

# Sub Hilbert spaces in a Bi-Disk

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**To cite this article:**

Niteesh Sahni. Sub Hilbert Spaces in a Bi-Disk, *Pure and Applied Mathematics Journals*. Vol. 2, No. 2, 2013, pp.98-100.

doi: 10.11648/j.pamj.20130202.17

**Abstract:** Recently, Sahni and Singh [7] have solved an open problem posed by Yousefi and Hesameddini [12] regarding Hilbert spaces contained algebraically in the Hardy space  $H^2(T)$ . In fact the result obtained by Sahni and Singh is much more general than the open problem. In the present note we examine the validity of the main results of [7] and [12] in two variables.

**Keywords:** Hardy Space, Beurling Type Result, Isometry, Wold Type Decomposition

## 1. Introduction

In [12], Yousefi and Hesameddini have characterized the invariant subspaces of all Hilbert spaces  $H$  that are vector subspaces of the Hardy space  $H^2(T)$ , with  $\langle \cdot, \cdot \rangle_H$  (the inner product on  $H$ ) satisfying the following axioms:

A1. If there are four functions  $f_1, f_2, g_1, g_2 \in H$  such that  $\langle f_1, g_1 \rangle_H = \langle f_2, g_2 \rangle_H$ , then we have  $\langle f_1, g_1 \rangle_{H^2} = \langle f_2, g_2 \rangle_{H^2}$ .

A2. If  $\phi$  is any inner function, then  $\phi f \in H$  and  $\langle \phi f, \phi g \rangle_H = \langle f, g \rangle_H$  for all  $f, g \in H$ .

The characterization of  $H$ , however is left as the following open question in [12]: Is every Hilbert space  $H$  satisfying axioms A1 and A2 of the form  $\phi H^2$ , for some  $\phi \in H^\infty$ , and does there exist a constant  $k$  such that  $\langle f, g \rangle_H = k \langle f, g \rangle_{H^2}$  for every  $f, g \in H$ ?

Sahni and Singh [7] have settled this open problem in the affirmative. They in fact prove a far more general version of the above problem. Below we record a special case of their main result which clearly generalized the open problem mentioned above:

**Theorem 1.1 ([7], Corollary 3.2).** Let  $H$  be a Hilbert space which is algebraically contained in  $H^2$  and invariant under multiplication by  $z$ . Further if the inner product on  $H$  satisfies the following conditions

- i.  $\langle zf, zg \rangle_H = \langle f, g \rangle_H$  for all  $f, g \in H$ ;
- ii. for any  $f_1, f_2, g_1, g_2 \in H$  such that  $\langle f_1, g_1 \rangle_H = \langle f_2, g_2 \rangle_H$ , then we have

$$\langle f_1, g_1 \rangle_{H^2} = \langle f_2, g_2 \rangle_{H^2}.$$

then there exists a unique inner function  $b \in H^\infty$  and a constant  $k$  such that  $H = bH^2$  and  $\langle f, g \rangle_H = k \langle f, g \rangle_{H^2}$  for

every  $f, g \in H$ .

In [7], the authors use the above characterization of  $H$  to derive the following invariant subspace characterization:

**Theorem 1.2 ([7], Theorem 3.3).** Let  $H$  be a Hilbert space satisfying all the conditions of Theorem 1.1. If  $M$  is a closed subspace of  $H$  and invariant under multiplication by  $z$ , then there exists a unique inner function  $\phi \in H^\infty$  such that  $M = \phi H$ .

It is further noted in [7] that the characterization obtained in [12] comes as a special case of Theorem 1.2. In the present paper, we obtain a two variable version of Theorem 1.1 for the Hardy space  $H_2(T_2)$  on the torus  $T_2$ , and also examine the validity of Theorem 1.2 in two variables.

## 2. Notations and Preliminary Results

Let  $T$  denote the unit circle in the complex plane, and  $T^2$  stands for the cartesian product of  $T$ , that is,  $T^2 = \{(e^{i\theta_1}, e^{i\theta_2}): 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi\}$ . We shall denote the coordinate functions  $e^{i\theta_1}$  and  $e^{i\theta_2}$  by  $z$  and  $w$  respectively. The Lebesgue space  $L^2(T^2)$  on the torus is a collection of complex valued functions  $f$  on the torus such that  $\int |f|^2 dm$  is finite, where  $dm$  is the normalized Lebesgue measure on  $T^2$ . It is well known that  $L^2(T^2)$  is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_2 = \int_{T^2} f g dm,$$

and the collection  $\{z^n w^m: m, n \in \mathbb{Z}\}$  is an orthonormal basis for  $L^2(T^2)$ . The Hardy space  $H^2(T^2)$  comprises of those  $L^2(T^2)$  functions  $f(z, w)$  that have the Fourier expansion of the form  $\sum_{m=0}^\infty \sum_{n=0}^\infty a_{mn} z^m w^n$ , and hence the norm of  $f$  is given by  $\|f\|_2 = (\sum_{m=0}^\infty \sum_{n=0}^\infty |a_{mn}|^2)^{1/2}$ .

The space  $H^2(T^2)$  turns out to be a closed subspace of  $L^2(T^2)$  and hence a Hilbert space under the above norm. The class of essentially bounded functions in  $H^2(T^2)$  is denoted by  $H^\infty$ . Let  $S_1$  and  $S_2$  be operators on  $H^2(T^2)$  defined by  $S_1f(z, w) = zf(z, w)$ , and  $S_2f(z, w) = wf(z, w)$  respectively.

A function  $\phi \in H^2(T^2)$  is called an inner function if  $|\phi| = 1$  a.e.. Equivalently, this means that  $\langle \phi f, \phi g \rangle_2 = \langle f, g \rangle_2$  for every  $f, g \in H^2(T^2)$ .

**Theorem 2.1** (Singh [11]). Let  $M$  be a Hilbert space which is a vector subspace of  $H^2(T^2)$  such that  $S_1(M) \subset M$ ,  $S_2(M) \subset M$ ; and  $S_1, S_2$  are doubly commuting isometries on  $M$ . Then there exists  $b \in H^\infty$  such that  $M = bH^2(T^2)$ , and  $\|bf\|_M = \|f\|_2$  for all  $f \in H^2(T^2)$ .

The following lemma provides a necessary and sufficient condition for an  $H^\infty$  function to be an inner function:

**Lemma 2.2.** A function  $\varphi \in H^\infty$  is inner if and only if  $\{z^m w^n \varphi : m, n = 0, 1, 2, \dots\}$  is an orthonormal set in  $H^2(T^2)$ .

Proof. Suppose  $\phi$  is inner. So  $|\phi| = 1$  a.e., and therefore

$$\langle z^{m_1} w^{n_1} \phi, z^{m_2} w^{n_2} \phi \rangle_2 = \langle z^{m_1} w^{n_1}, z^{m_2} w^{n_2} \rangle_2 = \begin{cases} 1 & \text{if } (m_1, n_1) = (m_2, n_2) \\ 0 & \text{if } (m_1, n_1) \neq (m_2, n_2) \end{cases}$$

Conversely, the orthonormality of the set  $\{z^m w^n \varphi : m, n = 0, 1, 2, \dots\}$  yields the following equations:

$$\langle \varphi, \varphi z^m w^n \rangle_2 = 0 = \overline{\langle \varphi, \varphi z^m w^n \rangle_2} \text{ for all } m > 0 \text{ and } n > 0,$$

$$\langle \varphi z^m, \varphi w^n \rangle_2 = 0 = \overline{\langle \varphi z^m, \varphi w^n \rangle_2} \text{ for all } m > 0 \text{ and } n > 0,$$

$$\langle \varphi, \varphi z^m \rangle_2 = 0 = \overline{\langle \varphi, \varphi z^m \rangle_2} \text{ for all } m > 0,$$

$$\text{and } \langle \varphi, \varphi w^n \rangle_2 = 0 = \overline{\langle \varphi, \varphi w^n \rangle_2} \text{ for all } n > 0.$$

From the above four equations we conclude that

$$\int |\varphi|^2 z^m w^n dm = 0 \text{ for all } (m, n) \neq (0, 0).$$

Therefore  $|\varphi|^2$  is a constant, say  $c$ .

Since  $1 = \|\varphi\|_2 = \int |\varphi|^2 dm$ , we have  $c = 1$ . This proves that  $\phi$  is inner.

### 3. Main Results

**Theorem 3.1.** Let  $H$  be a Hilbert space which is a vector subspace of  $H^2(T^2)$  such that  $S_1(H) \subset H$ ,  $S_2(H) \subset H$ ; and  $S_1, S_2$  are doubly commuting isometries on  $H$ . Further, assume that  $\langle \cdot, \cdot \rangle_H$  (the inner product on  $H$ ) satisfies the following property:

(P) For any functions  $f_1, f_2, g_1, g_2 \in H$  such that  $\langle f_1, g_1 \rangle_H = \langle f_2, g_2 \rangle_H$ , then we have  $\langle f_1, g_1 \rangle_{H^2} = \langle f_2, g_2 \rangle_{H^2}$ .

Then there exists a unique inner function  $b$  such that  $H = bH^2(T^2)$ , and  $\|bf\|_H = \|f\|_2$  for every  $f \in H^2(T^2)$ .

Proof. Since the operators  $S_1$  and  $S_2$  are doubly commuting isometries on the Hilbert space  $H$ , so all the conditions of Theorem 2.1 are satisfied and hence there exists  $b \in H^\infty$  such that  $H = bH^2(T^2)$ , and  $\|bf\|_H = \|f\|_2$  for all  $f \in H^2(T^2)$ . It remains to be shown that  $b$  is inner. The proof of Theorem 2.1, as given in [11], reveals that  $H$  has the Slocinski-Wold type decomposition:

$$H = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \oplus S_1^m S_2^n (L_1 \cap L_2).$$

Here  $L_k = H \ominus S_k(H)$ ,  $k=1, 2$ . It turns out that  $b$  is a unit

vector in  $H$  that spans  $L_1 \cap L_2$ . From the above Wold type decomposition we see that  $\langle b, bz^m w^n \rangle_H = 0$  for all  $(m, n) \neq (0, 0)$ . The arguments used to establish this fact are similar to the ones adopted in the proof of Lemma 2.2. This, in view of property (P), implies that  $\langle b, bz^m w^n \rangle_2 = 0$  for all  $(m, n) \neq (0, 0)$ . Thus the set  $\{bz^m w^n : m \geq 0 \text{ and } n \geq 0\}$  is orthogonal in  $H^2(T^2)$ . So the collection  $\{\frac{b}{\|b\|_2} z^m w^n : m \geq 0 \text{ and } n \geq 0\}$  is orthonormal in  $H^2(T^2)$ .

Hence by Lemma 2.2, the function  $\frac{b}{\|b\|_2}$  is inner, and we have  $H = bH^2(T^2) = \frac{b}{\|b\|_2} H^2(T^2)$ .

The above theorem clearly generalizes the two variable Beurling type result proved by Mandrekar [5]:

**Corollary 3.2 ([5], Theorem 2).** Let  $M$  be a closed subspace of  $H^2(T^2)$  which is invariant under  $S_1$  and  $S_2$  and that  $S_1, S_2$  doubly commute on  $M$ . Then there exists a unique inner function  $b$  such that  $M = bH^2(T^2)$ .

**Proof.** Since the norm on  $M$  is the same as that on  $H^2$ , so the property (P) stated in Theorem 3.1 is automatically satisfied and hence the result follows.

**Theorem 3.3.** Let  $H$  be a Hilbert space satisfying the conditions of Theorem 3.1, and  $M$  be a closed subspace of  $H$  which is invariant under  $S_1$  and  $S_2$ . Then there exists a unique inner function  $\phi$  such that  $M = \phi H$ .

**Proof.** By Theorem 3.1, there exists an inner function  $b$  such that  $H = bH^2(T^2)$ . Since  $M$  is a closed subspace of  $H$ , so  $M$  is also a Hilbert space satisfying the assumptions of Theorem 3.1. Thus there exists an inner function  $c$  such that  $M = cH^2(T^2)$ . Note that  $c = b\phi$  for some  $\phi \in H^2(T^2)$ . This implies that  $|\phi| = 1$  a.e. Further,  $M = \phi bH^2(T^2) = \phi H$ .

### Acknowledgements

The author thanks the Shiv Nadar University, Dadri, Uttar Pradesh (India), for support and facilities needed to complete the present work.

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