On the Paper: Numerical Radius Preserving Linear Maps on Banach Algebras

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To cite this article:

Mohammed El Azhari. On the Paper: Numerical Radius Preserving Linear Maps on Banach Algebras. *Mathematics Letters*.

Vol. 3, No. 5, 2017, pp. 50-51. doi: 10.11648/j.ml.20170305.11

Received: May 16, 2017; Accepted: September 8, 2017; Published: October 10, 2017

Abstract: We give an example of a unital commutative complex Banach algebra having a normalized state which is not a spectral state and admitting an extreme normalized state which is not multiplicative. This disproves two results by Golfarshchi and Khalilzadeh.

Keywords: Banach Algebra, Regular Norm, Normalized State, Spectral State

1. Preliminaries

Let $(A, \|\cdot\|)$ be a complex normed algebra with an identity e such that $\|e\| = 1$. Let $D(A, e) = \{f \in A': f(e) = \|f\| = 1\}$, where A' is the dual space of A . The elements of $D(A, e)$ are called normalized states on A . For $a \in A$, let $V(A, a) = \{f(a): f \in D(A, e)\}$, $V(A, a)$ is called the numerical range of a . Let $sp(a)$ be the spectrum of $a \in A$, and let $co(sp(a))$ be the convex hull of $sp(a)$. We say that a linear functional f on A is a spectral state if $f(a) \in co(sp(a))$ for all $a \in A$. We denote by $M(A)$ the set of all non-zero continuous multiplicative linear functionals on A .

2. Result

2.1. Counterexample

Golfarshchi and Khalilzadeh proved the following results [4]:

[4, Theorem 2]. Let A be a unital complex Banach algebra, and let f be a linear functional on A . Then f is a normalized state on A if and only if $f(a) \in co(sp(a))$ for all $a \in A$.

[4, Theorem 3]. Let A be a unital commutative complex Banach algebra. Then each extreme normalized state on A is multiplicative.

Here we give a counterexample disproving the above results. We also remark that Theorems 5 and 6 [4] are called into question since the authors used Theorem 3 [4] to prove these results.

Let $(A, \|\cdot\|)$ be a non-zero commutative radical complex

Banach algebra [6, p.316]. Let $A_e = \{a + \lambda e: a \in A, \lambda \in \mathbb{C}\}$ be the unitization of A with the identity e , and the norm $\|a + \lambda e\|_1 = \|a\| + |\lambda|$ for all $a + \lambda e \in A_e$. $(A_e, \|\cdot\|_1)$ is a unital commutative complex Banach algebra, and $M(A_e) = \{\varphi_\infty\}$, where φ_∞ is the continuous multiplicative linear functional on A_e defined by $\varphi_\infty(a + \lambda e) = \lambda$ for all $a + \lambda e \in A_e$.

(1). let a be a non-zero element of A , $V(A_e, a) = \{z \in \mathbb{C}: |z| \leq \|a\|\}$ by [2, Remark 3.8], and $sp(a) = \{\varphi_\infty(a)\} = \{0\}$, hence $co(sp(a)) = \{0\}$ is strictly included in $V(A_e, a)$ since $\|a\| \neq 0$. Therefore the direct implication of [4, Theorem 2] does not hold.

(2). By [1, Lemma 1.10.3], $D(A_e, e)$ is a non-empty weak* compact convex subset of A'_e , then $ext(D(A_e, e))$ is a non-empty set. Assume that each extreme normalized state on A_e is multiplicative, then $ext(D(A_e, e)) = \{\varphi_\infty\}$. Let a be a non-zero element of A , by [1, Corollary 1.10.15] there exists $f \in D(A_e, e)$ such that $f(a) \neq 0 = \varphi_\infty(a)$. Therefore $\overline{co}(ext(D(A_e, e))) = \{\varphi_\infty\}$ is strictly included in $D(A_e, e)$, which contradicts the Krein-Milman Theorem. This shows that [4, Theorem 3] is not valid.

2.2. Regular Norm and the Operator Seminorm

Let $(A, \|\cdot\|)$ be a non-unital complex Banach algebra, and let $A_e = \{a + \lambda e: a \in A, \lambda \in \mathbb{C}\}$ be the unitization of A with the identity e . Let $\|a + \lambda e\|_{op} = \sup\{\|(a + \lambda e)x\|, \|x(a + \lambda e)\|: x \in A, \|x\| \leq 1\}$ for all $a + \lambda e \in A_e$, $\|\cdot\|_{op}$ is an algebra seminorm on A_e . We say that $\|\cdot\|$ is regular if $\|\cdot\|_{op} = \|\cdot\|$ on A . If $\|\cdot\|$ is regular, it is well known that $(A_e, \|\cdot\|_{op})$ is a complex Banach algebra. The following

question was asked [3]: If $(A_e, \|\cdot\|_{op})$ is a complex Banach algebra, is the norm $\|\cdot\|$ regular?

Orenstein tried to give an answer to this question in the commutative case [5], but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

3. Conclusion

In this note, we show that Theorems 2 and 3 [4] are false by giving a counterexample. We also remark that Theorems 5 and 6 [4] and Theorem 1.1 [5] are called into question since the authors used Theorems 2 or 3 [4] to prove these results.

References

- [1] F. F. Bonsall and J. Duncan, Complete normed algebras, New York: Springer Verlag 1973.
- [2] A. K. Gaur and T. Husain, Spatial numerical ranges of elements of Banach algebras, International Journal of Mathematics and Mathematical Sciences, 12(4)(1989), 633-640.
- [3] A. K. Gaur and Z. V. Kovářík, Norms, states and numerical ranges on direct sums, Analysis, 11(2-3)(1991), 155-164.
- [4] F. Golfarshchi and A. A. Khalilzadeh, Numerical radius preserving linear maps on Banach algebras, International Journal of Pure and Applied Mathematics, 88(2)(2013), 233-238.
- [5] A. Orenstein, Regular norm and the operator seminorm on a non-unital complex commutative Banach algebra, arXiv: 1410.8790v2 [math.FA] 11 Dec 2015.
- [6] C. E. Rickart, General theory of Banach algebras, New York: Van Nostrand 1960.