On the Paper: Numerical Radius Preserving Linear Maps on Banach Algebras

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Abstract: We give an example of a unital commutative complex Banach algebra having a normalized state which is not a spectral state and admitting an extreme normalized state which is not multiplicative. This disproves two results by Golfarshchi and Khalilzadeh.

Keywords: Banach Algebra, Regular Norm, Normalized State, Spectral State

1. Preliminaries

Let $(A, \|\cdot\|)$ be a complex normed algebra with an identity $e$ such that $\|e\| = 1$. Let $D(A, e) = \{f \in A^*: f(e) = \|f\| = 1\}$, where $A^*$ is the dual space of $A$. The elements of $D(A, e)$ are called normalized states on $A$. For $a \in A$, let $V(A, a) = \{f(a): f \in D(A, e)\}$, $V(A, a)$ is called the numerical range of $a$. Let $sp(a)$ be the spectrum of $a \in A$, and let $co(sp(a))$ be the convex hull of $sp(a)$. We say that a linear functional $f$ on $A$ is a spectral state if $f(a) \in co(sp(a))$ for all $a \in A$. We denote by $M(A)$ the set of all non-zero continuous multiplicative linear functionals on $A$.

2. Result

2.1. Counterexample

Golfarshchi and Khalilzadeh proved the following results [4]:

[4, Theorem 2]. Let $A$ be a unital complex Banach algebra, and let $f$ be a linear functional on $A$. Then $f$ is a normalized state on $A$ if and only if $f(a) \in co(sp(a))$ for all $a \in A$.

[4, Theorem 3]. Let $A$ be a unital commutative complex Banach algebra. Then each extreme normalized state on $A$ is multiplicative.

Here we give a counterexample disproving the above results. We also remark that Theorems 5 and 6 [4] are called into question since the authors used Theorem 3 [4] to prove these results.

Let $(A, \|\cdot\|)$ be a non-zero commutative radical complex Banach algebra [6, p.316]. Let $A_e = \{a + \lambda e: a \in A, \lambda \in \mathbb{C}\}$ be the unitization of $A$ with the identity $e$, and the norm $\|a + \lambda e\|_e = \|a\| + \|e\|$ for all $a + \lambda e \in A_e$. $(A_e, \|\cdot\|_e)$ is a unital commutative complex Banach algebra, and $M(A_e) = \{\varphi_\infty\}$, where $\varphi_\infty$ is the continuous multiplicative linear functional on $A_e$ defined by $\varphi_\infty(a + \lambda e) = \lambda$ for all $a + \lambda e \in A_e$.

1. Let $a$ be a non-zero element of $A$, $V(A_e, a) = \{z \in \mathbb{C}: \|z\| \leq \|a\|\}$. By [2, Remark 3.8], and $sp(a) = \{\varphi_\infty(a)\}$, hence $co(sp(a)) = \{0\}$, strictly included in $V(A_e, a)$ since $\|a\| \neq 0$. Therefore the direct implication of [4, Theorem 2] does not hold.

2. By [1, Lemma 1.10.3], $D(A_e, e)$ is a non-empty weak* compact convex set of $A_e^*$, then $ext(D(A_e, e))$ is a non-empty set. Assume that each extreme normalized state on $A_e$ is multiplicative, then $ext(D(A_e, e)) = \{\varphi_\infty\}$. Let $a$ be a non-zero element of $A$, by [1, Corollary 1.10.15] there exists $f \in D(A_e, e)$ such that $f(a) \neq 0 = \varphi_\infty(a)$. Therefore $\overline{co}(ext(D(A_e, e))) = \{\varphi_\infty\}$ is strictly included in $D(A_e, e)$, which contradicts the Krein-Milman Theorem. This shows that [4, Theorem 3] is not valid.

2.2. Regular Norm and the Operator Seminorm

Let $(A, \|\cdot\|)$ be a non-unital complex Banach algebra, and let $A_e = \{a + \lambda e: a \in A, \lambda \in \mathbb{C}\}$ be the unitization of $A$ with the identity $e$. Let $\|a + \lambda e\|_{op} = sup\{\|a + \lambda e\|: x \in A, \|x\| \leq 1\}$ for all $a + \lambda e \in A_e$. $\|\cdot\|_{op}$ is an algebra seminorm on $A_e$. We say that $\|\cdot\|_{op}$ is regular if $\|\cdot\|_{op} = \|\cdot\|$ on $A$. If $\|\cdot\|_{op}$ is regular, it is well known that $(A_e, \|\cdot\|_{op})$ is a complex Banach algebra. The following
question was asked [3]: If \((A, \| \cdot \|)\) is a complex Banach algebra, is the norm \(\| \cdot \|\) regular?

Orenstein tried to give an answer to this question in the commutative case [5], but his proof is not correct since it is essentially based on the direct implication of [4, Theorem 2].

3. Conclusion

In this note, we show that Theorems 2 and 3 [4] are false by giving a counterexample. We also remark that Theorems 5 and 6 [4] and Theorem 1.1 [5] are called into question since the authors used Theorems 2 or 3 [4] to prove these results.

References


