

# Asymptotic method for certain over-damped nonlinear vibrating systems

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**Abstract:** Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended and applied to certain over-damped nonlinear system in which the linear equation has two almost equal roots. The method is illustrated by an example.

**Keywords:** Nonlinear System, Unperturbed Equation, Over-Damped Oscillatory System, Equal Roots

## 1. Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [1-3] method is widely used techniques to obtain approximate solutions of weakly nonlinear system. The method was originally developed for approximating periodic solutions of second order nonlinear differential systems was later extended by Popov [4] to damped oscillatory nonlinear systems. Murty, Dekshatulu and Krisna [5] extended the method to over-damped nonlinear system. Recently Shamsul [6] has presented a unified method for solving an  $n$ -th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with constant coefficients. In another recent article, Shamsul [7], Pinakee, *et al* [7,8] investigated over-damped nonlinear systems and found approximate solutions of *Duffing's* equation when one root of the unperturbed equation was double of the other. The aim of this article is to find an approximate solution of over-damped nonlinear differential systems based on the extended KBM (by Popov [4]) method in which one of the eigen-values is almost equal to the other eigen-value.

## 2. Materials and Method

Consider a nonlinear system governed by the differential equation,

$$\ddot{x} + k_1\dot{x} + k_2x = -\varepsilon f(x, \dot{x}), \quad (1)$$

Where the over-dots denote differentiation with respect to  $t$ ,  $k_1$  and  $k_2$  are constants,  $\varepsilon$  is a small parameter,  $f$  is the given nonlinear function. When  $\varepsilon = 0$  (1) has two roots, say  $\lambda_1$  and  $\lambda_2$ . Therefore, the solution of the unperturbed equation of (1) become

$$x(t, 0) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right), \quad (2)$$

Where  $a_0$  and  $b_0$  are arbitrary constant. We choose an approximate solution of (1) in the form of the asymptotic expansion

$$x(t, \varepsilon) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + \varepsilon u_1(a, b, t) + \varepsilon^2 \dots, \quad (3)$$

Where  $a$  and  $b$  satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a, b, t) + \varepsilon^2 A_2(a, b, t) + \varepsilon^3 \dots, \\ \dot{b} &= \varepsilon B_1(a, b, t) + \varepsilon^2 B_2(a, b, t) + \varepsilon^3 \dots, \end{aligned} \quad (4)$$

Herein solution (3) together with (4) is not considered in a usual form of the classical KBM method. But this solution was early introduced by Murty and Deekshatulu [5] to investigate an over-damped case of equation (1). Now it is being used to investigate various oscillatory and non-oscillatory problems (see [6-9] for details).

Differentiating  $x(t, \varepsilon)$  twice with respect to  $t$ , substituting the derivatives,  $\dot{x}$ ,  $\ddot{x}$  and  $x(t, \varepsilon)$  in the original equation (1) and equating the coefficient of  $\varepsilon$ , we obtain

$$\frac{1}{2} \left( \left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_1 t} + \left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_2 t} \right) + \frac{\partial B_1}{\partial t} \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \\ (e^{\lambda_1 t} + e^{\lambda_2 t}) B_1 + \left( \frac{\partial}{\partial t} - \lambda_1 \right) \left( \frac{\partial}{\partial t} - \lambda_2 \right) u_1 = -f^{(0)}(a, b, t) \quad (5)$$

where  $f^{(0)} = f(x_0, \dot{x}_0)$  and

$$x(t, 0) = \frac{1}{2} a_0 (e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0 \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)$$

In general  $f^{(0)}$  be expanded in power of  $\left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)$  as;

$$f^{(0)} = g_0(a, b, t) + g_1(a, b, t) \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + g_2(a, b, t) \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^2 + \dots \quad (6)$$

Substitute the expansions of  $f^{(0)}$  from (6) into (5) and equation the coefficients of  $\left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^0, \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^1$  and higher order terms of  $\left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)$ , we obtain

$$\frac{1}{2} \left( \left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_1 t} + \left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_2 t} \right) + (e^{\lambda_1 t} + e^{\lambda_2 t}) B_1 = -g_0(a, b, t) \quad (7)$$

$$\frac{\partial B_1}{\partial t} = -g_1(a, b, t) \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad (8)$$

and

$$\left( \frac{\partial}{\partial t} - \lambda_1 \right) \left( \frac{\partial}{\partial t} - \lambda_2 \right) u_1 = -g_2(a, b, t) \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^2 - (9)$$

$$u_1 = -\frac{3ab^2}{4} \left( \frac{e^{3\lambda_1 t}}{\lambda_1(3\lambda_1 - \lambda_2)} - \frac{e^{(2\lambda_1 + \lambda_2)t}}{\lambda_1(\lambda_1 + \lambda_2)} - \frac{e^{(\lambda_1 + 2\lambda_2)t}}{\lambda_2(\lambda_1 + \lambda_2)} + \frac{e^{3\lambda_2 t}}{\lambda_2(3\lambda_2 - \lambda_1)} \right) \\ - \frac{b^3}{2(\lambda_1 - \lambda_2)^3} \left( \frac{e^{3\lambda_1 t}}{\lambda_1(3\lambda_1 - \lambda_2)} - \frac{3e^{(2\lambda_1 + \lambda_2)t}}{\lambda_1(\lambda_1 + \lambda_2)} + \frac{3e^{(\lambda_1 + 2\lambda_2)t}}{\lambda_2(\lambda_1 + \lambda_2)} - \frac{e^{3\lambda_2 t}}{\lambda_2(3\lambda_2 - \lambda_1)} \right) \quad (12)$$

Now substituting the values of  $g_0$  and the values of  $B_1$  from (11) into (7) and simplifying, we obtain

$$\left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_1 t} + \left( \frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2) A_1 \right) e^{\lambda_2 t} \\ = -\frac{a^2}{4} \left( a - \frac{6b}{\lambda_1 + \lambda_2} \right) (e^{\lambda_1 t} + e^{\lambda_2 t})^3 - \frac{(\lambda_1 - \lambda_2) a^2 b}{\lambda_1 \lambda_2} (\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}) (e^{\lambda_1 t} + e^{\lambda_2 t}) \quad (13)$$

The particular solution of (7)-(9) gives three unknown functions  $A_1, B_2$  and  $u_1$ , which complete the determination of the first order solution of (1).

**Example:** Let us consider a *Duffing* equation with a large linear damping,

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3, \quad (10)$$

Here  $\lambda_1 + \lambda_2 = -2k$  and  $\lambda_1 \lambda_2 = \omega^2$ .

The function  $f^{(0)}$  becomes,

$$f^{(0)} = \frac{a^3}{8} (e^{\lambda_1 t} + e^{\lambda_2 t})^3 + \frac{3a^2 b}{4} (e^{\lambda_1 t} + e^{\lambda_2 t})^2 \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \\ + \frac{3ab^2}{4} (e^{\lambda_1 t} + e^{\lambda_2 t}) \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^2 + b^3 \left( \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right)^3$$

Therefore, the nonzero coefficients of  $g_r, r = 0, 1, 2, \dots$  are

$$g_0 = \frac{a^3}{8} (e^{\lambda_1 t} + e^{\lambda_2 t})^3, \quad g_1 = \frac{3a^2 b}{4} (e^{\lambda_1 t} + e^{\lambda_2 t})^2,$$

$g_2 = \frac{3ab^2}{4} (e^{\lambda_1 t} + e^{\lambda_2 t}), g_3 = b^3$ . Substituting the values of  $g_1$  in to (8) and the values of  $g_2, g_3$  into (9) and solving them, we obtain

$$B_1 = \frac{-3a^2 b}{4} \left( \frac{e^{2\lambda_1 t}}{2\lambda_1} + \frac{2e^{(\lambda_1 + \lambda_2)t}}{\lambda_1 + \lambda_2} + \frac{e^{2\lambda_2 t}}{2\lambda_2} \right) \quad (11)$$

and

It is noted that (13) has not always an exact solution. It has an exact solution when  $\lambda_1 = \lambda_2$ . However, we can find an approximate solution of (13) when  $(\lambda_1 - \lambda_2)^2 \leq \varepsilon$ . We

neglect the last term of (13), since  $\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}$  is order of  $\lambda_1 - \lambda_2$ . Therefore, we rewrite (13) as:

$$\left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_1 t} + \left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_2 t} = -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)(e^{\lambda_1 t} + e^{\lambda_2 t})^3 \quad (14)$$

Equation (14) has again not an exact solution unless  $\lambda_1 = \lambda_2$ . Now we can start with the following equation and a trial solution as:

$$\left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_1 t} + \left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_2 t} = -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)(e^{3\lambda_1 t} + he^{(2\lambda_1 + \lambda_2)t} + he^{(\lambda_1 + 2\lambda_2)t} + e^{3\lambda_2 t}) \quad (15)$$

and  $A_1 = l_1 e^{2\lambda_1 t} + l_2 e^{(\lambda_1 + \lambda_2)t} + l_3 e^{2\lambda_2 t}$  where  $l_1, l_2, l_3$  and  $h$  are unknown. Substituting  $A_1$  into (15) and equating the coefficients of  $e^{3\lambda_1 t}, \dots, e^{3\lambda_2 t}$ , we obtain a set of algebraic equations, whose solutions are

$$l_1 = -\frac{a^2}{4(3\lambda_1 - \lambda_2)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right),$$

$$l_3 = -\frac{a^2}{4(3\lambda_2 - \lambda_1)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)$$

$$A_1 \cong -\frac{a^2}{4(3\lambda_1 - \lambda_2)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)\left(\frac{e^{2\lambda_1 t}}{(3\lambda_1 - \lambda_2)} + \frac{2(\lambda_1 + \lambda_2)e^{(2\lambda_1 + \lambda_2)t}}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} + \frac{e^{2\lambda_2 t}}{(3\lambda_2 + \lambda_1)}\right) \quad (18)$$

Now substituting the values of  $A_1$  from (18) and  $B_1$  from (11) into (4) and then integrating with respect to  $t$  by assuming that  $a$  and  $b$  are constant in the right sides of (4), we obtain

$$a \cong a_0 - \frac{\varepsilon a_0^2}{4(3\lambda_1 - \lambda_2)}\left(a_0 - \frac{6b_0}{\lambda_1 + \lambda_2}\right)\left(\frac{e^{2\lambda_1 t} - 1}{2\lambda_1(3\lambda_1 - \lambda_2)} + \frac{2(e^{(\lambda_1 + \lambda_2)t} - 1)}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} + \frac{e^{2\lambda_2 t} - 1}{2\lambda_2(3\lambda_2 + \lambda_1)}\right)$$

$$b = b_0 - \frac{3\varepsilon a_0^2 b_0}{4}\left(\frac{e^{2\lambda_1 t} - 1}{4\lambda_1^2} + \frac{2(e^{(\lambda_1 + \lambda_2)t} - 1)}{(\lambda_1 + \lambda_2)^2} + \frac{e^{2\lambda_2 t}}{2\lambda_2^2}\right) \quad (19)$$

Hence the first order solution of (10) is

$$x(t, \varepsilon) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + \varepsilon u_1(a, b, t) \quad (20)$$

where  $a$  and  $b$  are given by (19) and  $u_1$  is given by (12).

**Discussion of Murty's Unified theory:** Murty found a unified solution (for un-damped, under damped and over damped case of (1) in the form

$$x(t, \varepsilon) = \rho \cosh \psi + \varepsilon u_1(\rho, \psi) + \dots \quad (21)$$

or

$$l_2 = -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right) * \frac{\lambda_1 + \lambda_2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)},$$

$$h = \frac{3(\lambda_1 + \lambda_2)^2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} \quad (16)$$

Now we should investigate the value of  $h$ . It is obvious that as  $\lambda_1 \rightarrow \lambda_2$ ,  $h \rightarrow 3$ . When  $\lambda_1 \neq \lambda_2$ , we obtain

$$3 - h = \frac{-12(\lambda_1 - \lambda_2)^2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} \quad (17)$$

Therefore, we neglect the right hand side of (17) when  $(\lambda_1 - \lambda_2)^2 \leq \varepsilon$ . Thus an approximate solution of (14) or (13) is

$$x(t, \varepsilon) = \rho \sinh \psi + \varepsilon u_1(\rho, \psi) + \dots \quad (22)$$

where  $\rho$  and  $\psi$  satisfy the first order differential equations

$$\left. \begin{aligned} \dot{\rho} &= -k\rho + \varepsilon A_1(\rho) + \dots \\ \dot{\psi} &= \omega_0 + \varepsilon B_1(\rho) + \dots \end{aligned} \right\} \quad (23)$$

It is interesting to note that such type of unified solutions can be found from (5). In this paper, we obtain a solution of (10) in the form of (21). We rewrite (3) as

$$x(t, \varepsilon) = a(t)e^{\lambda_1 t} + b(t)e^{\lambda_2 t} + \varepsilon u_1(a, b, t) + \dots \quad (24)$$

where  $a$  and  $b$  satisfy the first order differential equations

$$\left. \begin{aligned} \dot{a} &= \varepsilon \tilde{A}_1(a, b, t) + \dots \\ \dot{b} &= \varepsilon \tilde{B}_1(a, b, t) + \dots \end{aligned} \right\} \quad (25)$$

By comparing (3) and (24), we obtain

$$a(t) = \frac{\alpha(t)}{2} + \frac{\beta(t)}{\lambda_1 - \lambda_2}, \quad b(t) = \frac{\alpha(t)}{2} - \frac{\beta(t)}{\lambda_1 - \lambda_2} \quad (26)$$

Differentiating (26) with respect to  $t$  and utilizing relations of (4) and (25), the following relations between  $A_1, B_1$  and  $\tilde{A}_1, \tilde{B}_1$  can be found:

$$\left( \frac{\partial \tilde{A}_1}{\partial t} + (\lambda_1 - \lambda_2) \tilde{A}_1 \right) e^{\lambda_1 t} + \left( \frac{\partial \tilde{B}_1}{\partial t} + (\lambda_2 - \lambda_1) \tilde{B}_1 \right) e^{\lambda_2 t} + \left( \frac{\partial}{\partial t} - \lambda_1 \right) \left( \frac{\partial}{\partial t} - \lambda_2 \right) u_1 = -f^{(0)} \quad (29)$$

According to then unified theory, the roots of the linear equation of (10) are  $\lambda_1 = -k + \omega_0$  and  $\lambda_2 = -k - \omega_0$ , so that  $f^{(0)} = e^{-3kt} (a^3 e^{3\omega_0 t} + 3a^2 b e^{\omega_0 t} + 3ab^2 e^{-\omega_0 t} + b^3 e^{-3\omega_0 t})$ . Moreover, in accordance to KBM method,  $u_1$  does not contain terms with  $e^{\omega_0 t}$  and  $e^{-\omega_0 t}$ . Substituting the values of  $\lambda_1, \lambda_2$  and  $f^{(0)}$  into (29) and assuming that  $u_1$  excludes the terms with  $e^{\omega_0 t}$  and  $e^{-\omega_0 t}$ , we obtain

$$\left( \frac{\partial \tilde{A}_1}{\partial t} + 2\omega_0 \tilde{A}_1 = -3a^2 b e^{-2kt} \right) \quad (30)$$

$$\left( \frac{\partial \tilde{B}_1}{\partial t} + 2\omega_0 \tilde{B}_1 = -3ab^2 e^{-2kt} \right) \quad (31)$$

and

$$\left( \frac{\partial}{\partial t} + k - \omega_0 \right) \left( \frac{\partial}{\partial t} + k + \omega_0 \right) u_1 = -e^{-3kt} (a^3 e^{3\omega_0 t} + b^3 e^{\omega_0 t}) \quad (32)$$

Solving (30)-(32), we obtain

$$\tilde{A}_1 = \frac{3a^2 b e^{-2kt}}{2(k - \omega_0)}, \quad \tilde{B}_1 = \frac{3ab^2 e^{-2kt}}{2(k + \omega_0)} \quad (33)$$

and

$$u_1 = -\frac{e^{-3kt}}{4} \left( \frac{a^3 e^{3\omega_0 t}}{(k - \omega_0)(k - 2\omega_0)} + \frac{b^3 e^{-3\omega_0 t}}{(k + \omega_0)(k + 2\omega_0)} \right) \quad (34)$$

Substituting the values of  $\tilde{A}_1$  and  $\tilde{B}_1$  from (33) into (25), we obtain

$$\tilde{A}_1 = \frac{A_1}{2} + \frac{B_1}{\lambda_1 - \lambda_2}, \quad \tilde{B}_1 = \frac{A_1}{2} - \frac{B_1}{\lambda_1 - \lambda_2} \quad (27)$$

or,

$$A_1 = \tilde{A}_1 + \tilde{B}_1, \quad B_1 = \frac{1}{2}(\lambda_1 - \lambda_2)(\tilde{A}_1 + \tilde{B}_1) \quad (28)$$

Substituting the values of  $A_1$  and  $B_1$  from (28) into (5) and simplifying, we obtain

$$\dot{a} = \frac{3\varepsilon a^2 b e^{-2kt}}{2(k - \omega_0)}, \quad \dot{b} = \frac{3\varepsilon a b^2 e^{-2kt}}{2(k + \omega_0)} \quad (35)$$

Equations of (35) have exact solutions. These equations reduce to

$$\dot{r} = \frac{3\varepsilon k r^3 e^{-2kt}}{8\omega^2}, \quad \dot{\phi} = \frac{3\varepsilon \omega_0 r^2 e^{-2kt}}{8\omega^2} \quad (36)$$

under the transformations  $a = \frac{1}{2} r e^\phi, b = \frac{1}{2} r e^{-\phi}$ .  $u_1$  in (34) become

$$u_1 = \frac{r^3 e^{-3kt} \left( (k^2 + 2\omega_0^2) \cosh 3(\omega t + \phi) + 3k\omega \sinh 3(\omega t + \phi) \right)}{16\omega^2 (k^2 - 4\omega_0^2)} \quad (37)$$

On the other hand, under the transformations (24) becomes

$$x(t, \varepsilon) = r e^{-kt} \cosh(\omega_0 t + \phi) + \varepsilon u_1 \quad (38)$$

where  $u_1$  is given by (37),  $r$  and  $\phi$  given by (36). Replacing  $\rho = r e^{-kt}$  and  $\psi = \omega_0 t + \phi$ , we can show that (38) is a unified solution of (10)<sup>(1)</sup>. Similarly, we can find the second unified solution of the form (22) from (5).

### 3. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison con-

cerning the presented KBM method of this article, we refer to the works of Murty *et,al* [5] (who found an over-damped solution of a second order nonlinear system with constant coefficients), and Shamsul [6-7]. In our present paper, for different initial conditions, we have compared the perturbation solutions (16) of *Duffing's* equations (7) to those obtained by Runge-Kutta Fourth-order procedure.

First of all,  $x$  is calculated by (20) with initial conditions  $x(0)=1.00000$   $\dot{x} = 0.00000$  for  $\lambda_1 = -1.162$ ,  $\lambda_2 = -.8612$  and  $\varepsilon = .1$ . The corresponding numerical

solution is also computed by Runge-Kutta fourth-order method and is given in the third column of the Table 1. Moreover  $x$  is calculated by (38). All the results are shown in Table 1. Percentage errors have also been calculated and given in the fourth column and sixth column of the Table 1. From Table 1, we see that errors for unified solution (38) occur more than 22%, while for the asymptotic solution (20), percentage errors are less than 1.25%. However, when the deference of two roots is much smaller than unity errors occur only 1% (Table 3).

Table 1

$t$	$x_p$	$x_{nu}$	$E^{(1)}\%$	$x_u$	$E^{(2)}\%$
0.0	1.00000	1.00000	0.00000	1.00000	0.00000
1.0	0.707868	0.716179	-1.1605	0.715523	-0.0916
2.0	0.379374	0.381757	-0.6242	0.338067	-11.4445
3.0	0.183567	0.183943	-0.2044	0.152127	-17.2967
4.0	0.084622	0.084564	0.0686	0.067037	-20.7263
5.0	0.037945	0.037851	0.2483	0.029178	-22.9135

Table 2

$t$	$x_p$	$x_{nu}$	$E^{(1)}\%$
0.0	1.00000	1.00000	0.00000
1.0	0.708669	0.715368	-0.9364
2.0	0.378369	0.379474	-0.2912
3.0	0.181542	0.181205	0.1860
4.0	0.082642	0.082230	0.5010
5.0	0.036457	0.036198	0.7155

### 4. Conclusion

An asymptotic solution has been obtained for certain over-damped nonlinear systems, which has been found in the sense of extended KBM method, shows a good coincidence with the numerical solution.

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