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Geometric series of numbers approximating positive integers

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Abstract: The predictability of cycles in the series of Pythagorean triples led to an investigation that yielded numbers (x) that are associated with irrational square roots ($\sqrt{n}$). The cycles recur with geometric factors (cycle factors y) that are comprised of a positive integer x where $y = x + \sqrt{x^2 \pm 1}$. On raising the cycle factors to the positive integer powers (y^m), a series is generated where each consecutive member comes closer and closer to positive integers as the series progresses. A formula associates the square root ($\sqrt{n}$) with these series. Prime factorising the positive integers in the power series (x_m) produces predictable patterns among the prime factors in the series. In general, power series that have each consecutive member in the series come closer to positive integers are limited to $(x + \sqrt{x^2 \pm r})^m$ where x and r are positive integers and $r < (x + 1)^2 - x^2$ for the $+ r$ condition and $r < x^2 - (x - 1)^2$ for the $- r$ condition.

Keywords: Power Series of Irrational Numbers, Approximating Positive Integers, Factors That Relate Perfect Squares, Prime Factor Patterns

1. Introduction

In the previous paper [1], we reported series of right-angled triangles which provide series of rational numbers that have as their limits irrational square roots. A fascinating aspect of all these series, is the constants that are generated and the predictability of these series. We were attracted to the geometric nature of the recurrence of these phenomena, which are especially noticeable at the series with many subseries. The indices [2] of the triangles, and thus the triangles, recur with geometric regularity. This paper deals with a search into the nature of the recurrence of these series in an effort to rationalise this phenomenon.

2. The Cycle Factor of $\sqrt{2}$

The place to begin this investigation is with the series of rational numbers that leads to $\sqrt{2}$. Table 1 shows the indices [2] and the sides of the series of triangles that lead to the 45° triangle. The ratio of i_{n+1}/i_n and j_{n+1}/j_n are also provided for investigation. Both these ratios are converging to the same number, which we term the cycle factor (y). As the number develops in the table the part that remains unchanged and is the same as the limit is underlined. The li-mit is irrational since it is $1 + \sqrt{2}$ [1]. These converging ratios alternate between being greater than, and less than the limit. The i- and the j-series are out of phase in their alternations.

The cycle factor puzzled us, and we tried to determine its significance. We recorded it and looked at the cycle factors of the greater roots.

3. Cycle Factors (y) of the Roots of Numbers Adjacent to Perfect Squares

There are two series that lead to $\sqrt{3}$, its e-series, where the even-numbered leg of the series of triangles is approaching the irrational leg of the limiting triangle ($\sqrt{3}$/1/2-triangle), and the u-series where the uneven numbers approach the irrational leg of the same triangle. Both these series have the same indices, the only difference being that the i-indices of the e- and u-series are dislocated by one position. The quotient of the indices will yield the same result. Just as with $\sqrt{2}$ where the cycle factor is the sum of an integer and a square root, the cycle factor for $\sqrt{3}$ is $2 + \sqrt{3}$. It turns out that the cycle factors of all the numbers just short of or just greater than a perfect square is comprised of that square root plus the square root (x) of the adjacent perfect square, irrespective of what triangle series is used to obtain the cycle factor of that square root. Table 2 contains the cycle factors of all the numbers adjacent to a perfect square up to 101.

Table 1. *The series of triangles $(i,j)_n$ that have h/e and h/u converge to $\sqrt{2}$, and the cycle factors of their indices.*

n	*i*	*j*	*u*	*e*	*h*	i_{n+1}/i_n	j_{n+1}/j_n
1	1	2	3	4	5	3	2
2	3	4	21	20	29	2.33333333333333	2.5
3	7	10	119	120	169	2.42857142857143	2.4
4	17	24	697	696	985	2.41176470588235	2.41666666666667
5	41	58	4059	4060	5741	2.41463414634146	2.41379310344828
6	99	140	23661	23660	33461	2.41414141414141	2.41428571428571
7	239	338	137903	137904	195025	2.41422594142259	2.41420118343195
8	577	816	803761	803760	1136689	2.41421143847487	2.41421568627451
9	1393	1970	4684659	4684660	6625109	2.41421392677674	2.41421319796954
10	3363	4756	27304197	27304196	38613965	2.41421349985132	2.41421362489487
11	8119	11482	159140519	159140520	225058681	2.41421357310013	2.41421355164605
12	19601	27720	927538921	927538920	1311738121	2.41421356053263	2.41421356421356
13	47321	66922	5406093003	5406093004	7645370045	2.41421356268887	2.41421356205732
14	114243	161564	31509019101	31509019100	44560482149	2.41421356231892	2.41421356242727
15	275807	390050	183648021599	183648021600	259717522849	$1+\sqrt{2}$ = 2.414213562373095048801689	

Table 2. *Cycle factors of roots of numbers adjacent to perfect squares.*

Root	Cycle factor		Root	Cycle factor	
	radical	decimal		radical	decimal
$\sqrt{2}$	$1+\sqrt{2}$	2.41421356237309504880	$\sqrt{37}$	$6+\sqrt{37}$	12.0827625302982196890
$\sqrt{3}$	$2+\sqrt{3}$	3.73205080756887729353	$\sqrt{48}$	$7+\sqrt{48}$	13.9282032302755091741
$\sqrt{5}$	$2+\sqrt{5}$	4.23606797749978969641	$\sqrt{50}$	$7+\sqrt{50}$	14.0710678118654752440
$\sqrt{8}$	$3+\sqrt{8}$	5.82842712474619009760	$\sqrt{63}$	$8+\sqrt{63}$	15.9372539331937717715
$\sqrt{10}$	$3+\sqrt{10}$	6.16227766016837933200	$\sqrt{65}$	$8+\sqrt{65}$	16.0622577482985496524
$\sqrt{15}$	$4+\sqrt{15}$	7.87298334620741688517	$\sqrt{80}$	$9+\sqrt{80}$	17.9442719099991587856
$\sqrt{17}$	$4+\sqrt{17}$	8.12310562561766054982	$\sqrt{82}$	$9+\sqrt{82}$	18.0553851381374166266
$\sqrt{24}$	$5+\sqrt{24}$	9.89897948556635619639	$\sqrt{99}$	$10+\sqrt{99}$	19.9498743710661995473
$\sqrt{26}$	$5+\sqrt{26}$	10.0990195135927848300	$\sqrt{101}$	$10+\sqrt{101}$	20.0498756211208902702
$\sqrt{35}$	$6+\sqrt{35}$	11.9160797830996160426			

4. All Other Cycle Factors (y)

Having established the cycle factors of the roots of the numbers one greater or one less than a perfect square, is it possible to determine the cycle factors of all the other numbers? An investigation into the cycle factors in table 2 shows that the numbers are close to an even positive integer, because one component is an integer and the other the square root of a value very close to the square of that same integer. Studying the approximate cycle factors of the roots of numbers not adjacent to a perfect square as determined by the quotient of indeces as done previously for $\sqrt{2}$, we find that these cycle factors are also close to even integers. Taking the cue, we find that the approximate cycle factors are very close to the sum of the square root of a perfect square and the square root of the number just larger or just smaller than that perfect square (table 3), which turns out to be the actual cycle factors.

Table 3. *The exact and calculated cycle factors of the roots less than the square root of 102.*

Root	Cycle factor			Root	Cycle factor		
	radical	decimal			radical	decimal	
		exact	$index_n/index_{n-1}$			exact	$index_n/index_{n-1}$
$\sqrt{2}$	$1+\sqrt{2}$	2.41421356237	2.41421356243	$\sqrt{53}$	$182+\sqrt{33125}$	364.002747232	364.002747228
$\sqrt{3}$	$2+\sqrt{3}$	3.73205080757	3.73205080758	$\sqrt{55}$	$89+\sqrt{7920}$	177.994381845	177.994381613
$\sqrt{5}$	$2+\sqrt{5}$	4.23606797750	4.23606797748	$\sqrt{56}$	$15+\sqrt{224}$	29.9666295471	29.9666294756
$\sqrt{7}$	$8+\sqrt{63}$	15.9372539332	15.9372539327	$\sqrt{57}$	$151+\sqrt{22800}$	301.996688705	301.997183099
$\sqrt{8}$	$3+\sqrt{8}$	5.82842712475	5.82842712464	$\sqrt{58}$	$19603+\sqrt{384277608}$	39205.9999745	39205.9999743
$\sqrt{10}$	$3+\sqrt{10}$	6.16227766017	6.16227766011	$\sqrt{59}$	$530+\sqrt{280899}$	1059.99905660	1059.99920723
$\sqrt{11}$	$10+\sqrt{99}$	19.9498743711	19.9498743711	$\sqrt{61}$	$29718+\sqrt{883159525}$	59436.0000168	59436.0000140
$\sqrt{13}$	$18+\sqrt{325}$	36.0277563773	36.0277563765	$\sqrt{63}$	$8+\sqrt{63}$	15.9372539332	15.9372539332
$\sqrt{15}$	$4+\sqrt{15}$	7.87298334621	7.87298334438	$\sqrt{65}$	$8+\sqrt{65}$	16.0622577483	16.0622577483
$\sqrt{17}$	$4+\sqrt{17}$	8.12310562562	8.12310562666	$\sqrt{67}$	$48842+\sqrt{2385540963}$	97683.9999898	97683.9999897
$\sqrt{19}$	$170+\sqrt{28899}$	339.997058798	339.997059744	$\sqrt{69}$	$7775+\sqrt{60450624}$	15549.9999357	15549.9999347
$\sqrt{21}$	$55+\sqrt{3024}$	109.990908339	109.990908288	$\sqrt{71}$	$3480+\sqrt{12110399}$	6959.99985632	6959.99985553
$\sqrt{23}$	$24+\sqrt{575}$	47.9791576166	47.9791576106	$\sqrt{72}$	$17+\sqrt{288}$	33.9705627485	33.9705627484
$\sqrt{24}$	$5+\sqrt{24}$	9.89897948557	9.89897948448	$\sqrt{73}$	$1068+\sqrt{1140625}$	2136.00046816	2136.00054259
$\sqrt{26}$	$5+\sqrt{26}$	10.0990195136	10.0990195127	$\sqrt{74}$	$3699+\sqrt{13682600}$	7397.99986483	7397.99983509
$\sqrt{27}$	$26+\sqrt{675}$	51.9807621135	51.9807621096	$\sqrt{75}$	$26+\sqrt{675}$	51.9807621135	51.9807621136
$\sqrt{29}$	$70+\sqrt{4901}$	140.007142493	140.007142389	$\sqrt{77}$	$351+\sqrt{123200}$	701.998575496	701.998604002
$\sqrt{31}$	$1520+\sqrt{2310399}$	3039.99967105	3039.99919808	$\sqrt{79}$	$80+\sqrt{6399}$	159.993749756	159.993749763
$\sqrt{32}$	$17+\sqrt{288}$	33.9705627485	33.9705627485	$\sqrt{80}$	$9+\sqrt{80}$	17.9442719100	17.9442719100
$\sqrt{33}$	$23+\sqrt{528}$	45.9782505862	45.9782505792	$\sqrt{82}$	$9+\sqrt{82}$	18.0553851381	18.0553851397
$\sqrt{34}$	$35+\sqrt{1224}$	69.9857113691	69.9857113487	$\sqrt{83}$	$82+\sqrt{6723}$	163.993902212	163.993902206
$\sqrt{35}$	$6+\sqrt{35}$	11.9160797831	11.9160797828	$\sqrt{85}$	$378+\sqrt{142885}$	756.001322749	756.001347388
$\sqrt{37}$	$6+\sqrt{37}$	12.0827625303	12.0827625306	$\sqrt{87}$	$28+\sqrt{783}$	55.9821371593	55.9821370945
$\sqrt{39}$	$25+\sqrt{624}$	49.9799919936	49.9799919892	$\sqrt{88}$	$197+\sqrt{38808}$	393.997461913	393.997542998
$\sqrt{40}$	$19+\sqrt{360}$	37.9736659610	37.9736659366	$\sqrt{89}$	$500+\sqrt{250001}$	1000.00100000	1000.00100431
$\sqrt{41}$	$32+\sqrt{1025}$	64.0156211872	64.0156211884	$\sqrt{91}$	$1574+\sqrt{2477475}$	3147.99968234	3147.99968315
$\sqrt{43}$	$3482+\sqrt{12124323}$	6963.99985640	6963.99966113	$\sqrt{93}$	$12151+\sqrt{147646800}$	24301.9999589	24301.9999592
$\sqrt{45}$	$161+\sqrt{25920}$	321.996894380	321.997128087	$\sqrt{95}$	$39+\sqrt{1520}$	77.9871773792	77.9871774124
$\sqrt{47}$	$48+\sqrt{2303}$	95.9895822028	95.9895806861	$\sqrt{96}$	$49+\sqrt{2400}$	97.9897948557	97.9897948555
$\sqrt{48}$	$7+\sqrt{48}$	13.9282032303	13.9282032302	$\sqrt{97}$	$5604+\sqrt{31404817}$	11208.0000892	11208.0000872
$\sqrt{50}$	$7+\sqrt{50}$	14.0710678119	14.0710678119	$\sqrt{99}$	$10+\sqrt{99}$	19.9498743711	19.9498743607
$\sqrt{51}$	$50+\sqrt{2499}$	99.9899989998	99.9899976739	$\sqrt{101}$	$10+\sqrt{101}$	20.0498756211	20.0498756294

Glancing through the data, several aspects are easily noticed.

- The cycle factors seem to be quite random. The larger the cycle factor, the closer it is to a positive integer. The largest cycle factor is that of $\sqrt{67}$ and its decimals begin with four 9s.
- Most of them are just less than a positive integer, but a few are just larger than a positive integer.
- The powers of ten seem to be present as cycle factors, at least the first three. $\sqrt{24}$ and $\sqrt{26}$ are just below and above 10, respectively, $\sqrt{51}$ is just below 100, and $\sqrt{89}$ is just above 1000.
- In odd places, cycle factors are the same, such as for $\sqrt{7}$ and $\sqrt{63}$, which both have a cycle factor of 15.937253933. What do 7 and 63 have in common? $7 \times 9 = 63$. Likewise $\sqrt{11}$ and $\sqrt{99}$ have the same cycle factors. Once again 11 is a factor of 99. 63 and 99 are parts of the $9 \times$ table.

- Some of the index-derived approximations to the cycle factors are exactly the same as the exact cycle factors up to the 12 significant figures provided in the table, eg $\sqrt{32}$ and $\sqrt{50}$. On the other hand, some of the greater cycle factors vary at the last 7 of the 12 significant figures, because of limited indeces eg $\sqrt{45}$ and $\sqrt{57}$.

If table 2 were to be extended, all of the cycle factors of the square roots of numbers not adjacent to perfect squares in table 3 would be incorporated. We see therefore that the cycle factors of the square roots of numbers not adjacent to perfect squares make “premature” use of the cycle factors of square roots of numbers adjacent to perfect squares. What do they have in common? At the fourth bullet above we observed that numbers whose square roots share a cycle factor have the smaller number a factor of the larger. Let us look at a few more and observe this phenomenon closer. For $\sqrt{21}$ the cycle factor is $55 + \sqrt{3024}$. 21 is a factor of 3024. For $\sqrt{69}$ the cycle factor is $7775 + \sqrt{60450624}$. 69 is a factor of 60450624. A look at the complementing factor reveals an important fact. For 7 and 63, whose roots share a cycle factor, the complementing factor is 9, likewise for 11 and 99 (bullet four above). For 21 and 3024 the complementing factor is 144 and for 69 and 60450624 the complementing factor is 876096. All three these complementing factors are perfect squares (9 – 3, 144 – 12 and 876096 – 936). We therefore have a relationship $n \times z^2 = t$, where t is the number adjacent to a perfect square, n is a smaller number whose root shares the cycle factor with the root of t, and z^2 is the complementing factor which is a perfect square.

5. General validity of the equation $n \times z^2 \pm 1 = x^2$

The relationship $n \times z^2 = t$ was discovered and developed from the concept of a cycle factor of Pythagorean triples that represent a series of triangles that approximate a right-angled triangle with one irrational side. Not all the numbers are represented in this system of cycle factors, but with the development of the algebraic relationship that links these cycle factors, could the algebraic relationship be extended to all numbers? Does the relationship where a perfect square times a number n equal a number adjacent to another perfect square answer to all values of n, *ie* $x^2 = n \times z^2 \pm 1$? Our first impulse was to systematically run through all the perfect squares, looking at the numbers 1 less, and 1 greater, factorising these numbers, looking for those that may have factors that can be described by $n \times z^2$. We soon realised that that would be a monstrous task, and turned it round the other way. We systematically searched for each number, starting with n = 2 to 99, all the solutions where $n \times z^2 \pm 1$ is a perfect square where z is any number from 2 – 300 000. The results are reported in table 4 in the form of ordered pairs in which the first number represents z and the second number represents $x = \sqrt{n \times z^2 \pm 1}$. The ordered pairs which are bold are the ones that relate to cycle factors presented in table 3. With a little consideration, it is clear that n may not be a perfect square, since the product of two perfect squares is also a perfect square, and adding or subtracting 1 to a perfect square will not yield a perfect square, since for positive integers no two perfect squares are adjacent to each other. The table thus contains all the values less than 100 that are not perfect squares. The solutions where 1 is added is indicated by a + before the row, and - where 1 is subtracted.

Table 4. *All the solutions for n = 2 – 99 and z = 2 – 300 000 where a perfect square is 1 more or less than a smaller perfect square times the factor n.*

n	±1	$(z, \sqrt{n \times z^2 \pm 1})$
2	+	(2,3); (12,17); (70,99); (408,577); (2378,3363); (13860,19601); (80782,114243)
	-	(5,7); (29,41); (169,239); (985,1393); (5741,8119); (33461,47321); (195025,275807)
3	+	(4,7); (15,26); (56,97); (209,362); (780,1351); (2911,5042); (10864,18817); (40545,70226); (151316,262087)
5	+	(4,9); (72,161); (1292,2889); (23184,51841)
	-	(17,38); (305,682); (5473,2889); (98209,219602)
6	+	(2,5); (20,49); (198,485); (1960,4801); (19402,47525); (192060,470449)
7	+	(3,8); (48,127); (765,2024); (12192,32257); (194307,514088)
8	+	(6,17); (35,99); (204,577); (1189,3363); (6930,19601); (40391,114243); (235416,665857)
10	+	(6,19); (228,721); (8658,27379)
	-	(37,117); (1405,4443); (53353,168717)
11	+	(3,10); (60,199); (1197,3970); (23880,79201)
12	+	(2,7); (28,97); (390,1351); (5432,18817); (75658,262087)
13	+	(180,649); (233640,842401)
	-	(5,18); (6485,23382)
14	+	(4,15); (120,449); (3596,13455); (107760,403201)
15	+	(8,31); (63,127); (496,1921); (3905,15124); (30744,119071)
17	+	(8,33); (528,2177); (34840,143649)
	-	(65,268); (4289,17684); (283009,1166876)
18	+	(4,17); (136,577); (4620,19601); (156944,665857)
19	+	(39,170); (13260,57799)
20	+	(2,9); (36,161); (646,2889); (11592,51841)
21	+	(12,55); (1320,6049); (145188,665335)
22	+	(42,197); (16548,77617)
23	+	(5,24); (240,1151); (11515,55224)
24	+	(10,49); (99,485); (980,4801); (9701,47525); (96030,470449)
26	+	(10,51); (1020,5201); (104030,530451)
	-	(101,515); (10301,52525)
27	+	(5,26); (260,1351); (13515,70226)
28	+	(24,127); (6096,32257)
29	+	(1820,9801)
	-	(13,70); (254813,1372210)

30	+	(2,11); (44,241); (966,5291); (21208,116161)
31	+	(273,1520)
32	+	(3,17); (102,577); (3465,19601); (117708,665857)
33	+	(4,23); (184,1057); (8460,48599)
34	+	(6,35); (420,2449); (29394,171395)
35	+	(12,71); (143,846); (1704,10081); (20305,120126); (241956,1431431)
37	+	(12,73); (1752,10657); (255780,1555849)
	-	(145,882); (21169,128766)
38	+	(6,37); (444,2737); (32850,202501)
39	+	(4,25); (200,1249); (9996,62425)
40	+	(3,19); (114,721); (4329,27379); (164388,1039681)
41	+	(320,2049)
	-	(5,32); (20485,131168)
42	+	(2,13); (52,337); (1350,8749); (35048,227137)
43	+	(531,3482)
44	+	(30,199); (11940,79201)
45	+	(24,161); (7728,51841)
46	+	(3588,24335)
47	+	(7,48); (672,4607); (64505,442224)
48	+	(14,97); (195,1351); (2716,18817); (37829,262087)
50	+	(14,99); (2772,19601)
	-	(197,1393); (39005,275807)
51	+	(7,50); (700,4999); (69993,499850)
52	+	(90,649); (116820,842401)
53	+	(9100,66249)
	-	(25,182)
54	+	(66,485); (64020,470449)
55	+	(12,89); (2136,15841)
56	+	(2,15); (60,449); (1798,13455); (53880,403201)
57	+	(20,151); (6040,45601)
58	+	(2574,19603)
	-	(13,99)
59	+	(69,530); (73140,561799)
60	+	(4,31); (248,1921); (15372,119071)
61	-	(3805,29718)
62	+	(8,63); (1008,7937); (127000,999999)
63	+	(16,127); (255,2024); (4064,32257); (64769,514088)
65	+	(16,129); (4128,33281)
	-	(257,2072); (66305,534568)
66	+	(8,65); (1040,8449); (135192,1098305)
67	+	(5967,48842)
68	+	(4,33); (264,2177); (17420,143649)
69	+	(936,7775)
70	+	(30,251); (15060,126001)
71	+	(413,3480)
72	+	(2,17); (68,577); (2310,19601); (78472,665857)
73	+	(267000,2281249)
	-	(125,1068)
74	+	(430,3699)
	-	(5,43); (36985,318157)
75	+	(3,26); (156,1351); (8109,70226)
76	+	(6630,57799)
77	+	(40,351); (28080,246401)
78	+	(6,53); (636,5617); (67410,595349)
79	+	(9,80); (1440,12799); (230391,2047760)
80	+	(18,161); (323,2889); (5796,51841); (104005,930249)
82	+	(18,163); (5868,53137)
	-	(325,2943); (105949,959409)
83	+	(9,82); (1476,13447); (242055,2205226)
84	+	(6,55); (660,6049); (72594,665335)
85	+	(30996,285769)
	-	(41,378)
86	+	(1122,10405)
87	+	(3,28); (168,1567); (9405,87724)
88	+	(21,197); (8274,77617)
89	+	(53000,500001)
	-	(53,500)
90	+	(2,19); (76,721); (2886,27379); (109592,1039681)
91	+	(165,1574)
92	+	(120,1151); (276240,2649601)
93	+	(1260,12151)
94	+	(221064,2143295)
95	+	(4,39); (312,3041); (24332,237159)
96	+	(5,49); (490,4801); (48015,470449)
97	-	(569,5604)
98	+	(10,99); (1980,19601)
99	+	(20,199); (399,3970); (7960,79201); (158801,1580050)

Interesting consistencies that arise in table 4 between n and the calculated second number ($\sqrt{n \times z^2 \pm 1} = x$) in the ordered pair of the first solutions for n are.

- For n just below each perfect square, $x = 2n + 1$, and for n just above each perfect square, $x = 2n - 1$.
- The first number in the solution (z), both just greater than and just less than a perfect square, is two times the root of the perfect square.
- For the numbers n, 2 less than a perfect square, $x = n + 1$, and for numbers n, 2 greater than a perfect square, $x = n – 1$.
- Another progression is (see diagram below). for n =

2 (2 less than the perfect square 4), x = 3. Equidistant above the perfect square 4 (n = 6), x is 2 more, 5. But n = 6 is three less than perfect square 9 (x = 5), so three greater than perfect square 9 (n = 12), x = 7. 12, in turn, is four less than perfect square 16 (x = 7), so for n = 20 (four greater than perfect square 16), x = 9. Thus we see that for n = 30, x = 11, and for n = 42, x = 13, and for n = 56, x = 15, and for n = 72, x = 17, and for n = 90, x = 19.

diff	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	
n	2	**4**	6	**9**	12	**16**	20	**25**	30	**36**	42	**49**	56	**64**	72	**81**	90	**100**
x	3		5		7		9		11		13		15		17		19	

- For n = four less than an even perfect square, $x = n/2 + 1$, and for n = four greater than an even perfect square, $x = n/2 – 1$.
- Beginning with perfect square 36, for n = six less than a perfect square that is a multiple of 9, $x = n/3 + 1$, and for n = six greater than a perfect square that is a multiple of 9, $x = n/3 – 1$.
- Similarly, starting with perfect square 64, for n = eight less than a perfect square that is a multiple of 16, $x = n/4 + 1$, and for n = eight greater than a perfect square that is a multiple of 16, $x = n/4 – 1$.
- This will probably also be true for n = 10 less and greater than perfect squares that are multiples of 25, starting from 100, since for n = 90, x = 90/5 + 1 = 19.

Revisiting all the cycle factors (y) of roots under 102, we find the complementing perfect square factors for them all in table 5. Note that by default even the numbers adjacent to a perfect square comply with this formula since the complementing factor is 1, which is also a perfect square (marked in yellow). The "cycle factors" derived from table 4 without the help of a series of triangles is highlighted in red.

Table 5. *Relationship of numbers adjacent to perfect squares to numbers not adjacent to perfect squares whose roots share the same cycle factors or their powers.*

Rt	Cycle factor	$n\times z^2$	Rt	Cycle factor	$n\times z^2$	Rt	Cycle factor	$n\times z^2$
$\sqrt{2}$	$1+\sqrt{2}$	2×1^2	$\sqrt{38}$	$37+\sqrt{1368}$	38×6^2	$\sqrt{71}$	$3480+\sqrt{12110399}$	71×413^2
$\sqrt{3}$	$2+\sqrt{3}$	3×1^2	$\sqrt{39}$	$25+\sqrt{624}$	39×4^2	$\sqrt{72}$	$17+\sqrt{288}$	72×2^2
$\sqrt{5}$	$2+\sqrt{5}$	5×1^2	$\sqrt{40}$	$19+\sqrt{360}$	40×3^2	$\sqrt{73}$	$1068+\sqrt{1140625}$	73×125^2
$\sqrt{6}$	$5+\sqrt{24}$	6×2^2	$\sqrt{41}$	$32+\sqrt{1025}$	41×5^2	$\sqrt{74}$	$3699+\sqrt{13682600}$	74×430^2
$\sqrt{7}$	$8+\sqrt{63}$	7×3^2	$\sqrt{42}$	$13+\sqrt{168}$	42×2^2	$\sqrt{75}$	$26+\sqrt{675}$	75×3^2
$\sqrt{8}$	$3+\sqrt{8}$	8×1^2	$\sqrt{43}$	$3482+\sqrt{12124323}$	43×531^2	$\sqrt{76}$	$57799+\sqrt{3340724400}$	76×6630^2
$\sqrt{10}$	$3+\sqrt{10}$	10×1^2	$\sqrt{44}$	$199+\sqrt{39600}$	44×30^2	$\sqrt{77}$	$351+\sqrt{123200}$	77×400^2
$\sqrt{11}$	$10+\sqrt{99}$	11×3^2	$\sqrt{45}$	$161+\sqrt{25920}$	45×24^2	$\sqrt{78}$	$53+\sqrt{2808}$	78×6^2
$\sqrt{12}$	$7+\sqrt{48}$	12×2^2	$\sqrt{46}$	$24335+\sqrt{592192224}$	46×3588^2	$\sqrt{79}$	$80+\sqrt{6399}$	79×9^2
$\sqrt{13}$	$18+\sqrt{325}$	13×5^2	$\sqrt{47}$	$48+\sqrt{2303}$	47×7^2	$\sqrt{80}$	$9+\sqrt{80}$	80×1^2
$\sqrt{14}$	$15+\sqrt{224}$	14×4^2	$\sqrt{48}$	$7+\sqrt{48}$	48×1^2	$\sqrt{82}$	$9+\sqrt{82}$	82×1^2
$\sqrt{15}$	$4+\sqrt{15}$	15×1^2	$\sqrt{50}$	$7+\sqrt{50}$	50×1^2	$\sqrt{83}$	$82+\sqrt{6723}$	83×9^2
$\sqrt{17}$	$4+\sqrt{17}$	17×1^2	$\sqrt{51}$	$50+\sqrt{2499}$	51×7^2	$\sqrt{84}$	$55+\sqrt{3024}$	84×6^2
$\sqrt{18}$	$17+\sqrt{288}$	18×4^2	$\sqrt{52}$	$649+\sqrt{421200}$	52×90^2	$\sqrt{85}$	$378+\sqrt{142885}$	85×41^2
$\sqrt{19}$	$170+\sqrt{28899}$	19×39^2	$\sqrt{53}$	$182+\sqrt{33125}$	53×25^2	$\sqrt{86}$	$10405+\sqrt{108264024}$	86×1122^2
$\sqrt{20}$	$9+\sqrt{80}$	20×2^2	$\sqrt{54}$	$485+\sqrt{235224}$	54×66^2	$\sqrt{87}$	$28+\sqrt{783}$	87×3^2
$\sqrt{21}$	$55+\sqrt{3024}$	21×12^2	$\sqrt{55}$	$89+\sqrt{7920}$	55×12^2	$\sqrt{88}$	$197+\sqrt{38808}$	88×21^2
$\sqrt{22}$	$197+\sqrt{38808}$	22×42^2	$\sqrt{56}$	$15+\sqrt{224}$	56×2^2	$\sqrt{89}$	$500+\sqrt{250001}$	89×53^2
$\sqrt{23}$	$24+\sqrt{575}$	23×5^2	$\sqrt{57}$	$151+\sqrt{22800}$	57×20^2	$\sqrt{90}$	$19+\sqrt{360}$	90×2^2
$\sqrt{24}$	$5+\sqrt{24}$	24×1^2	$\sqrt{58}$	$19603+\sqrt{384277608}$	58×2574^2	$\sqrt{91}$	$1574+\sqrt{2477475}$	91×165^2
$\sqrt{26}$	$5+\sqrt{26}$	26×1^2	$\sqrt{59}$	$530+\sqrt{280899}$	59×69^2	$\sqrt{92}$	$1151+\sqrt{1324800}$	92×1260^2
$\sqrt{27}$	$26+\sqrt{675}$	27×5^2	$\sqrt{60}$	$31+\sqrt{960}$	60×4^2	$\sqrt{93}$	$12151+\sqrt{147646800}$	93×1260^2
$\sqrt{28}$	$127+\sqrt{16128}$	28×24^2	$\sqrt{61}$	$29718+\sqrt{883159525}$	61×3805^2	$\sqrt{94}$	$2143295+\sqrt{4593713457024}$	94×221064^2
$\sqrt{29}$	$70+\sqrt{4901}$	29×13^2	$\sqrt{62}$	$63+\sqrt{3968}$	62×8^2			
$\sqrt{30}$	$11+\sqrt{120}$	30×2^2	$\sqrt{63}$	$8+\sqrt{63}$	63×1^2	$\sqrt{95}$	$39+\sqrt{1520}$	95×4^2
$\sqrt{31}$	$1520+\sqrt{2310399}$	31×273^2	$\sqrt{65}$	$8+\sqrt{65}$	65×1^2	$\sqrt{96}$	$49+\sqrt{2400}$	96×5^2
$\sqrt{32}$	$17+\sqrt{288}$	32×3^2	$\sqrt{66}$	$65+\sqrt{4224}$	66×8^2	$\sqrt{97}$	$5604+\sqrt{31404817}$	97×569^2
$\sqrt{33}$	$23+\sqrt{528}$	33×4^2	$\sqrt{67}$	$48842+\sqrt{2385540963}$	67×5967^2	$\sqrt{98}$	$99+\sqrt{9800}$	98×10^2
$\sqrt{34}$	$35+\sqrt{1224}$	34×6^2	$\sqrt{68}$	$33+\sqrt{1088}$	68×4^2	$\sqrt{99}$	$10+\sqrt{99}$	99×1^2
$\sqrt{35}$	$6+\sqrt{35}$	35×1^2	$\sqrt{69}$	$7775+\sqrt{60450624}$	69×936^2	$\sqrt{101}$	$10+\sqrt{101}$	101×1^2
$\sqrt{37}$	$6+\sqrt{37}$	37×1^2	$\sqrt{70}$	$251+\sqrt{63000}$	70×30^2			

- An interesting observation is that the roots of the numbers that are 2 greater or less than a perfect square (excepting for 2) have as their z^2 factors the perfect square. For example, 79×9^2 and 83×9^2 respectively less than and greater than $9^2 = 81$.
- All values for n below 102 have solutions, even though some, such as $\sqrt{94}$, had to go a very long way before a solution was found.

6. Powers of Cycle Factors (y^m)

It was then found that raising the cycle factors in their powers (Table 6) produced interesting results. In the table, where a number exceeds 10^{13}, the cells are left blank. On the left of the table, the square roots whose cycle factors begin with a number just greater than an integer are highlighted in yellow (The rest begin with cycle factors that are just less than an integer.). The square roots whose cycle factors were not derived from series of triangles are highlighted in green. All the powers of cycle factors whose decimals when rounded off do not show whether the number is greater than or less than an integer are highlighted in yellow. All the other highlighting colours associate cycle factors or their powers with those of other square roots. Green, for instance at $\sqrt{2}$ shows powers of cycle factors that correlate with cycle factors at $\sqrt{8}$, and many other square roots in the table.

Table 6. *Powers of the cycle factors.*

Rt	Cycle factor (y)	y^2	y^3	y^4	y^5	y^6	y^7
$\sqrt{2}$	2.41421356237	5.82842712475	14.0710678119	33.9705627485	82.0121933088	197.994949366	478.002092041
$\sqrt{3}$	3.73205080757	13.9282032303	51.9807621135	193.994845224	723.998618782	2701.99962990	10083.9999008
$\sqrt{5}$	4.23606797750	17.9442719100	76.0131556175	321.996894380	1364.00073314	5777.99982693	24476.0000409
$\sqrt{6}$	9.89897948557	97.9897948557	969.998969071	9601.99989586	95049.9999895	940897.999999	9313930.00000
$\sqrt{7}$	15.9372539332	253.996062931	4047.99975296	64513.9999845	1028176.00000	16386302.0000	261152656.000
$\sqrt{8}$	5.82842712475	33.9705627485	197.994949366	1153.99913345	6725.99985132	39201.9999745	228485.999996
$\sqrt{10}$	6.16227766017	37.9736659610	234.004273426	1441.99930652	8886.00011254	54757.9999817	337434.000003
$\sqrt{11}$	19.9498743711	397.997487421	7939.99987406	158401.999994	3160100.00000	63043598.0000	1257711860.00
$\sqrt{12}$	13.9282032303	193.994845224	2701.99962990	37633.9999734	524173.999998	7300802.00000	101687054.000
$\sqrt{13}$	36.0277563773	1297.99922958	46764.0000214	1684802.00000	60699636.0000	2186871698.00	78788080764.0
$\sqrt{14}$	29.9666295471	897.998886413	26909.9999628	806401.999999	24165150.0000	724148098.000	21700277790.0
$\sqrt{15}$	7.87298334621	61.9838667697	487.997950811	3841.99973972	30247.9999669	238141.999996	1874888.00000
$\sqrt{17}$	8.12310562562	65.9848450049	536.001865665	4353.99977033	35368.0000283	287297.999997	2333752.00000
$\sqrt{18}$	33.9705627485	1153.99913345	39201.9999745	1331714.00000	45239074.0000	1536796802.00	52205852194.0
$\sqrt{19}$	339.997058798	115597.999991	39302980.0000	13362897602.0			
$\sqrt{20}$	17.9442719100	321.996894380	5777.99982693	103681.999990	1860498.00000	33385282.0000	599074578.000
$\sqrt{21}$	109.990908339	12097.9999173	1330670.00000	146361602.000	16098445550.0		
$\sqrt{22}$	393.997461913	155233.999994	61161802.0000	24097594754.0			
$\sqrt{23}$	47.9791576166	2301.99956560	110447.999991	5299202.00000	254251248.000	12198760702.0	585286262448.
$\sqrt{24}$	9.89897948557	97.9897948557	969.998969071	9601.99989586	95049.9999895	940897.999999	9313930.00000
$\sqrt{26}$	10.0990195136	101.990195136	1030.00097087	10401.9999039	105050.000010	1060902.00000	10714070.0000
$\sqrt{27}$	51.9807621135	2701.99962990	140451.999993	7300802.00000	379501252.000	19726764302.0	
$\sqrt{28}$	253.996062931	64513.9999845	16386302.0000	4162056194.00			
$\sqrt{29}$	140.007142493	19601.9999490	2744420.00000	384238402.000	53796120700.0		
$\sqrt{30}$	21.9544511501	481.997925302	10581.9999055	232321.999996	5100502.00000	111978722.000	2458431382.00
$\sqrt{31}$	3039.99967105	9241598.00000	28094454880.0				
$\sqrt{32}$	33.9705627485	1153.99913345	39201.9999745	1331714.00000	45239074.0000	1536796802.00	52205852194.0
$\sqrt{33}$	45.9782505862	2113.99952696	97197.9999897	4468994.00000	205476526.000	9447451202.00	434377278766.
$\sqrt{34}$	69.9857113691	4897.99979584	342789.999997	23990402.0000	1678985350.00	117504984098.	
$\sqrt{35}$	11.9160797831	141.992957397	1691.99940898	20161.9999504	240251.999996	2862862.00000	34114092.0000
$\sqrt{37}$	12.0827625303	145.993150364	1764.00056689	21313.9999531	257532.000004	3111698.00000	37597908.0000
$\sqrt{38}$	73.9864840178	5473.99981732	405001.999998	29964674.0000	2216980874.00	164026620002.	
$\sqrt{39}$	49.9799919936	2497.99959968	124849.999992	6240002.00000	311875250.000	15587522498.0	779064249650.
$\sqrt{40}$	37.9736659610	1441.99930652	54757.9999817	2079362.00000	78960998.0000	2998438562.00	113861704358.
$\sqrt{41}$	64.0156211872	4097.99975598	262336.000004	16793602.0000	1075052864.00	68820176898.0	
$\sqrt{42}$	25.9614813968	673.998516317	17497.9999429	454273.999998	11793626.0000	306180002.000	7948886426.00
$\sqrt{43}$	6963.99985640	48497294.0000	337735148452.				
$\sqrt{44}$	397.997487421	158401.999994	63043598.0000	25091193602.0			

√45	321.996894380	103681.999990	33385282.0000	10749957122.0			
√46	48669.9999795	2368768898.00					
√47	95.9895822028	9213.99989147	884447.999999	84897794.0000	8149303776.00	782248264702.	
√48	13.9282032303	193.994845224	2701.99962990	37633.9999734	524173.999998	7300802.00000	101687054.000
√50	14.0710678119	197.994949366	2786.00035894	39201.9999745	551614.000002	7761798.00000	109216786.000
√51	99.9899989998	9997.99989998	999699.999999	99960002.0000	9995000500.00	999400089998.	
√52	1297.99922958	1684802.00000	2186871698.00				
√53	364.002747232	132497.999992	48229636.0000	17555720002.0			
√54	969.998969071	940897.999999	912670090.000	885289046402.			
√55	177.994381845	31681.9999684	5639218.00000	1003749122.00	178661704498.		
√56	29.9666295471	897.998886413	26909.9999628	806401.999999	24165150.0000	724148098.000	21700277790.0
√57	301.996688705	91201.9999890	27542702.0000	8317804802.00			
√58	39205.9999745	1537110434.00					
√59	1059.99905660	1123598.00000	1191012820.00				
√60	61.9838667697	3841.99973972	238141.999996	14760962.0000	914941502.000	56711612162.0	
√61	59436.0000168	3532638098.00					
√62	125.992062992	15873.9999370	1999998.00000	251983874.000	31747968126.0		
√63	15.9372539332	253.996062931	4047.99975296	64513.9999845	1028176.00000	16386302.0000	261152656.000
√65	16.0622577483	257.996123973	4144.00024131	66561.9999850	1069136.00000	17172738.0000	275832944.000
√66	129.992307237	16897.9999408	2196610.00000	285542402.000	37118315650.0		
√67	97683.9999898	9542163854.00					
√68	65.9848450049	4353.99977033	287297.999997	18957314.0000	1250895426.00	82540140802.0	
√69	15549.9999357	241802498.000					
√70	501.998007960	252001.999996	126504502.000	63505008002.0			
√71	6959.99985632	48441598.0000	337153515120.				
√72	33.9705627485	1153.99913345	39201.9999745	1331714.00000	45239074.0000	1536796802.00	52205852194.0
√73	2136.00046816	4562498.00000	9745497864.00				
√74	7397.99986483	54730402.0000	404895506598.				
√75	51.9807621135	2701.99962990	140451.999993	7300802.00000	379501252.000	19726764302.0	
√76	115597.999991	13362897602.0					
√77	701.998575496	492801.999998	345946302.000	242853811202.			
√78	105.990565198	11233.9999110	1190698.00000	126202754.000	13376301226.0		
√79	159.993749756	25597.9999609	4095520.00000	655257602.000	104837120800.		
√80	17.9442719100	321.996894380	5777.99982693	103681.999990	1860498.00000	33385282.0000	599074578.000
√82	18.0553851381	325.996932486	5886.00016989	106273.999991	1918818.00000	34644998.0000	625528782.000
√83	163.993902212	26893.9999628	4410452.00000	723287234.000	118614695924.		
√84	109.990908339	12097.9999173	1330670.00000	146361602.000	16098445550.0		
√85	756.001322749	571537.999998	432083484.000	326655685442.			
√86	20809.9999519	433056098.000					
√87	55.9821371593	3133.99968092	175447.999994	9821954.00000	549853976.000	30782000702.0	
√88	393.997461913	155233.999994	61161802.0000	24097594754.0			
√89	1000.00100000	1000002.00000	1000003000.00				
√90	37.9736659610	1441.99930652	54757.9999817	2079362.00000	78960998.0000	2998438562.00	113861704358.
√91	3147.99968234	9909902.00000	31196368348.0				
√92	2301.99956560	5299202.00000	12198760702.0				
√93	24301.9999589	590587202.000					
√94	4286590.00000						
√95	77.9871773792	6081.99983558	474317.999998	36990722.0000	2884801998.00	224977565122.	

$\sqrt{96}$	97.9897948557	9601.99989586	940897.999999	92198402.0000	9034502498.00	885289046402.	
$\sqrt{97}$	11208.0000892	125619266.000					
$\sqrt{98}$	197.994949366	39201.9999745	7761798.00000	1536796802.00	304278004998.		
$\sqrt{99}$	19.9498743711	397.997487421	7939.99987406	158401.999994	3160100.00000	63043598.0000	1257711860.00
$\sqrt{101}$	20.0498756211	401.997512422	8060.00012407	161601.999994	3240100.00000	64963602.0000	1302512140.00

The following observations may be made with respect to raising cycle factors to their powers.

- Powers of the cycle factors come closer to positive integers as they increase.
- The rule that applies to the cycle factors that are not adjacent to a perfect square ($n \times z^2 = t$) also applies to the powers of a cycle factors. Eg the cycle factor of $\sqrt{7}$ ($8 + \sqrt{63}$) is the same as for the cycle factor of $\sqrt{7 \times 3^2} = \sqrt{63}$. Likewise the square of the cycle factor for these numbers ($127 + \sqrt{16128} = 253.996062931$) is also the cycle factor of $\sqrt{16128}$. But $16128 = 2^8 \cdot 3^2 \cdot 7$, therefore $16128 = 4032 \times 2^2 = 1792 \times 3^2 = 1008 \times 4^2 = 448 \times 6^2 = 252 \times 8^2 = 112 \times 12^2 = 63 \times 16^2 = 7 \times 48^2$. Therefore the square roots of 7, 63, 112, 252, 448, 1008, 1792, 4032 and 16128 share the same cycle factors and/or powers of cycle factors. A limited amount of this can be seen in table 6 (all colour coded with colours other than yellow).
- Consider all the bright green highlighted cycle factors and powers of cycle factors. $\sqrt{2}$ shares in the cycle factors and powers of cycle factors with $\sqrt{2 \times 2^2} = \sqrt{8}$, $\sqrt{2 \times 3^2} = \sqrt{18}$, $\sqrt{2 \times 4^2} = \sqrt{32}$, $\sqrt{2 \times 5^2} = \sqrt{50}$, $\sqrt{2 \times 6^2} = \sqrt{72}$ and $\sqrt{2 \times 7^2} = \sqrt{98}$. Likewise each other colour also relates different cycle factors and their powers (Binary colours were used when we ran out of colours).
- Looking at the multiple solutions for the ordered pairs in table 4 (z,$\sqrt{n \times z^2 \pm 1}$), we find that the respective solutions represent the respective powers of the cycle factors for each number n. Eg for $n = 7$, the first solution is for $z = 3$ and $x = 8$. This means that the cycle factor is the same as for the square root of $7 \times 3^2 = 63$, which is just short of 8^2. The cycle factor for $\sqrt{63}$ is $8 + \sqrt{63}$. As shown earlier, $(8 + \sqrt{63})^2 = 127 + \sqrt{16128} = 253.996062931$. $\sqrt{16128/7} = 48$. 16128 is 127^2-1, giving an ordered pair (48, 127), the second solution to $n = 7$ in table 4, etc. Note that when working with n just greater or less than a perfect square, the solution in table 4 starts with the first power of the cycle factor and not the cycle factor itself as for the square roots of numbers not adjacent to a perfect square.

Taking a further look at table 4 and its solutions to cycle factors and their powers, the following is apparent.

- Only for $n = 61$ and 97 there seem to be no addition solutions (solutions where 1 must be added to $n \times z^2$) below 300 000. Several numbers have addition and subtraction solutions. When they do, the values alternate, therefore, values of n that have more than one addition solution do not have any subtraction solutions even beyond 300 000. Only the numbers that directly follow perfect squares alternate beginning with an addition solution. All the other cases that have addition and subtraction solutions begin with a subtraction solution first. It turns out that 61 and 97 belong to that category except that their respective next solutions, which are addition solutions, are greater than 300 000.
- 4 greater than every uneven perfect square has subtraction and addition solutions beginning with subtraction, in contrast with the numbers that follow a perfect square which begin with an addition solution.
- All the numbers that have addition and subtraction solutions are the sum of two perfect squares. Not all the sums of two perfect squares, however, have both addition and subtraction solutions.
- Table 7 shows a matrix of all the numbers up to 100 that can be made up of the sum of two perfect squares, and whether they have addition and then subtraction (+ superscript), or subtraction and then addition (- superscript) solutions, as well as those that are in themselves perfect squares (* superscript) and thus not have any solutions. Those that are addition solutions only have no superscript. All the numbers below the stepped double line are a repetition of information above the stepped double line.

Table 7. *A matrix that indicates all the sums of two perfect squares less than or equal to 100 showing their relation to the formula $n + z^2 \pm 1 = x^2$.*

	1	4	9	16	25	36	49	64	81
1	2^+	5^+	10^+	17^+	26^+	37^+	50^+	65^+	82^+
4	5^+	8	13^-	20	29^-	40	53^-	68	85^-
9	10^+	13^-	18	25^*	34	45	58^-	73^-	90
16	17^+	20	25^*	32	41^-	52	65^-	80	97^-
25	26^+	29^-	34	41^-	50^+	61^-	74^-	89^-	
36	37^+	40	45	52	61^-	72	85^-	100^*	
49	50^+	53^-	58^-	65^-	74^-	85^-	98		
64	65^+	68	73^-	80	89^-	100^*			
81	82^+	85^-	90	97^-					

Patterns are visible for the first two rows/columns. The fourth row/column also has all the uneven sums that include 16 as addition and subtraction solutions, except 25 because it is a perfect square in itself. All the uneven numbered sums throughout seem to be a part of the addition and subtraction solution system except 45 (36 + 9), but besides these, there is no apparent pattern. This may become clear if the matrix is expanded. 2 × 25 is the only number away from the first row/column that is positive, because it duplicates 1 + 49 in the first column. The diagonal does not have addition and subtraction solutions, other than the first row/column and 50.

All these patterns provide a clue as to how these systems work and need further investigation.

7. Prime factorizing the root of the perfect square factor (z)

Taking another look at table 6, and applying the formula $n \times z^2 \pm 1 = x^2$ it turns out that for each power of a cycle factor, n is a factor of the square of the integer portion (x) of any power of the cycle factor plus or minus 1, and the remaining factor is a perfect square (z^2).

- The first observation that needs to be made is that for each power of the cycle factor, the integer portion (x) is like the integer portion of the cycle factor itself, and is equal to the x-component of the corresponding solution in table 4. For example. the integer portion of the powers of the cycle factor of 7 are in increasing order (the closest integer to half the cycle factor or its power). 8, 127, 2024, 32257, 514088, …. (table 6). The x-part of the solutions for 7 in table 4 are. 8, 127, 2024, 32257, 514088. When applying the formula $z = \sqrt{(x^2 \pm 1)/n}$ [where x is the integer portion of the cycle factor or its power, approximately half of the cycle factor (y) or power (y^m)] the z-part of the solution in table 4 is produced. 3, 48, 765, 12192, 194307, ….
- Because the calculated part ($z = \sqrt{(x^2 \pm 1)/n}$) renders a perfect-square factor, the root of the perfect square (z) is prime factorised for investigation (table 8). Very clear patterns are visible as the powers increase. Using the cycle factor of the $\sqrt{7}$ as an example to illustrate the patterns, we see the following.
    - Some factors appear in the solutions of every power (3), some in every second power (24), some every third power (5, 17 and a second 3), some in every fourth power (127 and a fifth 2), some in every fifth power (239 and 271), some in every sixth power (11 and 23), etc.
    - 2 behaves binary in that it occurs at least 4 times in every second power, at least 5 times in every fourth power, at least 6 times in every eighth power etc.
    - 3 behaves ternary in that it occurs in every power, at least 2 times in every third power, and at least 3 times in every ninth power.
    - In the power series of the cycle factor of $\sqrt{11}$, 5 occurs every second power, and at least twice in every tenth power, showing it is quinternary.

Table 8. *The prime factors of the roots of the perfect-square factors after processing the powers of cycle factors like the cycle factors themselves were processed.*

n	**power**	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
2	z	1	2	5	12	29	70	169	408	985	2378	5741
	factorised	-	2	5	$2^2\cdot 3$	29	$2\cdot 5\cdot 7$	13^2	$2^3\cdot 3\cdot 17$	$5\cdot 197$	$2\cdot 29\cdot 41$	5741
	power	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}				
	z	13860	33461	80782	195025	470832	1136689	2744210				
	factorised	$2^2\cdot 3^2\cdot 5\cdot 7\cdot 11$	33461	$2\cdot 13^2\cdot 239$	$5^2\cdot 29\cdot 269$	$2^4\cdot 3\cdot 17\cdot 577$	$137\cdot 8297$	$2\cdot 5\cdot 7\cdot 197\cdot 199$				
3	power	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9		
	z	1	4	15	56	209	780	2911	10864	40545		
	factorised	-	2^2	$3\cdot 5$	$2^3\cdot 7$	$11\cdot 19$	$2^2\cdot 3\cdot 5\cdot 13$	$41\cdot 71$	$2^4\cdot 7\cdot 97$	$3^2\cdot 5\cdot 17\cdot 53$		
	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}					
	z	151316	564719	2107560	7865521	29354524	109552575					
	factorised	$2^2\cdot 11\cdot 19\cdot 181$	$23\cdot 43\cdot 571$	$2^3\cdot 3\cdot 5\cdot 7\cdot 13\cdot 193$	$2131\cdot 3691$	$2^2\cdot 41\cdot 71\cdot 2521$	$3\cdot 5^2\cdot 11\cdot 19\cdot 29\cdot 241$					
5	power	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9		
	z	1	4	17	72	305	1292	5473	23184	98209		
	factorised	-	2^2	17	$2^3\cdot 3^2$	$5\cdot 61$	$2^2\cdot 17\cdot 19$	$13\cdot 421$	$2^4\cdot 3^2\cdot 7\cdot 23$	$17\cdot 53\cdot 109\cdot$		

	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}
	z	416020	1762289	7465176	31622993	133957148	567451585
	factorised	$2^{2}\cdot 5\cdot 11\cdot 61\cdot 31$	89·19801	$2^{3}\cdot 3^{3}\cdot 17\cdot 19\cdot 107$	233·135721	$2^{2}\cdot 13\cdot 29\cdot 211\cdot 421$	5·17·61·109441

6	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}
	z	2	20	198	1960	19402	192060	1901198	18819920
	factorised	2	$2^{2}\cdot 5$	$2\cdot 3^{2}\cdot 11$	$2^{3}\cdot 5\cdot 7^{2}$	2·89·109	$2^{2}\cdot 3^{2}\cdot 5\cdot 11\cdot 97$	2·13·83·881	$2^{4}\cdot 5\cdot 7^{2}\cdot 4801$

6	power	x^{9}	x^{10}	x^{11}	x^{12}	x^{13}
	z	186298002	1844160100	18255302998	180708869880	1788833395802
	factorised	$2\cdot 3^{3}\cdot 11\cdot 17\cdot 19\cdot 971$	$2^{2}\cdot 5^{2}\cdot 89\cdot 109\cdot 1901$	2·23·131·659·4597	$2^{3}\cdot 3^{2}\cdot 5\cdot 7^{2}\cdot 11\cdot 97\cdot 9601$	2·937·1117·854569

7	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}
	z	3	48	765	12192	194307	3096720	49353213	786554688
	factorised	3	$2^{4}\cdot 3$	$3^{2}\cdot 5\cdot 17$	$2^{5}\cdot 3\cdot 127$	3·239·271	$2^{4}\cdot 3^{2}\cdot 5\cdot 11\cdot 17\cdot 23$	3·7·13·293·617	$2^{6}\cdot 3\cdot 127\cdot 32257$

7	power	x^{9}	x^{10}	x^{11}	x^{12}
	z	12535521795	199781794032	3183973182717	50743789129440
	factorised	$3^{3}\cdot 5\cdot 17\cdot 19\cdot 71\cdot 4049$	$2^{4}\cdot 3\cdot 179\cdot 239\cdot 271\cdot 359$	3·419·2309·1097009	$2^{5}\cdot 3^{2}\cdot 5\cdot 11\cdot 17\cdot 23\cdot 127\cdot 64513$

8	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}
	z	1	6	35	204	1189	6930	40391	235416	1372105
	factorised	-	2·3	5·7	$2^{2}\cdot 3\cdot 17$	29·41	$2\cdot 3^{2}\cdot 5\cdot 7\cdot 11$	$13^{2}\cdot 239$	$2^{3}\cdot 3\cdot 17\cdot 577$	5·7·197·199

8	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
	z	7997214	46611179	271669860	1583407981	9228778026
	factorised	2·3·19·29·41·59	23·353·5741	$2^{2}\cdot 3^{2}\cdot 5\cdot 7\cdot 11\cdot 17\cdot 1153$	79·599·33461	$2\cdot 3\cdot 13^{2}\cdot 239\cdot 113\cdot 337$

10	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}
	z	1	6	37	228	1405	8658	53353	328776	2026009
	factorised	-	2·3	37	$2^{2}\cdot 3\cdot 19$	5·281	$2\cdot 3^{2}\cdot 13\cdot 37$	53353	$2^{3}\cdot 3\cdot 7\cdot 19\cdot 103$	17·37·3221

10	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
	z	12484830	76934989	474094764	2921503573	18003116202
	factorised	2·3·5·281·1481	76934989	$2^{2}\cdot 3^{2}\cdot 11\cdot 13\cdot 19\cdot 37\cdot 131$	857·3408989	2·3·53353·56239

11	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}
	z	3	60	1197	23880	476403	9504180	189607197	3782639760
	factorised	3	$2^{2}\cdot 3\cdot 5$	$3^{2}\cdot 7\cdot 19$	$2^{3}\cdot 3\cdot 5\cdot 199$	3·379·419	$2^{2}\cdot 3^{2}\cdot 5\cdot 7\cdot 19\cdot 397$	3·13·643·7561	$2^{4}\cdot 3\cdot 5\cdot 199\cdot 79201$

11	power	x^{9}	x^{10}	x^{11}	x^{12}
	z	75463188003	1505481120300	30034159217997	599177703239640
	factorised	$3^{3}\cdot 7\cdot 17\cdot 19\cdot 467\cdot 2647$	$2^{2}\cdot 3\cdot 5^{2}\cdot 379\cdot 419\cdot 31601$	3·11·273569·3326861	$2^{3}\cdot 3^{2}\cdot 5\cdot 7\cdot 19\cdot 23\cdot 71\cdot 97\cdot 199\cdot 397$

12	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}
	z	2	28	390	5432	75658	1053780	14677262	204427888
	factorised	2	$2^{2}\cdot 7$	2·3·5·13	$2^{3}\cdot 7\cdot 97$	2·11·19·181	$2^{2}\cdot 3\cdot 5\cdot 7\cdot 13\cdot 193$	2·41·71·2521	$2^{4}\cdot 7\cdot 31\cdot 97\cdot 607$

12	power	x^{9}	x^{10}	x^{11}	x^{12}
	z	2847313170	39657956492	552364077718	7693439131560
	factorised	$2\cdot 3^{2}\cdot 5\cdot 13\cdot 17\cdot 37\cdot 53\cdot 73$	$2^{2}\cdot 7\cdot 11\cdot 19\cdot 181\cdot 37441$	2·23·43·571·489061	$2^{3}\cdot 3\cdot 5\cdot 7\cdot 13\cdot 97\cdot 193\cdot 37633$

13	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}

	z	5	180	6485	233640	8417525	303264540	10925940965
	factorised	5	$2^{2}\cdot 3^{2}\cdot 5$	$5\cdot 1297$	$2^{3}\cdot 3^{2}\cdot 5\cdot 11\cdot 59$	$5^{2}\cdot 109\cdot 3089$	$2^{2}\cdot 3^{3}\cdot 5\cdot 433\cdot 1297$	$5\cdot 29\cdot 41\cdot 1837837$
	power	x^{8}	x^{9}	x^{10}	x^{11}			
	z	393637139280	14181862955045	510940703520900	18408047189707445			
	factorised	$2^{4}\cdot 3^{2}\cdot 5\cdot 7\cdot 11\cdot 17\cdot 59\cdot 7079$	$5\cdot 1297\cdot 5237\cdot 417581$	$2^{2}\cdot 3^{2}\cdot 5^{2}\cdot 61\cdot 109\cdot 131\cdot 211\cdot 3089$	$5\cdot 83621\cdot 141481\cdot 311189$			

14	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}
	z	4	120	3596	107760	3229204	96768360	2899821596
	factorised	2^{2}	$2^{3}\cdot 3\cdot 5$	$2^{2}\cdot 29\cdot 31$	$2^{4}\cdot 3\cdot 5\cdot 449$	$2^{2}\cdot 11\cdot 79\cdot 929$	$2^{3}\cdot 3^{2}\cdot 5\cdot 13\cdot 23\cdot 29\cdot 31$	$2^{2}\cdot 7\cdot 41\cdot 97\cdot 26041$
	power	x^{8}	x^{9}	x^{10}	x^{11}			
	z	86897879520	2604036564004	78034199040600	2338421934653996			
	factorised	$2^{5}\cdot 3\cdot 5\cdot 191\cdot 449\cdot 2111$	$2^{2}\cdot 17\cdot 29\cdot 31\cdot 71\cdot 379\cdot 1583$	$2^{3}\cdot 3\cdot 5^{2}\cdot 11\cdot 19\cdot 61\cdot 79\cdot 139\cdot 929$	$2^{2}\cdot 4027\cdot 5807\ 24999391$			

15	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}
	z	1	8	63	496	3905	30744	242047	1905632	15003009
	factorised	-	2^{3}	$3^{2}\cdot 7$	$2^{4}\cdot 31$	$5\cdot 11\cdot 71$	$2^{3}\cdot 3^{2}\cdot 7\cdot 61$	$13\cdot 43\cdot 433$	$2^{5}\cdot 17\cdot 31\cdot 113$	$3^{3}\cdot 7\cdot 163\cdot 487$
	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}				
	z	118118440	929944511	7321437648	57641556673	453811015736				
	factorised	$2^{3}\cdot 5\cdot 11\cdot 19\cdot 71\cdot 199$	$26839\cdot 34649$	$2^{4}\cdot 3^{2}\cdot 7\cdot 31\cdot 61\cdot 23\cdot 167$	$53\cdot 131\cdot 1613\cdot 5147$	$2^{3}\cdot 13\cdot 43\cdot 433\cdot 234361$				

17	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}
	z	1	8	65	528	4289	34840	283009	2298912	18674305
	factorised	-	2^{3}	$5\cdot 13$	$2^{4}\cdot 3\cdot 11$	4289	$2^{3}\cdot 5\cdot 13\cdot 67$	283009	$2^{5}\cdot 3\cdot 7\cdot 11\cdot 311$	$5\cdot 13\cdot 287297$
	power	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}				
	z	151693352	1232221121	10009462320	81307919681	660472819768				
	factorised	$2^{3}\cdot 4289\cdot 4421$	1232221121	$2^{4}\cdot 3^{2}\cdot 5\cdot 11\cdot 13\cdot 67\cdot 1451$	$70901\cdot 1146781$	$2^{3}\cdot 127\cdot 2297\cdot 283009$				

18	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}
	z	4	136	4620	156944	5331476	181113240	6152518684
	factorised	2^{2}	$2^{3}\cdot 17$	$2^{2}\cdot 3\cdot 5\cdot 7\cdot 11$	$2^{4}\cdot 17\cdot 577$	$2^{2}\cdot 19\cdot 29\cdot 41\cdot 59$	$2^{3}\cdot 3\cdot 5\cdot 7\cdot 11\cdot 17\cdot 1153$	$2^{2}\cdot 13^{2}\cdot 113\cdot 239\cdot 337$
	power	x^{8}	x^{9}	x^{10}	x^{11}			
	z	209004522016	7100001229860	241191037293224	8193395266739756			
	factorised	$2^{5}\cdot 17\cdot 577\cdot 665857$	$2^{2}\cdot 3^{2}\cdot 5\cdot 7\cdot 11\cdot 73\cdot 179\cdot 197\cdot 199$	$2^{3}\cdot 17\cdot 19\cdot 29\cdot 41\cdot 59\cdot 241\cdot 5521$	$2^{2}\cdot 23\cdot 43\ 89\cdot 353\cdot 5741\cdot 11483$			

19	power	x	x^{2}	x^{3}	x^{4}	x^{5}
	z	39	13260	4508361	1532829480	521157514839
	factorised	$3\cdot 13$	$2^{2}\cdot 3\cdot 5\cdot 13\cdot 17$	$3^{2}\cdot 11\cdot 13\cdot 31\cdot 113$	$2^{3}\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 23\cdot 359$	$3\cdot 13\cdot 269\cdot 431\cdot 115259$
	power	x^{6}	x^{7}	x^{8}		
	z	177192022215780	60244766395850361	20483043382566906960		
	factorised	$2^{2}\cdot 3^{2}\cdot 5\cdot 11\cdot 13\cdot 17\cdot 31\cdot 113\cdot 115597$	$3\cdot 13\cdot 39187721\cdot 39418919$	$2^{4}\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 23\cdot 353\cdot 359\cdot 18927617$		

20	power	x	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}
	z	2	36	646	11592	208010	3732588	66978574	1201881744
	factorised	2	$2^{2}\cdot 3^{2}$	$2\cdot 17\cdot 19$	$2^{3}\cdot 3^{2}\cdot 7\cdot 23$	$2\cdot 5\cdot 11\cdot 31\cdot 61$	$2^{2}\cdot 3^{3}\cdot 17\cdot 19\cdot 107$	$2\cdot 13\cdot 29\cdot 211\cdot 421$	$2^{4}\cdot 3^{2}\cdot 7\cdot 23\cdot 47\cdot 1103$
	power	x^{9}	x^{10}	x^{11}	x^{12}				
	z	21566892818	387002188980	6944472508822	124613502969816				
	factorised	$2\cdot 17\cdot 19\cdot 53\cdot 109\cdot 5779$	$2^{2}\cdot 3^{2}\cdot 5\cdot 11\cdot 31\cdot 41\cdot 61\cdot 2521$	$2\cdot 89\cdot 199\cdot 9901\cdot 19801$	$2^{3}\cdot 3^{3}\cdot 7\cdot 17\cdot 19\cdot 23\cdot 107\cdot 103681$				

21	power	x	x^2	x^3	x^4	x^5	x^6		
	z	12	1320	145188	15969360	1756484412	193197315960		
	factorised	$2^2\cdot3$	$2^3\cdot3\cdot5\cdot11$	$2^2\cdot3^2\cdot37\cdot109$	$2^4\cdot3\cdot5\cdot11\cdot23\cdot263$	$2^2\cdot3\cdot19\cdot29\cdot421\cdot631$	$2^3\cdot3^2\cdot5\cdot11\cdot37\cdot109\cdot12097$		
	power	x^7	x^8	x^9					
	z	21249948271188	2337301112514720	257081872428348012					
	factorised	$2^2\cdot3\cdot7\cdot13\cdot43\cdot139\cdot337\cdot9661$	$2^5\cdot3\cdot5\cdot11\cdot17\cdot23\cdot31\cdot263\cdot138863$	$2^2\cdot3^3\cdot37\cdot53\cdot109\cdot379\cdot3511\cdot8369$					
22	power	x	x^2	x^3	x^4	x^5			
	z	42	16548	6519870	2568812232	1012105499538			
	factorised	$2\cdot3\cdot7$	$2^2\cdot3\cdot7\cdot197$	$2\cdot3^2\cdot5\cdot7\cdot79\cdot131$	$2^3\cdot3\cdot7\cdot197\cdot77617$	$2\cdot3\cdot7\cdot19\cdot8191\cdot154841$			
	power	x^6	x^7	x^8					
	z	398766998005740	157113185108762022	61902196165854230928					
	factorised	$2^2\cdot3^2\cdot5\cdot7\cdot13\cdot79\cdot131\cdot197\cdot11941$	$2\cdot3\cdot7^2\cdot139\cdot1301\cdot6733\cdot438899$	$2^4\cdot3\cdot7\cdot47\cdot197\cdot77617\cdot256357391$					
23	power	x	x^2	x^3	x^4	x^5	x^6		
	z	5	240	11515	552480	26507525	1271808720		
	factorised	5	$2^4\cdot3\cdot5$	$5\cdot7^2\cdot47$	$2^5\cdot3\cdot5\cdot1151$	$5^2\cdot11\cdot41\cdot2351$	$2^4\cdot3^2\cdot5\cdot7^2\cdot13\cdot47\cdot59$		
	power	x^7	x^8	x^9	x^{10}				
	z	61020311035	2927703120960	140468729495045	6739571312641200				
	factorised	$5\cdot108193\cdot112799$	$2^6\cdot3\cdot5\cdot31\cdot1151\cdot127\cdot673$	$5\cdot7^2\cdot17\cdot19\cdot47\cdot73\cdot89\cdot5813$	$2^4\cdot3\cdot5^2\cdot11\cdot41\cdot1879\cdot2351\cdot2819$				
24	power	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
	z	1	10	99	980	9701	96030	950599	9409960
	factorised	-	$2\cdot5$	$3^2\cdot11$	$2^2\cdot5\cdot7^2$	$89\cdot109$	$2\cdot3^2\cdot5\cdot11\cdot97$	$13\cdot83\cdot881$	$2^3\cdot5\cdot7^2\cdot4801$
	power	x^9	x^{10}	x^{11}	x^{12}	x^{13}			
	z	93149001	922080050	9127651499	90354434940	894416697901			
	factorised	$3^3\cdot11\cdot17\cdot19\cdot971$	$2\cdot5^2\cdot89\cdot109\cdot1901$	$23\cdot131\cdot659\cdot4597$	$2^2\cdot3^2\cdot5\cdot7^2\cdot11\cdot97\cdot9601$	$937\cdot1117\cdot854569$			
26	power	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
	z	1	10	101	1020	10301	104030	1050601	10610040
	factorised	-	$2\cdot5$	101	$2^2\cdot3\cdot5$	10301	$2\cdot5\cdot101\cdot103$	$197\cdot5333$	$2^3\cdot3\cdot5\cdot7\cdot17\cdot743$
	power	x^9	x^{10}	x^{11}	x^{12}	x^{13}			
	z	107151001	1082120050	10928351501	110365635060	1114584702101			
	factorised	$37\cdot53\cdot101\cdot541$	$2\cdot5^2\cdot11\cdot191\cdot10301$	$4973\cdot4917537$	$2^2\cdot3^2\cdot5\cdot17\cdot101\cdot103\cdot3467$	$13\cdot85737284777$			
27	power	x	x^2	x^3	x^4	x^5	x^6		
	z	5	260	13515	702520	36517525	1898208780		
	factorised	5	$2^2\cdot5\cdot13$	$3\cdot5\cdot17\cdot53$	$2^3\cdot5\cdot7\cdot13\cdot193$	$5^2\cdot11\cdot19\cdot29\cdot241$	$2^2\cdot3\cdot5\cdot13\cdot17\cdot37\cdot53\cdot73$		
	power	x^7	x^8	x^9	x^{10}				
	z	98670339035	5128959421040	266607219555045	13858446457441300				
	factorised	$5\cdot41\cdot71\cdot2017\cdot3361$	$2^4\cdot5\cdot7\cdot13\cdot97\cdot193\cdot37633$	$3^2\cdot5\cdot17\cdot53\cdot46817\cdot140453$	$2^2\cdot5^2\cdot11\cdot13\cdot19\cdot29\cdot61\cdot181\cdot241\cdot661$				

8. Criteria for Series of Powers to Approach Integers

A last phenomenon to consider in this paper is the approach of series of the integer powers of numbers to integers as the series progresses. As seen in table 6 the powers of the cycle factors more and more approximate integers as the powers increase. We thought that this could be unique to the root of a perfect square plus the root of a number adjacent to the perfect square (y). The example used to investigate this scenario in table 9 is $2 + \sqrt{5}$ (the cycle factor of $\sqrt{5}$). Testing this concept on a number 2 greater than the perfect square, 4, it turned out to work for that number also *ie* 2 + $\sqrt{6}$, except that the approach to the integers was slower. Testing further numbers (2 + $\sqrt{7}$, 2 + $\sqrt{8}$, etc) it was found that these series would approach integers as long as the root of a non-perfect square was within the bounds set by the next perfect square, *ie* in this example not greater than 2 + $\sqrt{8}$. 2 + $\sqrt{9}$, of course is 5 and is not irrational, neither its powers. The power series of 2 + $\sqrt{10}$ does not approach integers but is randomly spaced between the integers as the series progresses. However, the series of the powers of 3 + $\sqrt{10}$ (cycle factor of *ia* $\sqrt{10}$) approaches integers etc.

Table 9. *The behavior of the decimal portion of powers of different numbers.*

y=	2 + $\sqrt{6}$	2 +832040/196418	2 + $\sqrt{5}$	2 + $\sqrt{5.5}$
y	4.4494897427831780981972841	4.236067977476606013705464876	4.236067977499789696409173669	4.345207879911714777282815057
y^2	19.797958971132712392789136	17.94427190980274347383383903	17.94427190999915878563669467	18.88083151964685910913126023
y^3	88.090815370097205767551113	76.01315561624838192003129769	76.01315561749642483895595237	82.04113789845500860244926349
y^4	391.95917942265424785578273	321.9968943729357986169436657	321.9968943799848581414605041	356.4857988732903230734939443
y^5	1744.0183484308114029582331	1364.000733100110488250780517	1364.000733137435857404797969	1549.004902340843805197649672
y^6	7759.9917525685541075444980	5777.999826739992926535141491	5777.999826929728287760652380	6730.748307673310705400839606
y^7	34528.003707135839236094458	24476.00003991866180591544118	24476.00004085634900844740749	29246.50058420450852939983293
y^8	153631.99833368046515946683	103681.9999858155737712118222	103681.9999903551243215502823	127082.1247983280001757005911
y^9	683584.00074899353911005623	439203.9999806432709863611507	439204.0000022768462946485368	552198.2500696187634969021140
y^{10}	3041599.9996633350867591586	1860497.999897638847720127136	1860497.999999462509500144430	2399416.187475967054251159343
y^{11}	13533568.000151327425256747	7881195.999525661735976614070	7881196.000000126884295226255	10425962.12500829636224999054
y^{12}	60217471.999931979874545304	33385281.99780738827807014617	33385281.99999997004668104945	45302972.78124713603037670118
y^{13}	267937024.00003057434869471	141422323.9899380878681459155	141422324.0000000070710194241	196850834.3125009886648817905
y^{14}	1192183039.9999862571438695	599074577.9540983343166722372	599074577.9999999983307587457	855357796.4218746587050922139
y^{15}	5304606208.0000061772728672	2537720635.791668676418882018	2537720636.000000000394054407	3716707437.156250117817691541
y^{16}	23602790911.999997223379208	10749957121.05866063969476624	10749957121.99999999990697637	16149866443.25781245932840449
y^{17}	105020376064.00000124806257	45537549119.76319888529377806	45537549124.00000000002195990	70174526928.76562501404015526
y^{18}	467287086079.99999943900868	192900153598.9968943809622086	192900153617.9999999999948160	304922907379.9492187451532278
y^{19}	2079189096448.0000002521598	817138163511.0294168596350440	817138163596.0000000000012237	1324953419912.945312501673144
y^{20}	9251330557951.9999998866567	3461452807623.114561821679999	3461452808001.999999999999711	5757198040721.705078124422418
y^{21}	41163700424704.000000050947	14662949393918.76630460604744	14662949395604.00000000000006	25016222292756.23828125019939
y^{22}	183157462814719.99999997710	62113250382939.29434208889059	62113250390417.99999999999998	108700686232107.5107421874312
y^{23}	814957252108288.00000001029	263115950924155.6805608021695	263115950957276.0000000000000	472327078367564.4003906250238
y^{24}	3626143934062591.9999999954	1114577054073122.078698540038	1114577054219522.000000000000	2052359342818418.867675781242
y^{25}	16134490240466944.000000002	4721424167189363.980695930378	4721424167835364.000000000000	8917927988825022.071289062503
y^{26}	71790248829992959.999999999	20000273722715018.00496004394	20000273725560978.00000000000	38750250969527716.58666992188
y^{27}	319429975800905728.00000000	84722519057559915.99979103221	84722519070079276.00000000000	168377895861348399.4536132813
y^{28}	1421300400863608832.0000000	358890349950881042.0046336407	358890350005878082.0000000000	731636959899685172.6944580078
y^{29}	6324061555056246784.0000000	1520283918852300004.019669146	1520283919093591604.000000000	3179114683390763289.958251953
y^{30}	28138847021952204800.000000	6440026025322871098.089408896	6440026026380244498.000000000	13813914173412580918.87469482
y^{31}	125203511197921312768.00000	27280388019986160476.40395370	27280388024614569596.00000000	60024328718736468610.43615723
y^{32}	557091738835589660672.00000	115561578104599807363.8217762	115561578124838522882.0000000	260818186135064745820.0566711
y^{33}	2478773977738201268224.0000	489526700435556943452.2002600	489526700523968661124.0000000	1133309237618363686195.880920

The series of an approximation to the cycle factor of $\sqrt{5}$, 832040/196418 (j_{10}/j_9 of the *e*-series of triangles having triangle 2/$\sqrt{5}$/3 as limit [1]) started approaching integers, but at (2 + 832040/196418)^8 the series was at its closest approach and then started drifting away again as the series progressed. Clearly the cycle factor must be correct and not approximated to have the series of powers continue approaching integers.

Then we wondered whether any square root in the defined bounds would approach integers as the power series continued? We tested this concept with 2 + $\sqrt{5.5}$. Its powers were randomly dispersed between integers with the progression of the powers.

The following rule seems to apply generally. A series $(x + \sqrt{x^2 \pm r})^m$ will come closer and closer to integers as m progresses from 1 – ∞ on condition that $r < (x + 1)^2 – x^2$ for the + condition and $r < x^2 – (x – 1)^2$ for the – condition, and x and r are positive integers. The equation $x^2 = (n \times z^2) \pm 1$ does not apply to any other numbers but the ones which may be defined as $y = x + \sqrt{x^2 \pm 1}$, x being a positive integer.

9. Conclusion

In this paper it has been shown that cycle factors (y) for series of right-angled triangles [1] that have a right-angled triangle with one irrational leg as the limit, are defined by $y = x + \sqrt{x^2 \pm 1}$ where x is a positive integer. Clearly, as x increases, $x + \sqrt{x^2 \pm 1}$ will come closer and closer to an integer. Powers of the cycle factors also obey the same rule, *ie* $(x + \sqrt{x^2 \pm 1})^m = x' + \sqrt{x'^2 \pm 1}$ for any positive integer value of m. Therefore a series of powers of $x + \sqrt{x^2 \pm 1}$ will approximate a positive integer closer and closer as the series progresses. It has been shown that the cycle factor of a root of n, that is not adjacent to a perfect square, neither is a perfect square, can be determined by finding the smallest natural number solution (z) to the equation $x^2 = (n \times z^2) \pm 1$ and x is the integer portion of the cycle factor (y). For 1 – 100, each non-perfect square has such a solution. The powers of the cycle factors also adhere to the same equation, rendering the greater solutions (z) to that equation. Prime factorising z for the series of increasing powers, reveals discrete patterns in the repetition of the prime factors.

The following questions arise from this work.

- Do all the positive integers (n) that are not perfect squares have a solution for the equation $x^2 = (n \times z^2) \pm 1$? The integers below 102 that are not perfect squares all have solutions to this equation.
- Why do the prime factors of z in the equation $x^2 = (n \times z^2) \pm 1$ in the series of powers of cycle factors display such predictable behavior? Do these patterns apply to the greater prime factors also?
- Why does a series $(x + \sqrt{x^2 \pm r})^m$ come closer and closer to integers as m progresses from 1 – ∞ on condition that $r < (x + 1)^2 – x^2$ for the positive condition and $r < x^2 – (x – 1)^2$ for the negative condition, and x and r are positive integers, but not to any other value of r, be it outside the range or non-integer?
- What determines when $x^2 = (n \times z^2) \pm 1$ is + 1 and especially when it is – 1?
- When roots share cycle factors (y), they are sometimes exactly the same, and sometimes one cycle factor is a power of the other. What determines whether they are the same or not, and when powers are involved, what determines which power?

These phenomena are being further investigated to procure a better understanding for their existence. Furthermore, the concept will be studied in terms of cubes and cube roots, and higher powers.

References

[1] MW Bredenkamp, *Applied and Computational Mathematics*, 2013, 2, 42-53.

[2] All rational right-angled triangles may be raised or reduced to a relatively prime right-angled triangle defined by a pair of positive integer indices (i,j) where i is an uneven number and j is an even number and the even leg (e), the uneven leg (u) and the hypotenuse (h) of the triangle are algebraically defined by the indices (i,j) as follows.

$$u = i^2 + ij$$

$$e = j^2/2 + ij$$

$$h = i^2 + ij + j^2/2$$

[3] MW Bredenkamp, *Applied and Computational Mathematics*, 2013, 2, 36-41.