D-Effect Algebra Can Be Made into a D-Total Algebra

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Abstract: In this paper we prove that every *D-effect algebra* $(E, \Delta, 0, 1)$ can be made into a *D-total algebra* $(E, \underline{\Delta}, \neg, 1)$ in such a way that two elements are compatible in $(E, \Delta, 0, 1)$ if and only if they commute in $(E, \underline{\Delta}, \neg, 1)$ where $x \Delta y = (x' + y')'$.

Keywords: D-Basic Algebra, Weak D-Basic Algebra, Antitone Involution, D-Effect Algebra, D-Total Algebra

1. Introduction

Chajda, I., Halas, R., Kuhr, J. (2009) introduced similar results for general effect algebras in the context of commutative directoids; they proved that every effect algebra $(E, +, 0, 1)$ can be made into a total algebra $(E, \oplus, \neg, 0)$ in such a way that two elements are compatible in $(E, +, 0, 1)$ if and only if they commute in $(E, \oplus, \neg, 0)$.

In the present paper we introduce and study the concept of a D-basic algebra, this being an algebra $(A, \underline{\Delta}, \neg, 1)$ of type $(2, 1, 0)$ with the property that the underlying poset $(A, \leq)$, defined by $x \geq y$ if and only if $\neg x \underline{\Delta} y = \neg 1$, is a bounded lattice and, for each $a \in A$, the mapping $(x \to \neg x \underline{\Delta} a)$ is an antitone involution on the principal ideal $(a] = \{x \in A \mid a \geq x\}$. The name '*D-basic algebra*' is used because these algebras capture common features of many known structures such as Boolean algebras, orthomodular lattices, lattice D-effect algebras. we have special attention to lattice D-effect algebras, which were originally defined as partial algebras $(E, \Delta, 0, 1)$, but where the presence of the meet operation allowe one to replace partial Δ by D-total $\underline{\Delta}$. The intent of the present paper is to establish similar results for D-effect algebras in the context of commutative directoids; we prove that every *D-effect algebra* $(E, \Delta, 0, 1)$ can be made into a D-total algebra $(E, \underline{\Delta}, \neg, 1)$.

We first recall several relevant notions.

Definition 1.1 [8]: A *commutative directoids* is a commutative, idempo-tent groupoid $(A, .)$ satisfying the equation x. $((x. y). z) = (x. y). z$.

Remark 1.2 For instance, every semilattice is a commutative directoid. It can easily be seen that the stipulation

$$x \geq y \text{ if and only if } x. y = y \qquad (1)$$

defines a partial order on A such that, for every $x, y \in A$, x. y is a lower bound of $\{x, y\}$. Thus the poset $(A, \leq)$ is downwards directed. *Conversely*, we may associate a commutative directoid to an arbitrary downwards directed set by letting x. y = y. x be some lower bound of $\{x, y\}$, such that whenever x, y are comparable, x. y = y. x is the least of x, y.

Like in semilattices, we could define the dual order by $x \geq y$ if and only if x. y = x, in which case x. y is an upper bound of $\{x, y\}$. But we shall be concerned with the partial order given by (1). Accordingly, we shall write $\sqcap$ instead of. in order to emphasize that $x \sqcap y$ is less than or equal to x, y.

Definition 1.3 [3] An *antitone involution* on a poset $(P, \leq)$ is a mapping $\beta: P \to P$ such that, for all $x, y \in P$,

$$x \geq y \Rightarrow \beta (y) \geq \beta (x), \qquad (2)$$

$$\beta (\beta(x)) = x. \qquad (3)$$

By a commutative directoid with sectional antitone involutions we shall mean a system $(A, \sqcap, (\beta_a)_{a \in A}, 0, 1)$ where

$(A, \sqcap)$ is a commutative directoid with a least element 0 and a greatest element 1, and every section (a] is equipped with an antitone involution β_a.

In particular, if $(A, \sqcap)$ is a semilattice, then the underlying poset is a lattice in which $\beta_1(\beta_1(x) \sqcap \beta_1(y))$ is the supremum of $\{x, y\}$, and hence we may say that $(A, \sqcap, (\beta_a)_{a \in A}, 0, 1)$ is a

lattice with sectional antitone involutions.

2. Weak D-Basic Algebra

Definition 2.1 A *Weak D-basic algebra* is an algebra (A, $\triangle$, ¬, 1) of type (2, 1, 0) satisfying the following identities and quasi-identity (where 1 is an abbreviation for ¬ 0):

(DW1) x $\triangle$ 1 = x,
(DW2) ¬ ¬x = x,
(DW3) ¬ (¬x $\triangle$ y) $\triangle$ y = ¬ (¬y $\triangle$ x) $\triangle$ x,
(DW4) x $\triangle$ (¬ (¬ (¬ (x $\triangle$ y) $\triangle$ y) $\triangle$ z) $\triangle$ z) = 0,
(DW5) ¬ x $\triangle$ (y $\triangle$ x) = 0,
(DW6) ¬ x $\triangle$ y = 0 & ¬ y $\triangle$ z = 0 ⇒ ¬ (¬ z $\triangle$ x)$\triangle$(¬ y $\triangle$ x) =0.

These algebras contains the equations x $\triangle$ 0 = 0= 0 $\triangle$ x. Specifically, if (A, $\triangle$, ¬, 0) is a *weak D-basic algebra* and if we put

$$x \sqcap y = ¬ (¬ x \triangle y) \triangle y,$$

then (A, ⊓) is a commutative directoid with a least element 0 and a greatest element 1, such that the underlying order ≤ is given by:

x ≥ y if and only if x ⊓ y = y if and only if ¬ x $\triangle$ y = 0 (4)

For each a ∈ A, (x → ¬ x $\triangle$ a) is an antitone involution on (a] = {x ∈ A | a ≥ x}. Conversely, if (A,⊓, $(\beta_a)_{a \in A}$, 0, 1) is a commutative directoid with sectional antitone involutions, then we can define $\triangle$ and ¬ as x $\triangle$ y = $\beta_y(\beta_1(x)⊓ y)$ and ¬ x = $\beta_1(x)$, respectively, and (A, $\triangle$, ¬, 0) becomes a *weak D-basic algebra* in which x $\triangle$ y = ¬ (¬ x $\triangle$ y) $\triangle$ y and $\beta_a(x) =$ ¬x $\triangle$ a. In every *Weak D-basic algebra*, in addition to the 'meat-like' operation ⊓, we can introduce the dual 'join-like' operation ⊔ by

$$x \sqcup y = ¬ (¬ x \sqcap ¬ y). \quad (5)$$

Then we have x ≥ y if and only if x ⊔ y = x, and the structure (A, ⊔, ⊓) is a λ-lattice in the sense of [9], i.e., both (A, ⊓) and (A, ⊔) are commutative directoids and the absorption laws
x ⊓ (x ⊔ y) = x = x ⊔ (x ⊓ y) are satisfied.

Definition 2.2 A *D-basic algebra* is an algebra (A, $\triangle$, ¬, 0) of type (2, 1, 0) satisfying the identities (again, 1 = ¬ 0)

(DB1) x $\triangle$ 1 = x,
(DB2) ¬ ¬ x = x,
(DB3) ¬ (¬ x $\triangle$ y) $\triangle$ y = ¬ (¬ y $\triangle$ x) $\triangle$x,
(DB4) ¬ (¬ (¬ (x $\triangle$ y) $\triangle$ y) $\triangle$ z) $\triangle$ (x $\triangle$ z) = 0.
Originally, we required x $\triangle$ 0 = 0 = 0 $\triangle$ x.

Every *D-basic algebra* is a *Weak D-basic algebra*, and the above assignment between *weak D-basic algebras* and commutative directoids with sectional antitone involutions, restricted to *D-basic algebras*, furnishes a one-to-one correspondence between *D-basic algebras* and lattices with sectional antitone involutions. In other words, a weak *D-basic algebra* (A, $\triangle$, ¬, 0) is a *D-basic algebra* if and only if (A,⊓, $(\beta_a)_{a \in A}$, 0, 1) is a lattice with sectional antitone involutions.

The axioms (DW3) - (DW6) may be rewritten in terms of ≥ and ⊓ as follows:
(DW3') x ⊓ y = y ⊓ x,
(DW4') x ≥ (x ⊓ y)⊓ z,
(DW5') x ≥ y $\triangle$ x,
(DW6') x ≥ y & y ≥ z ⇒ ¬ z $\triangle$ x ≥ ¬ y $\triangle$ x.
Moreover, in every weak D-basic algebra we have

$$x \triangle 0 = 0 = 0 \triangle x, \quad (6)$$

$$1 \triangle x = x, \quad (7)$$

$$¬ (x \sqcap y) \triangle y = ¬ x \triangle y. \quad (8)$$

Indeed, 0 $\triangle$ x = ¬ 1 $\triangle$ (x $\triangle$ 1) = 0 by (DW1) and (DW5), so x $\triangle$ 0 = x $\triangle$ (0 $\triangle$ ¬ x) = 0 by (DW2) and (DW5). Further, 1 $\triangle$ x = ¬ 0 $\triangle$ x = (¬ 1 $\triangle$ x) $\triangle$ x = (¬ x $\triangle$ 1) $\triangle$ 1 = ¬ ¬ x = x and ¬ (x ⊓ y) $\triangle$ y = ¬ (¬ (¬ x $\triangle$ y) $\triangle$ y) $\triangle$ y = (¬ x $\triangle$ y) ⊓ y = ¬ x $\triangle$ y.

Proposition 2.3 A *D-algebra* A = (A, $\triangle$, ¬, 0) satisfying (DW1)–(DW4) is a *Weak D-basic algebra* if and only if it satisfies the identity

$$¬ (¬ ((x \sqcap y) \sqcap z) \triangle x) \triangle (¬ y \triangle x) = 0. \quad (9)$$

Proof. Let A be a *Weak D-basic algebra*. We have x ≥ x ⊓ y and x ⊓ y ≥ (x ⊓ y) ⊓ z, i.e., ¬ x $\triangle$ (x ⊓ y) = 0 and ¬ (x ⊓ y) $\triangle$ ((x ⊓ y) ⊓ z) = 0, which yields

$$¬ (¬ ((x \sqcap y) \sqcap z) \triangle x) \triangle (¬ (x \sqcap y) \triangle x) = 0$$

by (DW6). But ¬ (x ⊓ y) $\triangle$ x = ¬ (y ⊓ x) $\triangle$ x = ¬ y $\triangle$x by (8), hence A fulfils (9). *Conversely*, assume that A satisfies (DW1)–(DW4) and (9). We first observe that x ⊓ 1 = ¬ (¬ x $\triangle$ 1) $\triangle$ 1= ¬ ¬ x = x, whence 0 $\triangle$ x = ¬1 $\triangle$ ((1 ⊓ 1) ⊓ x) = 0 by (DW4') and (8). This yields 1 $\triangle$ x = ¬0 $\triangle$ x = ¬ ¬1 $\triangle$ x) $\triangle$ x = 1 ⊓ x = x and so x ⊓ 0 = ¬ (¬0 $\triangle$ x) $\triangle$ x = ¬ x $\triangle$ x = ¬ x $\triangle$ ((x ⊓ 1) ⊓ 1) = 0. Further, ¬x $\triangle$ y = 0 entails x ⊓ y = ¬ (¬x $\triangle$ y) $\triangle$ y = ¬ 0 $\triangle$ y = 1 $\triangle$ y = y.

Now, if we substitute 0 and ¬y for z and y, respectively, by (9) we obtain 0 = ¬ (¬ ((x ⊓ ¬ y) ⊓ 0) $\triangle$ x) $\triangle$ (¬ ¬ y $\triangle$ x) = ¬ (¬ 0 $\triangle$ x) $\triangle$ (y $\triangle$ x) = ¬ x $\triangle$ (y $\triangle$ x), which is (DW5). Finally, if ¬ x $\triangle$ y = 0 and ¬ y $\triangle$z = 0, then (x ⊓ y) ⊓ z = z and by (9) we have

$$0 = ¬ (¬ ((x \sqcap y) \sqcap z) \triangle x) \triangle (¬ y \triangle x)$$

$$= ¬ (¬ z \triangle x) \triangle (¬ y \triangle x),$$

which settles (DW6). Thus A is a *Weak D-basic algebra*.

Another central concept is that of a *D-effect algebra*. We have a *D-effect algebra* is a system (E, Δ, 0, 1) where 0, 1 are distinguished elements of E and Δ is a partial binary operation on E such that

(DEA1) x Δ y = y Δ x if one side is defined,
(DEA2) (x Δ y) Δ z = x Δ (y Δ z) if one side is defined,
(DEA3) for every x ∈ E there exists a unique x' ∈ E with x' Δ x = 0,
(DEA4) if x Δ 0 is defined then x = 1.

Every *D-effect algebra* bears a natural partial order given by x ≥ y if and only if y = x Δ z for some z ∈ E.

The poset $(E, \leq)$ is bounded, 0 is the bottom element and 1 is the top element. If, moreover, $(E, \leq)$ is a lattice, then $(E, \Delta, 0, 1)$ is called a *lattice D-effect algebra*. In every *D-effect algebra*, a partial binary operation ∇ can be defined as follows:

$x \nabla y$ exists and equals z if and only if $z = (x' \Delta y)'$.

(Thus $x \nabla y$ is defined if and only if $y \geq x$.) The system $(E, \leq, \nabla, 0, 1)$ so obtained is called a *D-poset*.

When doing calculations, the following properties of *D-effect algebras* and *D-posets* will be useful:

Remark 2.4

(1) $x \Delta 1 = x$, $x \nabla 1 = x$, $x \nabla x = 1$, $0 \nabla x = x'$;
(2) $x \geq y$ if and only if $y' \geq x'$;
(3) $x \Delta y$ is defined if and only if $x \geq y'$ if and only if $y \geq x'$; in this case, $x \Delta y = (x' \nabla y)' = (y' \nabla x)'$;
(4) If $x \Delta y$ is defined, then so is $x_1 \Delta y_1$ for all $x_1 \geq x$ and $y_1 \geq y$;
(5) $x \Delta y = z$ if and only if $x' = y \Delta z'$ if and only if $y' = x \Delta z'$;
(6) If $x \geq y$, then $x \nabla y = (x' \Delta y)' \geq x$ and $x \nabla (x \nabla y) = y$;
(7) $x \geq y \geq z$ implies $z \nabla y \geq z \nabla x$ and $(z \nabla x) \nabla (z \nabla y) = y \nabla x$; in particular $x' \nabla y' = y \nabla x$;
(8) $x \geq y \geq z$ implies $y \nabla x \geq z \nabla x$ and $(z \nabla x) \nabla (y \nabla x) = z \nabla y$.

3. The Relation Between D-Effect Algebras and Weak D-Basic Algebras

Theorem 3.1 Let $A = (A, \boxdot, \neg, 0)$ be a *Weak D-basic algebra*. Define the partial binary relation Δ on A as follows: $x \Delta y$ is defined if and only if $x \geq \neg y$, and in this case $x \Delta y = x \boxdot y$. Then $\mathcal{E}(A) = (A, \Delta, 0, 1)$ is a D-effect algebra if and only if A satisfies the quasi-identity

$$x \geq \neg y \ \& \ x \boxdot y \geq \neg z \Rightarrow (x \boxdot y) \boxdot z = x \boxdot (z \boxdot y). \ (10)$$

Moreover, over *Weak D-basic algebras*, (10) is equivalent to the identity

$$(x \boxdot y) \boxdot (\neg (x \boxdot y) \sqcup z) = (x \sqcup \neg y) \boxdot ((\neg (x \boxdot y) \sqcup z) \boxdot y). \ (11)$$

Proof. Suppose that $\mathcal{E}(A)$ is a *D-effect algebra*. If $x \geq \neg y$ and $x \boxdot y \geq \neg z$, then $x \Delta y$ and $(x \Delta y) \Delta z$ exist, hence $y \Delta z$ and $x \Delta (y \Delta z)$ also exist and $(x \boxdot y) \boxdot z = (x \Delta y) \Delta z = x \Delta(y \Delta z) = x \Delta (z \Delta y) = x \boxdot (z \boxdot y)$.

Conversely, let A satisfy (10). We shall verify that $\mathcal{E}(A)$ is a *D-algebra*:

(EA1) Assume that $a \Delta b$ is defined. Since $a \geq \neg b$ if and only if $b \geq \neg a$, it follows that $b \Delta a$ is defined, too. By (10) we have $(1 \geq \neg a \ \& \ 1 \boxdot a \geq \neg b) \Rightarrow a \boxdot b = b \boxdot a$, so that $a \geq \neg b$ entails $a \Delta b = b \Delta a$.

(EA2) Let $(a \Delta b) \Delta c$ be defined, i.e., $a \geq \neg b$ and $a \Delta b = a \boxdot b \geq \neg c$. Since $b \geq a \boxdot b \geq \neg c$, also $b \Delta c = c \Delta b$ exists. Further, by (W6), $b \geq a \boxdot b \geq \neg c$ implies $c \boxdot b = \neg\neg c \boxdot b \geq \neg (a \boxdot b) \boxdot b = \neg a \sqcap b = \neg\neg a$, so $a \Delta (c \Delta b) = a \Delta (b \Delta c)$ is defined. Analogously, if $a \Delta (b \Delta c)$ exists, then so does $(a \Delta b) \Delta c$. By (10) we have $(a \Delta b) \Delta c = (a \boxdot b) \boxdot c = a \boxdot$

$(c \boxdot b) = a \Delta (c \Delta b) = a \Delta (b \Delta c)$.

(EA3) Clearly, we have $\neg a \Delta a = 0$. If $b \Delta a = 0$, then $b \geq \neg a$ since $b \Delta a$ is defined, and $\neg b \geq a$ (i.e., $b \leq \neg a$) since $b \boxdot a = 0$. Thus $b = \neg a$.

(EA4) Finally, if $a \Delta 0$ is defined, then $a \geq \neg 0 = 1$, so $a = 1$.

It remains to show that (10) and (11) are equivalent over *weak D-basic algebras*.

In any *weak D-basic algebra*, $x \sqcup \neg y \geq \neg y$ and, using (8),

$$(x \sqcup \neg y) \boxdot y = \neg (\neg x \sqcap y) \boxdot y = x \boxdot y \geq (x \boxdot y) \sqcap \neg z = \neg (\neg (x \boxdot y) \sqcup z).$$

Therefore, if (10) holds, then we have

$$(x \boxdot y) \boxdot (\neg (x \boxdot y) \sqcup z) = ((x \sqcup \neg y) \boxdot y) \boxdot (\neg (x \boxdot y) \sqcup z) = (x \sqcup \neg y) \boxdot ((\neg (x \boxdot y) \sqcup z) \boxdot y),$$

which is (11). On the other hand, (11) evidently implies (10).

Corollary 3.2 Let $A = (A, \boxdot, \neg, 0)$ be a *D-basic algebra* and let $\mathcal{E}(A) = (A, \Delta, 0, 1)$ be as in Theorem 3.1. Then $\mathcal{E}(A)$ is a *lattice D-effect algebra* if and only if A satisfies the quasi-identity (10).

However, as the following example shows, this is not true for *Weak D-basic algebras* since many different *Weak D-basic algebras* can determine the same *D-effect algebra*.
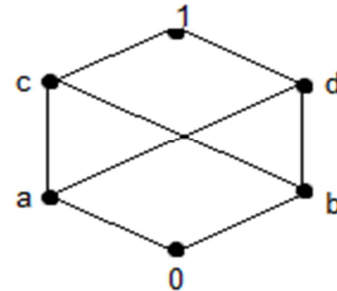
Example 3.3 Let $(A, \leq)$ be the poset

Figure 1. Weak D-basic algebra.

and let the sections $[1] = A$, $(c]$ and $(d]$ be equipped with the following antitone involutions:

β_1: $1 \to 0, 0 \to 1, d \to a, a \to d, b \to c, c \to b$;
β_c: $c \to 0, 0 \to c, b \to a, a \to b$,
β_d: $d \to 0, 0 \to d, b \to b, a \to a$,

the other sections admit unique antitone involutions. There are three possible ways in which we can associate a commutative directoid to $(A, \geq)$, and consequently, there are three *weak D-basic algebras* with the underlying poset $(A, \leq)$:

In (*table 1.*) for $c \sqcap_1 d = a$ we get $A_1 = (A, \boxdot_1, \neg, 1)$ where

Table 1. Show the first way to weak D-basic algebra when $c \sqcap_1 d = a$.

$\boxdot_1$	0	a	b	c	d	1	$\neg$
0	0	0	0	0	0	0	1
a	0	0	0	a	0	a	d
b	0	0	0	0	b	b	c
c	0	a	0	a	b	c	b
d	0	0	b	b	a	d	a
1	0	b	b	c	d	1	0

In (*table 2.*) for $c \sqcap_2 d = b$ we get $A_2 = (A, \boxdot_2, \neg, 1)$ where

Table 2. Show the second way to weak D-basic algebra when c $\sqcap_2$ d = b.

$\boxed{\Delta}_2$	0	A	B	c	d	1	¬
0	0	0	0	0	0	0	1
a	0	0	0	b	0	a	d
b	0	0	0	0	a	b	c
c	0	a	0	a	b	c	b
d	0	0	B	b	a	d	a
1	0	a	B	c	d	1	0

In (*table 3*.) for c $\sqcap_3$ d = 0 we get A$_3$ = (A, $\boxed{\Delta}_3$, ¬, 1) where

Table 3. Show the third way to weak D-basic algebra when c $\sqcap_3$d = 0.

$\boxed{\Delta}_3$	0	a	B	c	d	1	¬
0	0	0	0	0	0	0	1
a	0	0	0	C	0	a	d
b	0	0	0	0	d	b	c
c	0	a	0	a	b	c	b
d	0	0	b	b	a	d	a
1	0	a	b	c	d	1	0

All these *Weak D-basic algebras* induce the same *D-effect algebra* $\mathcal{E}$ (A$_1$) = $\mathcal{E}$ (A$_2$) = $\mathcal{E}$ (A$_3$) = (A, Δ, 0, 1) that clear in (*table 4*.) where

Table 4. *Show that the three ways in Weak D-basic algebras induce the same D-effect algebra.*

Δ	0	A	b	c	d	1	¬
0	.	.	.	.	.	0	1
a	.	.	.	.	0	a	d
b	.	.	.	0	.	b	c
c	.	.	0	a	b	c	b
d	.	0	.	b	a	d	a
1	0	a	b	c	d	1	0

Where for any x, y ∈ A we have x Δ y =. means x Δ y does not exist. Let E = (E, Δ, 0, 1) be a *D-effect algebra*. Since the underlying poset (E, ≤) is bounded, it can be organized into a commutative directoid (E, $\sqcap$). We shall simply say that the pair (E, $\sqcap$) is a *D-effect algebra* with an associated commutative directoid.

Theorem 3.4 Let (E, $\sqcap$) be a *D-effect algebra* E = (E, Δ, 0, 1) with an associated commutative directoid. Define

x $\boxed{\Delta}$ y = (x' $\sqcap$ y)' Δ y and ¬ x = x'.

Then DB (E, $\sqcap$) = (E, $\boxed{\Delta}$, ¬, 1) is a *Weak D-basic algebra* satisfying (10). *Moreover*, $\mathcal{E}$ (DB (E, $\sqcap$)), the *D-effect algebra* assigned to DB (E, $\sqcap$) by *Theorem 3.1*, is just E.

Proof. First, we prove that for each a ∈ E, β_a: x → x' Δ a is an anititone involution on [a]. For all x ∈ [a], we have x' Δ a is defined since x' ≥ a', and x' Δ a ≤ a. Thus β_a is well defined. We also have β_a (β_a (x)) = (x' Δ a)' Δ a = x because, by (v), (x' Δ a)' Δ a = x if and only if (x' Δ a)'' = a Δ x'. Finally, if a ≥ x ≥ y, then β_a(y) = y' Δ a ≥ x' Δ a = β_a(x), proving that β_a is an anititone involution.

We know that if we put x$\boxed{\Delta}_1$y = β_y (β_1 (x) $\sqcap$ y) and ¬ x = x', then (E, $\boxed{\Delta}_1$, ¬, 1) becomes a *Weak D-basic algebra*. But x$\boxed{\Delta}_1$y = β_y (β_1 (x) $\sqcap$ y) = ((x' Δ 1) $\sqcap$ y)' Δ y = (x' $\sqcap$ y)' Δ y = x Δ y. Therefore, DB (E, $\sqcap$) is a *Weak D-basic algebra*.

Now, we prove that DB (E, $\sqcap$) satisfies the quasi-identity (10). It is obvious that whenever x Δ y is defined in E (i.e., x

≥ y' = ¬y).

Then x $\boxed{\Delta}$ y = (x' $\sqcap$ y)' Δ y = x Δ y. Hence if x ≥ ¬y and x $\boxed{\Delta}$ y ≥ ¬z, then (x $\boxed{\Delta}$ y) $\boxed{\Delta}$ z = (x Δ y) Δ z = x Δ (y Δ z) = x Δ (z Δ y) = x $\boxed{\Delta}$ (z $\boxed{\Delta}$ y), which settles (10). The last assertion is clear.

Example 3.5 Let E be the *D-effect algebra* we have obtained in Example 3.3. If we put c $\sqcap_1$ d = a then DB (E, $\sqcap_1$) is just the *Weak D-basic algebra* A$_1$ from Example 3.3. Analogously, if c $\sqcap_2$ d = b then DB (E, $\sqcap_2$) = A$_2$, and for c $\sqcap_3$ d = 0 we have DB (E, $\sqcap_3$) = A$_3$.

There is a one-to-one correspondence between *Weak D-basic algebras* satisfying (10) (respectively, (11)) and pairs (E, $\sqcap$) where E = (E, Δ, 0, 1) is a *D-effect algebra* with an associated commutative directoid (E, $\sqcap$). Namely, the assignment

$$A \rightarrow (\mathcal{E} \ (A), \sqcap),$$

where $\mathcal{E}$ (A) is as in Theorem 3.1 and x $\sqcap$ y = ¬ (¬ x $\boxed{\Delta}$ y) $\boxed{\Delta}$ y, is a bijection the inverse of which is

$$(E, \sqcap) \rightarrow DB \ (E, \sqcap),$$

where DB (E, $\sqcap$) is defined in Theorem 3.4.

Let E = (E, Δ, 0, 1) be a D-effect algebra. When constructing (E, $\sqcap$), we so far have not taken care of existing infima we only required that x $\sqcap$ y = y $\sqcap$ x is min{x, y} provided x, y are comparable. Of course, this means that DB (E, $\sqcap$) need not be a *D-basic algebra* even though E is a *lattice D-effect algebra*. The situation can be improved if we define $\sqcap$ in such a way that the following condition holds:

$$\text{If } \inf\{x, y\} \text{ exists, then } x \sqcap y = y \sqcap x = \inf\{x, y\}. \quad (12)$$

Corollary 3.6 Let (E, $\sqcap$) be a *D-effect algebra* with an associated commutative directoid that satisfies the condition (12). Then DB (E, $\sqcap$) is a *weak D-basic algebra*, and if E is *a lattice D-effect algebra*, then DB (E, $\sqcap$) is a *D-basic algebra*.

Proof. By *Theorem 3.4*, DB (E, $\sqcap$) is a *Weak D-basic algebra*. Further, we know that DB (E, $\sqcap$) is a *D-basic algebra* if and only if the corresponding commutative directoid with sectional antitone involutions (E, $\sqcap$, (β_a)$_{a \in A}$, 0, 1), where β_a (x) = x' Δ a for x ≤ a, is actually a lattice with sectional antitone involutions, which is the case when (E, $\sqcap$) is a semilattice. Hence, if E is a *lattice D-effect algebra*, then, owing to (12), (E, $\sqcap$) is a semilattice, and it follows that DB (E, $\sqcap$) is a *D-basic algebra*.

4. Compatibility in D-Effect Algebra

Difination 4.1 We shall say that two elements x, y in a *D-effect algebra* E are said to be *compatible* (in symbols x ↔ y) if there exist u, v ∈ E such that u ≥ x, y ≥ v and x ∇ u = v ∇ y. This is equivalent to the existence of z ∈ E with x, y ≥ z, z ∇ x ≥ y and z ∇ y ≥ x. But z ∇ x ≥ y ≥ z implies z ∇ y ≥ z ∇ (z ∇ x) = x, and conversely, z ∇ y ≥ x ≥ z entails z ∇ x ≥ z ∇ (z ∇ y) = y. Therefore,

$$x \leftrightarrow y \text{ if and only if there is } z \text{ such that } x, y \geq z \text{ and } z \nabla x \geq y. \quad (13)$$

In general we have:

Proposition 4.2 Let $(E, \sqcap)$ and DB $(E, \sqcap)$ be as in *Theorem 3.4*. For every x, y $\in$ E, if x $\boxed{\Delta}$ y = y $\boxed{\Delta}$ x, then x $\leftrightarrow$ y.

Proof. Let z = x $\boxed{\Delta}$ y = y $\boxed{\Delta}$ x, i.e., $(x' \sqcap y)' \Delta y = (y' \sqcap x)' \Delta x$. Then x, y $\geq$ z and z ∇ x = $((y' \sqcap x)' \Delta x) \nabla x = (y' \sqcap x)' \geq$ y, so that x $\leftrightarrow$ y.

The reverse implication fails to be true. Let E be the *D-effect algebra* from Examples 3.3 and 3.5. It can easily be seen that every two elements are compatible, while $\boxed{\Delta}_i$ in A2 and A3 is not commutative (for instance, a $\leftrightarrow$ c, but a $\boxed{\Delta}_i$ c $\neq$ c $\boxed{\Delta}_i$ a for i = 2, 3).

In order to overcome this disadvantage, we define the 'meet-like' operation $\sqcap$ in a *D-effect algebra* E = (E, Δ, 0, 1) in the following way:

$$\text{If } x \leftrightarrow y, \text{ then } x \sqcap y = y \sqcap x = z \text{ where } z \leq x, y \text{ and } z \nabla x \geq y. \text{ At the same time, } x' \sqcap y' = y' \sqcap x' = (x \nabla (z \nabla y))'. \tag{14}$$

We must show that the condition is correct.

If z $\leq$ x, y and z ∇ x $\geq$ y, then x ∇ (z ∇ y) = y ∇ (z ∇ x). Indeed, if we put w = x ∇ (z ∇ y) then w Δ (z ∇ y) = x and w Δ (z ∇ y) Δ (z ∇ x) = x Δ (z ∇ x) = z, whence w Δ (z ∇ x) = z ∇ (z ∇ y) = y. So w = y ∇ (z ∇ x) and (x ∇ (z ∇ y))' = (y ∇ (z ∇ x))'.

Obviously, w' $\leq$ x', y'.

We have w' ∇ x' = x∇w = x∇(x∇ (z∇y)) = z∇y = y' ∇ z' $\geq$ y'. by using remark 2.4(ii)

Finally, (x' ∇ (w' ∇ y'))' = z. Indeed, (x' ∇ (w' ∇ y'))' = x Δ (w' ∇ y') = x Δ (y ∇ w), thus (x' ∇ (w' ∇ y'))' = z if and only if y ∇ w = z ∇ x, which is true since y ∇ w = y ∇ (y ∇ (z ∇ x)) = z ∇ x.

Also observe that the condition (14) is 'compatible' with (12) in the sense that we may take z = inf $\{x, y\}$ whenever inf $\{x, y\}$ exists. More precisely, if x $\sqcap$ y = inf $\{x, y\}$ and x $\leftrightarrow$ y, then (x $\sqcap$ y) ∇ x $\geq$ y, and if, in addition, also inf $\{x', y'\}$ exists, then inf $\{x', y'\}$ = x' $\sqcap$ y' = (x ∇ ((x $\sqcap$ y) ∇ y))'.

Indeed, x $\leftrightarrow$ y yields the existence of z with z $\leq$ x, y and z ∇ x $\geq$ y. Since x $\sqcap$ y = inf$\{x, y\}$, we have z $\leq$ x $\sqcap$ y $\leq$ x, whence y $\leq$ z ∇ x $\geq$ (x $\sqcap$ y) ∇ x.

Further, assume that inf$\{x', y'\}$ exists (equivalently, sup$\{x, y\}$ exists). We have to show that x ∇((x $\sqcap$ y) ∇ y) = sup$\{x, y\}$. Let w = x ∇ ((x $\sqcap$ y) ∇ y). By what we have established above we know that w = y ∇ ((x $\sqcap$ y) ∇ x) and w $\geq$ x, y. Thus w $\geq$ sup$\{x, y\}$ $\geq$ x, y, whence

$$\sup \{x, y\} \nabla w \geq x \nabla w = x \nabla (x \nabla ((x \sqcap y) \nabla y)) = (x \sqcap y) \nabla y,$$

$$\sup \{x, y\} \nabla w \geq y \nabla w = y \nabla (y \nabla ((x \sqcap y) \nabla x)) = (x \sqcap y) \nabla x.$$

It is known that sup $\{(x \sqcap y)\nabla y, (x \sqcap y)\nabla x\}$ = 1 if x $\sqcap$ y is inf$\{x, y\}$, and consequently, the above inequalities imply sup$\{x, y\}\nabla$ w = 1, so that sup$\{x, y\}$ = w as desired.

Summarizing, we have proved that in every *D-effect algebra* E = (E, Δ, 0, 1), the operation $\sqcap$ can always be defined in such a way that it obeys the requirements of the condition (14). The next result says, that x, y are compatible in E if and only if x, y commute in DB (E, $\sqcap$) = (E, $\boxed{\Delta}$, $\neg$, 0).

Theorem 4.3 Let $(E, \sqcap)$ be a *D-effect algebra* with an associated commutative directoid satisfying condition (14). Then DB $(E, \sqcap)$ is a *Weak D-basic algebra* such that, for all x, y $\in$ E, the following are equivalent:

(i) x $\leftrightarrow$ y,

(ii) (x $\sqcap$ y) ∇ y = x ∇ (x $\sqcup$ y),

(iii) x $\boxed{\Delta}$ y = y $\boxed{\Delta}$ x.

Proof. (i) $\Leftrightarrow$ (ii): Let x $\leftrightarrow$ y. Then (x $\sqcap$ y) ∇ y $\geq$ x and x $\sqcup$ y = (x' $\sqcap$ y')' = x ∇ ((x $\sqcap$ y) ∇ y), whence x ∇ (x $\sqcup$ y) = x ∇(x∇((x $\sqcap$ y)∇y)) = (x $\sqcap$ y) ∇ y. *Conversely*, if (x $\sqcap$ y) ∇ y = x ∇ (x $\sqcup$ y), then certainly x $\leftrightarrow$ y because x $\sqcap$ y $\leq$, y and (x $\sqcap$ y) ∇ y $\geq$ x.

(i) $\Leftrightarrow$ (iii): By Proposition 4.2 we know that x $\boxed{\Delta}$ y = y $\boxed{\Delta}$ x implies x $\leftrightarrow$ y. Hence, Then alsox' $\leftrightarrow$ y, which means (x' $\sqcap$ y) ∇ y = x' ∇ (x' $\sqcup$ y) by (ii). We then have (x $\boxed{\Delta}$ y)' = ((x' $\sqcap$ y)' Δ y)' = (x' $\sqcap$ y) ∇y = x' ∇(x' $\sqcup$ y) = (x' $\sqcup$ y)' ∇ x = (y' $\sqcap$ x) ∇ x = ((y' $\sqcap$ x)' Δ x)' = (y $\boxed{\Delta}$ x)', thus x $\boxed{\Delta}$ y = y$\boxed{\Delta}$ x.

Definition 4.4 By a *block* of a *Weak D-basic algebra* (A, $\boxed{\Delta}$, $\neg$, 0) we mean a subset DB of A which is maximal with respect to the property that x $\boxed{\Delta}$ y = y $\boxed{\Delta}$ x for all x, y $\in$ DB. It is evident that every element of A is contained in a block.

Theorem 4.5 Let $(E, \sqcap)$ be a *D-effect algebra* with an associated commutative directoid satisfying the condition (14). Assume that for all x, y, z $\in$ E, if x $\leftrightarrow$ y, x $\leftrightarrow$ z and y Δ z is defined, then x $\leftrightarrow$ y Δ z. Then a block DB of DB (E, $\sqcap$) is a subalgebra of DB (E, $\sqcap$) if and only if x $\sqcap$ y $\in$ DB for all x, y $\in$ DB.

Proof. Let DB be a block of DB (E, $\sqcap$). In view of Theorem 4.3, DB is a maximal set of pairwise compatible elements (i.e., DB is a maximal subset of E such that x $\leftrightarrow$ y for all x, y $\in$ DB). Since x $\leftrightarrow$ 0 and x $\leftrightarrow$ 1 for each x $\in$ E (this follows at once from (13)), it is plain that 0, 1 $\in$ DB. Also, x $\leftrightarrow$ y if and only if x' $\leftrightarrow$ y, hence x $\in$ DB if and only if $\neg$ x = x' $\in$ DB.

Suppose DB is closed under $\sqcap$. If x, y $\in$ DB, then also (x' $\sqcap$ y)' $\in$ DB. Thus (x' $\sqcap$ y)' $\leftrightarrow$ z and y $\leftrightarrow$ z for every z $\in$ DB, whence x $\boxed{\Delta}$ y = (x' $\sqcap$ y)' Δ y $\in$ DB, proving that DB is a subalgebra of DB (E, $\sqcap$). *Conversely*, if DB is a subalgebra of DB (E, $\sqcap$), then it is automatically closed with respect to $\sqcap$ because x $\sqcap$ y = $\neg$ ($\neg$ x $\boxed{\Delta}$ y) $\boxed{\Delta}$ y. The condition that x $\leftrightarrow$ y and x $\leftrightarrow$ z together yield x $\leftrightarrow$ y Δ z (if y Δ z exists) holds in *lattice D-effect algebras*, however, the next example shows that the operation Δ in *Theorem 4.5* cannot be omitted.

Example 4.6 Let E be the set consisting of the following pairs of integers: 0 = (0, 0), a = (1, 2), b = (1, 1), c = (2, 1), d = (2, 3), e = (3, 3), f = (3, 2), g = (2, 2) and 1 = (4, 4). If we equip E with Δ defined as the restriction to E of $\boxed{\Delta}$, then E = (E, Δ, 0) becomes a *D-effect algebra*. The underlying poset of E is as follows (notice that (x, y) $\geq$ (u, v) if and only if (x, y) = (u, v), or x > u & y > v) where

$(x, y) \triangle (u, v) = ((x, y)' + (u, v)')'$ and $+$ is the usual point in addition and $((x, y)' = (4 - x, 4 - y))$:
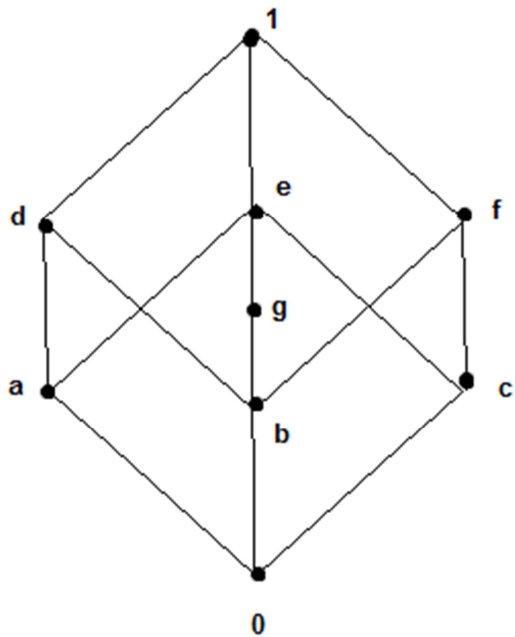

Figure 2. D-effect algebra.

Table 5. *Show that E \ {g} is a block of the assigned weak D-basic algebra DB (E, ⊓),*

△	0	A	b	C	d	e	f	g	1	¬
0	0	0	0	0	0	0	0	0	0	1
a	0	a	0	0	a	a	0	b	a	f
b	0	0	0	0	b	0	b	0	b	e
c	0	0	0	C	0	c	c	b	c	d
d	0	A	b	0	d	a	b	g	d	c
e	0	A	0	C	a	g	c	b	e	b
f	0	0	b	C	b	c	f	b	f	a
g	0	A	0	C	a	b	b	0	g	G
1	0	A	b	C	d	e	f	g	1	0

It is obvious that $a \leftrightarrow e$, but a is not compatible with $g = e \triangle e$. Indeed, the only common lower bound of a, g is 0, and $0 \nabla a = f \ngeq g$ as well as $0 \nabla g = g \ngeq a$, thus $a \leftrightarrow g$ by (13).

In accordance with the conditions (12) and (14), we put $f \sqcap e = c (= f \triangle e)$ and $e \sqcap d = a (= e \triangle d)$; in the other cases $\sqcap$ coincides with inf. A direct inspection shows that $E \setminus \{g\}$

is a block of the assigned *weak D-basic algebra* DB(E, ⊓) (see the table below (*table 5*.) which is closed under ⊓, but it is not closed under $\triangle$ as $e \Delta e = g$. On the other hand, {0, b, e, g, 1} is both a block and a subalgebra of DB (E, ⊓).

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