
\[ S(\alpha) = t_{a_{r+}} - f_{a_{r-}} \geq \min_j \left( f_{a_{r_j}} \right)^2 - \max_j \left( f_{a_{r_j}} \right)^2 = S(\alpha_{\min}) \]  
(19)

if
\[ S(\alpha) < S(\alpha_{\max}), S(\alpha) > S(\alpha_{\min}) \]

then
\[ \alpha_{\min} < \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) \leq \alpha_{\max} \]  
(20)

If
\[ \alpha = \left( t_{a_{r}}, 1 - f_{a_{r}} \right) \]

then
\[ \epsilon \Rightarrow S(\alpha) = \epsilon(\alpha) \]
\[ \epsilon \Rightarrow \lambda_{a_{r+}}^2 - \eta_{a_{r-}}^2 = \epsilon \left( \lambda_{a_{r+}} \right)^2 - \epsilon \left( \eta_{a_{r-}} \right)^2 \]
\[ \epsilon \Rightarrow \lambda_{a_{r+}} = \epsilon \left( \lambda_{a_{r+}} \right), \eta_{a_{r-}} = \epsilon \left( \eta_{a_{r-}} \right) \]  
(21)

Thus from equation (21) we have
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) = \alpha_{\max} \]  
(22)

If
\[ \epsilon \Rightarrow S(\alpha) = \epsilon(\alpha) \]
\[ \epsilon \Rightarrow t_{a_{r+}} - f_{a_{r-}} = \epsilon \left( f_{a_{r+}} \right)^2 - \epsilon \left( f_{a_{r-}} \right)^2 \]
\[ \epsilon \Rightarrow t_{a_{r+}} = \epsilon \left( f_{a_{r+}} \right), f_{a_{r-}} = \epsilon \left( f_{a_{r-}} \right) \]
\[ \epsilon \Rightarrow t_{a_{r+}} + f_{a_{r-}} = \epsilon \left( f_{a_{r+}} \right)^2 + \epsilon \left( f_{a_{r-}} \right)^2 \]  
(23)

Thus from equation (23) we have
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) = \epsilon(\alpha_{\min}) \]  
(24)

Thus from equation (20) to (24), we have equation (10) always holds.

**Theorem 5:** (Monotonicity): Let \( \left\langle u_j, \alpha_j \right\rangle, \left\langle u_j, \alpha_j' \right\rangle (j = 1, \ldots, n) \) be two set of 2-tuples, where \( \alpha_{j(j)} < \alpha'_{j(j)} \) for all \( j \), then
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) \]
\[ \leq \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha'_1 \right\rangle, \left\langle u_2, \alpha'_2 \right\rangle, \ldots, \left\langle u_n, \alpha'_n \right\rangle \right) \]  
(25)

Proof: As we know that
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha_1 \right\rangle, \left\langle u_2, \alpha_2 \right\rangle, \ldots, \left\langle u_n, \alpha_n \right\rangle \right) = \omega_1 \alpha_{\sigma(1)} \oplus \omega_2 \alpha_{\sigma(2)} \oplus \ldots \oplus \omega_n \alpha_{\sigma(n)} \]
and
\[ \text{I-PFOWA}_\omega \left( \left\langle u_1, \alpha'_1 \right\rangle, \left\langle u_2, \alpha'_2 \right\rangle, \ldots, \left\langle u_n, \alpha'_n \right\rangle \right) = \omega_1 \alpha'_{\sigma(1)} \oplus \omega_2 \alpha'_{\sigma(2)} \oplus \ldots \oplus \omega_n \alpha'_{\sigma(n)} \]

Since \( \alpha_{\sigma(j)} \leq \alpha'_{\sigma(j)} \) for all \( j \), thus equation (25) always holds.

### 3.2. Induced Pythagorean Fuzzy Hybrid Averaging Operator

**Definition 4:** An induced Pythagorean fuzzy hybrid
averaging (I-PFHA) operator can be defined as:

\[
\text{I-PFHA}_{\alpha,\omega}(\{u_1, \alpha_1\}, \{u_2, \alpha_2\}, \ldots, \{u_n, \alpha_n\}) = \left(\prod_{j=1}^{n} (1-\omega_j f_{\alpha_j}(u_j))^{\alpha_j}\right)^{1-\prod_{j=1}^{n} (1-\omega_j f_{\alpha_j}(u_j))^{\alpha_j}}
\]

(26)

where \(\alpha_{\sigma(j)}\) is the weighted Pythagorean fuzzy value \(\alpha_{\sigma(j)} = n \omega_j \alpha_{\sigma(j)}, j = 1, \ldots, n\) of the Pythagorean fuzzy ordered weighted averaging pair \(\{u_j, \alpha_j\}\) having the \(j^{th}\) largest \(u_j, u_j\) in \(\{u_j, \alpha_j\}\) is the order inducing variable and \(\alpha_j\) is the Pythagorean fuzzy argument variable \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) is the weighted vector of \(\alpha_j\) \((j = 1, \ldots, n)\) and also \(\omega_j \in [0, 1], \sum \omega_j = 1\).

Theorem 6: Let \(\{u_j, \alpha_j\} (j = 1, 2, \ldots, n)\) be a collection of 2-tuples, then their aggregated value by using the I-PFHA operator is also a Pythagorean fuzzy value, and

\[
\text{I-PFHA}_{\alpha,\omega}(\{u_1, \alpha_1\}, \{u_2, \alpha_2\}, \ldots, \{u_n, \alpha_n\}) = \left(\prod_{j=1}^{n} (1-\omega_j f_{\alpha_j}(u_j))^{\alpha_j}\right)^{1-\prod_{j=1}^{n} (1-\omega_j f_{\alpha_j}(u_j))^{\alpha_j}}
\]

(27)

where \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) is the weighted vector of \(\alpha_j\) \((j = 1, 2, 3, \ldots, n)\) with \(\omega_j \in [0, 1]\) and \(\sum \omega_j = 1\).

Proof: Proof is similar to Theorem 1.

4. Conclusion

In this paper, we have familiarized the idea of induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator and induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator and also discussed some of their basic properties.

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