Explicit Equations to Transform from Cartesian to Elliptic Coordinates

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Abstract: Explicit equations are obtained to convert Cartesian coordinates to elliptic coordinates, based on which a function in elliptic coordinates can be readily mapped in physical space. Application to Kirchhoff vortex shows that its elliptical core induces two counter-rotating irrotational eddies.

Keywords: Elliptic Coordinates, Cartesian Coordinates, Kirchhoff Vortex

1. Introduction

Figure 1. Coordinate lines for an elliptic coordinate system with $a=1, b=0.5$.

The elliptical coordinate system (ξ, η) as a two-dimensional orthogonal coordinate system has many dynamical and engineering applications, such as Kirchhoff vortex [1], insect aerodynamics [2], hydrodynamic wave

diffraction [3], and theoretical physics [4]. Its coordinate lines are confocal ellipses and hyperbolae and the transformation from elliptic to Cartesian coordinates is given by

$$\begin{aligned} x &= c \cosh(\xi) \cos(\eta) \\ y &= c \sinh(\xi) \sin(\eta) \\ c^2 &= a^2 - b^2 \\ \xi &\geq 0, \quad 0 \leq \eta < 2\pi \end{aligned} \quad (1)$$

where a and b denote the semi-major and semi-minor axes of the ellipse and c is the elliptical eccentricity (Figure 1).

In the meantime, explicit equations to transform from Cartesian to elliptic coordinates have not been found in the existing literature [5, 6, 7]. Such a conversion relation would be useful in mapping an elliptic-coordinate solution, for example the flow field of Kirchhoff vortex, in physical space.

2. Cartesian to Elliptic Coordinates

In order to invert the functional relation (1), we first eliminate ξ and have

$$\frac{x^2}{\cos^2(\eta)} - \frac{y^2}{\sin^2(\eta)} = c^2$$

which means curves of constant η are hyperbolae. The focus distance is c and the eccentricity is $e = \sec(\eta)$.

Let $p = \sin^2(\eta)$, we have

$$\frac{x^2}{1-p} - \frac{y^2}{p} = c^2$$

which becomes

$$c^2 p^2 + (x^2 + y^2 - c^2)p - y^2 = 0 \quad (2)$$

Then eliminating η from (1) we have

$$\frac{x^2}{\cosh^2(\xi)} + \frac{y^2}{\sinh^2(\xi)} = c^2$$

It shows that curves of constant ξ are ellipses. The focus distance is c and the eccentricity is $e = \cosh^{-1}(\xi)$.

Let $q = -\sinh^2(\xi)$, we have

$$\frac{x^2}{1-q} - \frac{y^2}{q} = c^2$$

which leads to

$$c^2 q^2 + (x^2 + y^2 - c^2)q - y^2 = 0 \quad (3)$$

It is essentially the same as (2). Therefore (p, q) constitute the two roots of a quadratic equation. Since $0 \leq p \leq 1$, $q \leq 0$, we have $p \geq q$, and the two roots are

$$p = \frac{-B + \sqrt{B^2 + 4c^2 y^2}}{2c^2}, \quad q = \frac{-B - \sqrt{B^2 + 4c^2 y^2}}{2c^2} \quad (4)$$

in which $B = x^2 + y^2 - c^2$.

From the definition of p we obtain

$$\eta_0 = \arcsin(\sqrt{p}) \quad (5)$$

It has four cases depending on which quadrant the Cartesian point (x, y) is located, i.e.,

$$\begin{aligned} \eta &= \eta_0, & x \geq 0, y \geq 0 \\ \eta &= \pi - \eta_0, & x < 0, y \geq 0 \\ \eta &= \pi + \eta_0, & x \leq 0, y < 0 \\ \eta &= 2\pi - \eta_0, & x > 0, y < 0 \end{aligned} \quad (6)$$

Based on the definition of q , we can solve ξ from quadratic equation

$$e^{4\xi} + (4q - 2)e^{2\xi} + 1 = 0,$$

which has two roots

$$e^{2\xi} = 1 - 2q \pm 2\sqrt{q^2 - q}$$

Since $q \leq 0$, both roots are real and denoted as (ξ_1, ξ_2) .

They clearly satisfy $e^{2\xi_1} \cdot e^{2\xi_2} = 1$, which leads to $\xi_2 = -\xi_1 < 0$. Because in elliptical coordinates only non-negative ξ value is considered, we obtain

$$\xi = \frac{1}{2} \ln(1 - 2q + 2\sqrt{q^2 - q}) \quad (7)$$

Eqs. (4-7) are explicit equations to derive elliptic coordinates from Cartesian grid. They can easily be realized via computation software such as Matlab.

3. Application to Kirchhoff Vortex

Kirchhoff vortex is a rotating elliptical region of uniform vorticity ω embedded in an irrotational ideal fluid [8]. It is the simplest example of non-smooth weak solutions to the Euler equations and has wide application in vortex dynamics [9-11]. It has a discontinuity of vorticity across its elliptical boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are semi-major and semi-minor axes, respectively. The boundary corresponds to contour $\xi = \operatorname{arcsinh}(b/c)$, and the vortex rotates with constant angular velocity

$$\Omega = \frac{ab}{(a+b)^2} \omega$$

around the origin. From Act. 159 of Lamb [1], the streamfunction outside the core is

$$\psi = \frac{1}{4} \Omega (a+b)^2 e^{-2\xi} \cos(2\eta) + \frac{1}{2} \omega ab \xi \quad (8)$$

The instantaneous streamlines in a unsteady flow are given by the curves $\psi = \text{const}$. In a rotating frame with angular velocity Ω , Kirchhoff vortex looks steady and its streamfunction outside the elliptic core is related to the inertial-frame streamfunction (8) by

$$\psi_R = \psi - \frac{1}{2} \Omega [x(\xi, \eta)^2 + y(\xi, \eta)^2] \quad (9)$$

From (9) it is straightforward to make a conformal mapping of steady streamfunction on a uniform mesh of elliptic coordinates (Figure 2).

Figure 2. Mapping of streamfunction (9) on a uniform mesh of (ξ, η) , with $a=1, b=0.5, \omega=10$.

In order to view the flow field in physical space, we use the conversion equations (4-7) to plot streamfunction (9) on a uniform Cartesian mesh in Figure 3, which shows the elliptical core of Kirchhoff vortex induces two irrotational eddies that rotate in opposite directions.

Figure 3. Mapping of streamfunction (9) on a uniform mesh of xy -plane, with $a=1, b=0.5, \omega=10$. The boundary of vortex core corresponds to $\xi=0.55$.

4. Conclusion

This study obtains explicit equations that convert Cartesian coordinates to elliptic coordinates. The conversion relation can be easily realized with computer software and used to map a known function of elliptic-coordinates, such as the streamfunction of Kirchhoff vortex, on a uniform Cartesian mesh.

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