Pure and Applied Mathematics Journal
2013; 2(6): 184-190
Published online January 10, 2014 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20130206.13

A nonexistence of solutions to a supercritical problem

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To cite this article:
Kamal Ould Bouh. A Nonexistence of Solutions to a Supercritical Problem. *Pure and Applied Mathematics Journal.* Vol. 2, No. 6, 2013, pp. 184-190. doi: 10.11648/j.pamj.20130206.13

Abstract: In this paper, we study the nonlinear elliptic problem involving nearly critical exponent (P_ϵ) : $-\Delta u = K\, u^{\frac{n+2}{n-2}+\epsilon}$ $in\ \Omega\ ;\ u>0\ \ in\ \ \Omega\ and\ u=0\ on\ \partial\Omega$ where Ω is a smooth bounded domain in IR^n, $n \geq 3$, K is a C^3positive function and ϵ is a small positive real parameter. We prove that, for ε small, (P_ε) has no positive solutions which blow up at one critical point of the function *K*.

Keywords: Nonlinear Elliptic Equations, Critical Exponent, Variational Problem

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. We consider the following nonlinear elliptic problem

$$(P_\epsilon)\quad \begin{cases} -\Delta u = K\, u^{p+\epsilon} & in\ \ \Omega \\ u\ >0 & in\ \ \Omega \\ u=0 & on\ \partial\Omega \end{cases}$$

where K is a C^3positive function, $p+1 = 2n/n-2$ is the critical Sobolev exponent and ϵ is a small positive real parameter.

Problem(P_ϵ) is in some sense related to the limiting problem(when $\epsilon = 0$) and the interest to it comes from its resemblance to the scalar curvature problem in differential geometry, which consists in finding suitable conditions on a given function K defined on M such that K is the scalar curvature for a metric $\bar{g}$ conformally equivalent to g, where (M, g) is a n-dimensional Riemannian manifold without boundary.

Note that the limiting problem has bee1.5

n widely studied in various works see for example [1], [2], [7] and [10].

In another view point, it is interesting to study the problem (P_ϵ) with $\epsilon< 0$ and $\epsilon > 0$ and to understand what happens to the solutions of (P_ϵ) (if they exist) as $\epsilon \rightarrow 0$!!

When $\epsilon \in (1-p, 0)$, the mountain pass lemma proves the existence of solutions of (P_ϵ) (see [3]). Note that, many works have been devoted to the study of positive solutions of (P_ϵ) with $\epsilon < 0$. In sharp contrast to this, very little study has been made concerning the sign-changing solutions of (P_ϵ) with $\epsilon < 0$ and even less for $\epsilon > 0$.

When $\epsilon > 0$, problem(P_ϵ) becomes more delicate since we loose the Sobolev embedding which is an important difficulty to overcome.

Concerning the supercritical case, $\epsilon > 0$ and K is a constant, it was proved in [4] that (P_ϵ) has no positive solution which blows up at a single point. This result shows that the situation is different from the subcritical one. However, del Pino et al [6] gave an existence result for two blow up points, provided that Ω satisfies some geometrical conditions. In sharp contrast to this, it proved in [5] for the case K is a constant and [8] for the case K is a non constant function that, for ϵ small, (P_ϵ) has no sign-changing solutions which blow up at two points.

In this paper, we consider the case K is a non constant function and we look to understand the influence of the function K in the study of the positive solutions of (P_ϵ) which blows up at a single point.

It is well known that problem (P_ϵ) has a variational structure. Setting

$$J(u) = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega K|u|^{p+1+\epsilon}\right)^{\frac{2}{p+1+\epsilon}}},\ u\ \epsilon\ H_0^1(\Omega)\,,u\ \not\equiv 0\,,$$

the positive critical points of J are solutions to (P_ϵ) , up to a multiplicative constant. J satisfies the Palais-Smale condition in the subcritical case, whereas this condition fails in the critical case. Such a failure is due to the function

$$\delta_{(a,\lambda)}(x) = C_0\, \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-2}{2}}}\ ,\ C_0 = (n(n-2))^{\frac{n-2}{4}}, \lambda > 0\,, a \in \mathbb{R}^n \quad (1)$$

which are the only solutions of

$-\Delta u = u^{\frac{n+2}{n-2}}$, $u > 0\ in\ \mathbb{R}^n$, with $u\,\epsilon\ \ L^{p+1}(\mathbb{R}^n)$ and $\nabla u\ \epsilon\ L^2(\mathbb{R}^n)$ and are also the only minimizers of the

Sobolev inequality on the whole space, that is

$$S = |u|^{-2}_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}, s.t\ \nabla u \in L^2, u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n), u \neq 0\} \quad (2)$$

We have the following nonexistence result for (P_ϵ):

Theorem 1

Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. Assume that $a_0 \in \Omega$ is a critical point of K satisfying one of the following conditions:

(i) $n = 3$,

(ii) $n = 4, and\ c_1 H(a_0, a_0) - \frac{c_3 \Delta K(a_0)}{16K(a_0)} > 0$,

(iii) $n \geq 5, and\ -\Delta K(a_0) > 0$.

Then the problem (P_ϵ) has no solution u_ϵ such that $u_\epsilon = \alpha_\epsilon P\delta_{a_\epsilon,\lambda_\epsilon} + v_\epsilon$ with $|u_\epsilon|^\epsilon$ is bounded and $v_\epsilon \to 0\ in\ H_0^1(\Omega)$ $\alpha_\epsilon \to K(a)^{(2-n)/4}$, $a_\epsilon \in \Omega$, $a_\epsilon \to a_0$ and $\lambda_\epsilon d(a_\epsilon, \partial\Omega) \to +\infty$ $as\ \epsilon \to 0$.

2. Preliminary Results

We need to introduce some notations:

$P\delta_{a,\lambda}$ is defined as the only function in $H_0^1(\Omega)$ such that $\Delta P\delta_{a,\lambda} = \Delta\delta_{a,\lambda}$. Writing

$$P\delta_{a,\lambda} = \delta_{a,\lambda} - \theta_{a,\lambda} \quad (3)$$

we have

$$\Delta\theta_{a,\lambda} = 0 \text{ in } \Omega\ ;\ \theta_{a,\lambda} = \delta_{a,\lambda} \text{ on } \partial\Omega \quad (4)$$

We note that projections $P\delta_{a,\lambda}$ of $\delta_{a,\lambda}$'s on $H_0^1(\Omega)$ are approximate solutions to the limiting problem as $a_\epsilon \in \Omega$ and $\lambda_\epsilon d(a_\epsilon, \partial\Omega)$ goes to infinity.

Let G be the Green's function for the Laplace operator with Dirichlet boundary conditions, that is, for any $x \in \Omega$.

$$\begin{cases} -\Delta G(x,.) = c_n\,\delta_x\ in\ \Omega \\ G(x,.) = 0\ on\ \ \partial\Omega \end{cases}$$

with δ_x the Dirac mass at x and $c_n = (n-2)|S^{n-1}|$

We denote by H the regular part of G , i.e .

$$H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2)$$

$$\text{for } (x_1, x_2) \in \Omega \times \Omega$$

The maximum principle provides us with the uniform estimate

$$\theta_{a,\lambda}(x) = C_0 \frac{H(x,a)}{\lambda^{\frac{n-2}{2}}} + O\left(\frac{1}{\lambda^{\frac{n+2}{2}}(d(a,\partial\Omega))^n}\right) as \lambda d(a,\partial\Omega) \to +\infty \quad (5)$$

Corresponding estimates hold for the derivatives of $\theta_{a,\lambda}$ with respect to a, λ and x .

Note that $H(x,x) = O(d(x,\partial\Omega)^{2-n})$ as $d(x,\partial\Omega) \to 0$ [9]. From [9] we also know that

$$\int_\Omega |\nabla\theta_{a,\lambda}|^2 = O(\lambda d(a,\partial\Omega)^{2-n}) \text{as} \lambda d(a,\partial\Omega) \to +\infty \quad (6)$$

Next, we recall that for u_ϵ satisfying the assumption of the theorem, there is a unique way to choose $a_\epsilon, \lambda_\epsilon$ and v_ϵ such that

$$u_\epsilon = \alpha_\epsilon P\delta_{a_\epsilon,\lambda_\epsilon} + v_\epsilon \quad (7)$$

with

$$\begin{cases} \alpha_\epsilon \in \mathbb{R},\ \ \alpha_\epsilon \to K(\alpha_\epsilon)^{(2-n)/4} \\ a_\epsilon \in \Omega,\ \lambda_\epsilon \in \mathbb{R}_+^*,\ \lambda_\epsilon d(a_\epsilon, \partial\Omega) \to +\infty \\ v_\epsilon \to 0 \quad in \quad H_0^1(\Omega),\ v_\epsilon \in E_{a_\epsilon,\lambda_\epsilon} \end{cases} \quad (8)$$

and for any $(a, \lambda) \in \Omega \times \mathbb{R}_+^*$, $E_{(a,\lambda)}$ denotes the subspace of $H_0^1(\Omega)$ defined by

$$E_{(a,\lambda)} = \{ w \in H_0^1(\Omega) / (\mathrm{w}, P\delta_{(a,\lambda)})_{H_0^1} = \left(w, \frac{\partial P\delta_{(a,\lambda)}}{\partial\lambda}\right)_{H_0^1}$$

$$= \left(w, \frac{\partial P\delta_{(a,\lambda)}}{\partial a_i}\right)_{H_0^1} = 0\ , 1 \leq i \leq n\ \}$$

For the proof of this fact, see [1], [9]. In the following, we always assume that u_ϵ, satisfying the assumption of the theorem, is written as in (8). In order to simplify the notations, we set

$$\delta_{a_\epsilon,\lambda_\epsilon} = \delta_\epsilon, P\delta_{a_\epsilon,\lambda_\epsilon} = P\delta_\epsilon \text{ and } \theta_{a_\epsilon,\lambda_\epsilon} = \theta_\epsilon$$

Lemma 2

Let u_ϵ satisfying the assumption of the theorem 1. Then

$$(i)\ \int_\Omega |\nabla u_\epsilon|^2 \to S^{n/2} \quad ; \quad (ii)\ \int_\Omega K\, {u_\epsilon}^{p+1+\epsilon} \to S^{n/2}$$

as $\epsilon \to o$, S, S denoting the Sobolev constant defined by (2).

Proof.

We have

$\int_\Omega |\nabla u_\epsilon|^2 = \int_\Omega |\nabla(\alpha_\epsilon P\delta_\epsilon + v_\epsilon)|^2 = {\alpha_\epsilon}^2 \int_\Omega |\nabla P\delta_\epsilon|^2 + \int_\Omega |\nabla v_\epsilon|^2$ since $v_\epsilon \in E_{a_\epsilon,\lambda_\epsilon}$

From the fact that δ_ϵ satisfies $-\Delta\delta_\epsilon = \delta_\epsilon^P\ in\ \mathbb{R}^n$ and is a minimizer for S, we deduce that $\int_{\mathbb{R}^n} |\nabla\delta_\epsilon|^2 = S^{n/2}$

On the other hand, an explicit computation provides us with

$$\int_\Omega |\nabla\delta_{a,\lambda}|^2 = \int_{\mathbb{R}^n} |\nabla\delta_{a,\lambda}|^2 + O\left(\frac{1}{(\lambda d(a,\partial\Omega))^n}\right) as\ \lambda d(a,\partial\Omega) \to +\infty.$$

Taking account of (6), claim (i) is a consequence of (8). Claim (ii) follows from the fact that u_ϵ solves (P_ϵ).

3. Estimating v_ϵ

As usual in this type of problems, we first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon.

Lemma 3

Let u_ϵ satisfying the assumption of the theorem. λ_ϵ occurring in (7) satisfies

$$\lambda_\epsilon^\epsilon \to 1, as \quad \epsilon \to 0 .$$

Proof.

According to Lemma 2, we have

$$\int_\Omega K\, u_\epsilon{}^{p+1+\epsilon} = S^{n/2} + o(1)\ as \quad \epsilon \to 0 \quad (9)$$

and

$$\int_\Omega K\, u_\epsilon{}^{p+1+\epsilon} = \int_\Omega K\,(\alpha_\epsilon\, P\delta_\epsilon + v_\epsilon)^{p+\epsilon}\alpha_\epsilon\, P\delta_\epsilon$$

$$+ \int_\Omega K\, u_\epsilon{}^{p+\epsilon} v_\epsilon$$

$$= \alpha_\epsilon{}^{p+1+\epsilon}\int_\Omega K\ \ P\delta_\epsilon{}^{p+\epsilon+1}$$

$$- \int_\Omega \Delta\, u_\epsilon\, v_\epsilon$$

$$+ O\left(\int_\Omega P\delta_\epsilon{}^{p+\epsilon}|v_\epsilon| + \int_\Omega |v_\epsilon|^{p+\epsilon} P\delta_\epsilon\right)$$

$$= \alpha_\epsilon{}^{p+1+\epsilon}\int_\Omega K\ \ P\delta_\epsilon{}^{p+\epsilon+1}$$

$$+ O\left(\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}\int_\Omega P\delta_\epsilon{}^{p}|v_\epsilon| + \lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}\int_\Omega |v_\epsilon|^{p+\epsilon} P\delta_\epsilon{}^{1-\epsilon} + |v_\epsilon|_{H_0^1}\right)$$

$$= \alpha_\epsilon{}^{p+1+\epsilon}\int_\Omega K\ \ P\delta_\epsilon{}^{p+\epsilon+1}$$

$$+ O\left(\lambda_\epsilon^{\epsilon(n-2)/2}|v_\epsilon|_{L^{p+1}} + \lambda_\epsilon^{\epsilon(n-2)/2}|v_\epsilon|_{L^{p+1}}^{p+\epsilon} + |v_\epsilon|_{H_0^1}\right)$$

Thus

$$\int_\Omega K u_\epsilon{}^{p+1+\epsilon} = \alpha_\epsilon{}^{p+1+\epsilon}\int_\Omega K\ \ P\delta_\epsilon{}^{p+\epsilon+1} + o\left(\lambda_\epsilon^{\epsilon(n-2)/2} + 1\right) (10)$$

We observe that

$$\int_\Omega K\ \ P\delta_\epsilon{}^{p+1+\epsilon} = \int_\Omega K\ \ (\delta_\epsilon - \theta_\epsilon)^{p+\epsilon+1} = \int_\Omega K\ \ \delta_\epsilon{}^{p+1+\epsilon} + O\left(\int_\Omega\ \delta_\epsilon{}^{p+\epsilon}\theta_\epsilon\right)$$

$$= C_0^{p+\epsilon+1}\, K(a_\epsilon)\int_{\mathrm{B}}\left(\frac{\lambda_\epsilon}{1+\lambda_\epsilon^2|x-a_\epsilon|^2}\right)^{\frac{(p+1+\epsilon)(n-2)}{2}} +$$

$$O\left(|\theta_\epsilon|_{L^\infty}\int_\Omega\left(\frac{\lambda_\epsilon}{1+\lambda_\epsilon^2|x-a_\epsilon|^2}\right)^{\frac{(p+1+\epsilon)(n-2)}{2}} + \frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}}{(\lambda_\epsilon d_\epsilon)^n}\right)$$

where $B = B(a_\epsilon, d_\epsilon)$ .Using Proposition 1 of [9], we obtain

$$\int_\Omega K\ \ P\delta_\epsilon{}^{p+1+\epsilon}$$

$$= \lambda_\epsilon^{\frac{\epsilon(n-2)}{2}} K(a_\epsilon)\left(C_0^{p+\epsilon+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{\frac{(p+1+\epsilon)(n-2)}{2}}} - C_0^{p+\epsilon+1}\int_{\mathbb{R}^n/B}\frac{dx}{(1+|x|^2)^{\frac{(p+1+\epsilon)(n-2)}{2}}}\right)$$

$$+ O\left(\frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}}{(\lambda_\epsilon d_\epsilon)^{n-2}}\right)$$

$$= \lambda_\epsilon^{\epsilon(n-2)/2} K(a_\epsilon) C_0^{p+\epsilon+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(p+1+\epsilon)(n-2)/2}}$$

$$+ O\left(\frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}}{(\lambda_\epsilon d_\epsilon)^{n-2}}\right)$$

We note that

$$C_0^{p+\epsilon+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(p+1+\epsilon)(n-2)/2}} = C_0^{p+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^n} + O(\epsilon) = S^{n/2} + O(\epsilon).$$

Therefore

$$\int_\Omega K\ \ P\delta_\epsilon{}^{p+1+\epsilon} = \lambda_\epsilon^{\epsilon(n-2)/2} K(a_\epsilon)(S^{n/2} + O(\epsilon) + o(1) \quad (11)$$

so (10) and (11) provide us with

$$\int_\Omega K u_\epsilon{}^{p+1+\epsilon} = \alpha_\epsilon{}^{p+1+\epsilon}\,\lambda_\epsilon^{\epsilon(n-2)/2} K(a_\epsilon)(S^{n/2} + o(1)) + o(1) \qquad (12)$$

Combination of (9) and (12) proves the lemma.

Next, we recall the following estimate [10] :

Remark 4

$$\delta_\epsilon^\epsilon(x) - C_0^\epsilon\lambda_\epsilon^{\epsilon(n-2)/2} = O(\epsilon\log(1+\lambda_\epsilon^2|x-a_\epsilon|^2)) \quad in \quad \Omega.$$

We are now able to study the v_ϵ -part of u_ϵ .

Lemma 5

Let u_ϵ satisfying the assumption of the theorem. v_ϵ occurring in (7) satisfies

$$|v_\epsilon|_{H_0^1(\Omega)} \le C + C\begin{cases} \dfrac{\nabla K(a_\epsilon)}{\lambda_\epsilon} + \dfrac{1}{(\lambda_\epsilon d_\epsilon)^{n-2}} & if \quad n < 6 \\ \dfrac{\nabla K(a_\epsilon)}{\lambda_\epsilon} + \dfrac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^4} & if \quad n = 6 \\ \dfrac{\nabla K(a_\epsilon)}{\lambda_\epsilon} + \dfrac{1}{(\lambda_\epsilon d_\epsilon)^{(n+2)/2}} & if \quad n > 6 \end{cases}$$

with C independent of ϵ .

Proof.

Multiplying (P_ϵ) by v_ϵ and integrating on Ω , we obtain

$$0=\int_\Omega \nabla u_\epsilon . \nabla v_\epsilon - \int_\Omega K u_\epsilon^{p+\epsilon}\, v_\epsilon$$

Thus

$$0=\int_\Omega |\nabla v_\epsilon|^2 - \int_\Omega K\big[(\alpha_\epsilon P\delta_\epsilon)^{p+\epsilon} + (p+\epsilon)(\alpha_\epsilon P\delta_\epsilon)^{p-1+\epsilon} v_\epsilon + O(\delta_\epsilon^{p-2+\epsilon} v_\epsilon^2 \chi_{|v_\epsilon|<\delta_\epsilon} + |v_\epsilon|^{p+\epsilon})\big]\, v_\epsilon .$$

Using the assumption that $|u_\epsilon|^\epsilon$ is bounded, we find

$$0 = Q_\epsilon(v_\epsilon, v_\epsilon) - f_\epsilon(v_\epsilon) + O\left(|v_\epsilon|_{H_0^1}^{\min(3,p+1)} + |v_\epsilon|_{H_0^1}^{p+1}\right) \qquad (13)$$

with

$$Q_\epsilon(v,v) = |v_\epsilon|_{H_0^1}^2 - (p+\epsilon)\int K(\alpha_\epsilon P\delta_\epsilon)^{p-1+\epsilon}\, v^2$$

and

$$f_\epsilon(v) = \int K(\alpha_\epsilon P\delta_\epsilon)^{p+\epsilon}\, v .$$

We observe that

$$\begin{aligned} Q_\epsilon(v,v) &= |v_\epsilon|_{H_0^1}^2 - p\int_\Omega K(\alpha_\epsilon P\delta_\epsilon)^{p-1+\epsilon}\, v^2 + O\left(\epsilon|v_\epsilon|_{H_0^1}^2\right) \\ &= |v_\epsilon|_{H_0^1}^2 - p\alpha_\epsilon^{p-1+\epsilon} K(a_\epsilon)\int_\Omega \left({\delta_\epsilon}^{p-1+\epsilon} + O\big({\delta_\epsilon}^{p-1+\epsilon}\theta_\epsilon\big)\right) v^2 + o\left(|v|_{H_0^1}^2\right) \\ &= |v_\epsilon|_{H_0^1}^2 - p\alpha_\epsilon^{p-1+\epsilon} K(a_\epsilon) C_0^\epsilon \lambda_\epsilon^{\epsilon(n-2)/2}\int_\Omega {\delta_\epsilon}^{p-1} v^2 + O\left(\int_\Omega \left({\delta_\epsilon}^{p-1+\epsilon} - C_0^\epsilon\lambda_\epsilon^{\epsilon(n-2)/2}{\delta_\epsilon}^{p-1}|v|^2\right)\right) + o\left(|v|_{H_0^1}^2\right) \end{aligned}$$

Using Remark 4, we find

$$Q_\epsilon(v,v) = Q_0(v,v) + o\left(|v|_{H_0^1}^2\right) \qquad \text{with}$$

$$Q_0(v,v) = |v|_{H_0^1}^2 - \int_\Omega {\delta_\epsilon}^{p-1}\, v^2 .$$

According to [1], Q_0 is coercive, that is, there exists some constant c >0 independent of ϵ, for small enough, such that

$$Q_0(v,v) \ge c|v|_{H_0^1}^2 \qquad \forall\, v\in E_{(a_\epsilon,\lambda_\epsilon)}. \qquad (14)$$

We also observe that

$$\begin{aligned} f_\epsilon(v) &= \alpha_\epsilon^{p+\epsilon}\int_\Omega K\left({\delta_\epsilon}^{p+\epsilon} + O\big({\delta_\epsilon}^{p-1+\epsilon}\theta_\epsilon\big)\right) v \\ &= \alpha_\epsilon^{p+\epsilon}\left[C_0^\epsilon\lambda_\epsilon^{\epsilon(n-2)/2}\int_\Omega K{\delta_\epsilon}^{p} v + O\left(\epsilon\int_\Omega K\log(1+\lambda_\epsilon^2|x-a_\epsilon|^2)\,\delta_\epsilon^2|v| + \int_\Omega {\delta_\epsilon}^{p-1}\theta_\epsilon|v|\right)\right] \end{aligned}$$

The last equality follows from Remark 4. Therefore we can write,

with $B = B(a_\epsilon, d_\epsilon)$

$$f_\epsilon(v) = O\left(\epsilon|v|_{H_0^1} + \int_B {\delta_\epsilon}^{p-1}\theta_\epsilon|v_\epsilon| + \int_{\mathbb{R}^n\setminus B} {\delta_\epsilon}^{p}|v|\right)$$

$$f_\epsilon(v) = O\left(\left(\epsilon + \frac{|\nabla K(a_\epsilon)|}{\lambda_\epsilon^2}\right)|v|_{H_0^1} + |v|_{H_0^1}|\theta_\epsilon|_{L^\infty}\left(\int_B \delta_\epsilon^{\frac{8n}{n^2-4}}\right)^{\frac{n+2}{2n}} + |v_\epsilon|_{H_0^1}\left(\int_{\mathbb{R}^n\setminus B}\delta_\epsilon^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}}\right)$$

We notice that

$$\int_{\mathbb{R}^n\setminus B} \delta_\epsilon^{2n/(n-2)} = O\left(\frac{1}{(\lambda_\epsilon d_\epsilon)^n}\right)$$

and

$$\left(\int_B \delta_\epsilon^{\frac{8n}{n^2-4}}\right)^{(n+2)/2n} \le C\begin{cases} \dfrac{d_\epsilon^{(n-6)/2}}{\lambda_\epsilon^2} & if \quad n>6 \\ \dfrac{\log(\lambda_\epsilon d_\epsilon)}{\lambda_\epsilon^2} & if \quad n=6 \\ \dfrac{1}{\lambda_\epsilon^{(n-2)/2}} & if \quad n<6 \end{cases}$$

Using (5), we obtain

$$|f_\epsilon(v)| \le \; C|v|_{H_0^1} + C\begin{cases} \left(\frac{|\nabla K(a_\epsilon)|}{\lambda_\epsilon} + \frac{1}{(\lambda_\epsilon d_\epsilon)^{n-2}}\right)|v|_{H_0^1} & if \quad n<6 \\ \left(\frac{|\nabla K(a_\epsilon)|}{\lambda_\epsilon} + \frac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^4}\right)|v|_{H_0^1} & if \quad n=6 \\ \left(\frac{|\nabla K(a_\epsilon)|}{\lambda_\epsilon} + \frac{1}{(\lambda_\epsilon d_\epsilon)^{(n+2)/2}}\right)|v|_{H_0^1} & if \quad n>6 \end{cases} \qquad (15)$$

Combining (13), (14) and (15), we obtain the desired estimate.

4. Proof of Theorem

Let us start by proving the following crucial result :

Proposition 6

Let u_ϵ satisfying the assumption of the theorem. Then,

$$\left|\alpha_\epsilon c_1 \frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{n-2}} - \alpha_\epsilon \frac{c_3}{n^2}\frac{\Delta K(a_\epsilon)}{K(a_\epsilon)\lambda_\epsilon^2} + \alpha_\epsilon c_2 \varepsilon\right| \le c\left(\epsilon^2 + \frac{1}{\lambda_\epsilon^3} + |v|_{H_0^1}^2\right) + \begin{cases} \frac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n} & (n \ge 4) \\ \frac{1}{(\lambda_\epsilon d_\epsilon)^2} & (n = 3) \end{cases} \quad (16)$$

where $a_\epsilon, \lambda_\epsilon$ and $d_\epsilon = (a_\epsilon, \partial\Omega)$ are given in (7) and c_1, c_2, c_3 are positive constants defined by

$$c_1 = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+2)/2}} \quad ,$$

$$c_2 = \frac{n-2}{2} c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \log(1+|x|^2) \frac{|x|^2-1}{(1+|x|^2)^{n+1}} dx$$

and $$c_3 = c_0^{\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} dx .$$

Proof.

Multiplying (P_ϵ) by $\lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda}$ and integrating on Ω, we obtain

$$0 = -\int_\Omega \Delta u_\epsilon \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} - \int_\Omega K u_\epsilon^{p+\epsilon} \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda}$$
$$= \int_\Omega \nabla(\alpha_\epsilon P\delta_\epsilon + v_\epsilon)\nabla\left(\lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda}\right) - \int_\Omega K(\alpha_\epsilon P\delta_\epsilon + v_\epsilon)^{p+\epsilon} \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda}$$
$$= \alpha_\epsilon \int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} - \int_\Omega K\big[(\alpha_\epsilon P\delta_\epsilon)^{p+\epsilon} + (p+\epsilon)(\alpha_\epsilon P\delta_\epsilon)^{p-1+\epsilon} v + O\big(\delta_\epsilon^{p-2+\epsilon}|v_\epsilon|^2 + |v_\epsilon|^{p+\epsilon}\big)\big] \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} . \quad (17)$$

We estimate each term of the right hand side in (17). First, we have

$$\int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} = \int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial\lambda} - \int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial \theta_\epsilon}{\partial\lambda}$$

whence

$$\int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} = \int_{\mathbb{R}^n} \delta_\epsilon^p \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial\lambda} - \int_{\mathbb{R}^n\setminus\Omega} \delta_\epsilon^p \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial\lambda} - \int_B \delta_\epsilon^p \lambda_\epsilon \frac{\partial \theta_\epsilon}{\partial\lambda} - \int_{\Omega\setminus B} \delta_\epsilon^p \lambda_\epsilon \frac{\partial \theta_\epsilon}{\partial\lambda}$$
$$= O\left(\frac{1}{(\lambda_\epsilon d_\epsilon)^n}\right) - \lambda_\epsilon \frac{\partial\theta_\epsilon}{\partial\lambda}(a_\epsilon) \int_B \delta_\epsilon^p + O\left(\lambda_\epsilon \int_B \delta_\epsilon^p |x - a_\epsilon|^2 \sup_B \left|D_x^2 \frac{\partial\theta_\epsilon}{\partial\lambda}\right|\right)$$

with B = (a , d) . According to [9], we have

$$\lambda_\epsilon \frac{\partial\theta_\epsilon}{\partial\lambda}(a_\epsilon) = -\frac{n-2}{2} \frac{c_0}{\lambda_\epsilon^{(n-2)/2}} H(a_\epsilon, a_\epsilon) + O\left(\frac{1}{\lambda_\epsilon^{(n+2)/2} d_\epsilon^n}\right)$$

and

$$\sup_B \left|D_x^2 \frac{\partial\theta_\epsilon}{\partial\lambda}\right| = O\left(\frac{1}{\lambda_\epsilon^{n/2} d_\epsilon^n}\right)$$

Therefore, estimating the integrals we obtain

$$\int_\Omega \delta_\epsilon^p \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} = \frac{n-2}{2} c_1 \frac{H(a_\epsilon, a_\epsilon)}{\lambda_\epsilon^{n-2}} + O\left(\frac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n}\right) \quad (18)$$

Secondly, we compute

$$\int_\Omega K(P\delta_\epsilon)^{p+\epsilon} \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda} = \int_\Omega K\left[\delta_\epsilon^{p+\epsilon} - (p+\epsilon)\delta_\epsilon^{p-1+\epsilon}\theta_\epsilon + O\big(\theta_\epsilon^2 \delta_\epsilon^{p-2+\epsilon} + \theta_\epsilon^{p+\epsilon}\big)\right] \lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial\lambda}$$
$$= \int_B K\delta_\epsilon^{p+\epsilon} \lambda_\epsilon \frac{\partial\delta_\epsilon}{\partial\lambda} - \int_B K\delta_\epsilon^{p+\epsilon} \lambda_\epsilon \frac{\partial\theta_\epsilon}{\partial\lambda} - (p+\epsilon)\int_B K\delta_\epsilon^{p-1+\epsilon}\theta_\epsilon \lambda_\epsilon \frac{\partial\delta_\epsilon}{\partial\lambda}$$
$$+ O\left(\int_\Omega \theta_\epsilon^2 \delta_\epsilon^{p-1+\epsilon} + \int_\Omega \delta_\epsilon^{p-1+\epsilon}\theta_\epsilon \left|\lambda_\epsilon \frac{\partial\theta_\epsilon}{\partial\lambda}\right| + \int_\Omega \theta_\epsilon^{p+\epsilon}\delta_\epsilon + \frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}}{(\lambda_\epsilon d_\epsilon)^n}\right) \quad (19)$$

and we have to estimate each term of the right hand side of (18). Using the fact that $\lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda} = \frac{n-2}{2}\left(\frac{1-\lambda_\epsilon^2|x-a_\epsilon|^2}{1+\lambda_\epsilon^2|x-a_\epsilon|^2}\right)\delta_\epsilon$, we derive that

$$\int_B K{\delta_\epsilon}^{p+\epsilon}\lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda}$$

$$= \frac{n-2}{2}\lambda_\epsilon^{\frac{\epsilon(n-2)}{n}} K(a_\epsilon)c_0^{p+1+\epsilon}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{n+\frac{\epsilon(n-2)}{n}}}\frac{1-|x|^2}{1+|x|^2}dx$$

$$+O\left(\frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{n}}}{(\lambda_\epsilon d_\epsilon)^n}\right)$$

$$= \lambda_\epsilon^{\frac{\epsilon(n-2)}{n}}\left(c_2K(a_\epsilon)\epsilon - \alpha_\epsilon \frac{c_3}{n^2}\frac{\Delta K(a_\epsilon)}{\lambda_\epsilon^2} + O\left(\epsilon^2+\frac{1}{\lambda_\epsilon^3}\right)\right) + O\left(\frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{n}}}{(\lambda_\epsilon d_\epsilon)^n}\right) \qquad (20)$$

For the other terms in (19), we write

$$\int_B K{\delta_\epsilon}^{p+\epsilon}\lambda_\epsilon\frac{\partial\theta_\epsilon}{\partial\lambda} = K(a_\epsilon)\lambda_\epsilon\frac{\partial\theta_\epsilon}{\partial\lambda}(a_\epsilon)\int_B {\delta_\epsilon}^{p+\epsilon} + O\left(\int_B {\delta_\epsilon}^{p+\epsilon}\frac{|x-a_\epsilon|^2}{\lambda_\epsilon^{(n-2)/2}d_\epsilon^n}\right)$$

$$= \frac{n-2}{2}K(a_\epsilon)c_0^{p+1+\epsilon}\frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{(n-2)/2}}\int_B\left(\frac{\lambda_\epsilon}{1+\lambda_\epsilon^2|x-a_\epsilon|^2}\right)^{(p+\epsilon)(n-2)/2}$$

$$+O\left(\frac{\lambda_\epsilon^{\epsilon(n-2)/2}\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n}\right)$$

$$= \frac{n-2}{2}c_1K(a_\epsilon)\frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{n-2}}\lambda_\epsilon^{\epsilon(n-2)/2} + O\left(\frac{\lambda_\epsilon^{\epsilon(n-2)/2}\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n}\right) \quad . \qquad (21)$$

and

$$\int_B K{\delta_\epsilon}^{p-1+\epsilon}\theta_\epsilon\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda} = \theta_\epsilon(a_\epsilon)K(a_\epsilon)\int_B {\delta_\epsilon}^{p-1+\epsilon}\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda} + O\left(\int_B {\delta_\epsilon}^{p+\epsilon}\frac{|x-a_\epsilon|^2}{\lambda_\epsilon^{(n-2)/2}d_\epsilon^n}\right)$$

Using (5), we obtain

$$\int_B K{\delta_\epsilon}^{p-1+\epsilon}\theta_\epsilon\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda} = \frac{n-2}{2}K(a_\epsilon)c_1\frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{n-2}}\lambda_\epsilon^{\epsilon(n-2)/2} + O\left(\frac{\lambda_\epsilon^{\epsilon(n-2)/2}\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n}\right) \quad . \qquad (22)$$

(19), (20), (21) and additional integral estimates of the same type provide us with the expansion

$$\int_\Omega K(P\delta_\epsilon)^{p+\epsilon}\lambda_\epsilon\frac{\partial P\delta_\epsilon}{\partial\lambda} = \frac{n-2}{2}\lambda_\epsilon^{\epsilon(n-2)/2}\left[c_2K(a_\epsilon)\epsilon + 2c_1K(a_\epsilon)\frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{n-2}} - c_3\frac{\Delta K(a_\epsilon)}{\lambda_\epsilon^2}\right] + O\left(c_3+\frac{1}{\lambda_\epsilon^3}+\frac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^n} + \frac{1}{(\lambda_\epsilon d_\epsilon)^2}\quad (if \quad n=3)\right) . \qquad (23)$$

We note that

$$\int_\Omega K(P\delta_\epsilon)^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial P\delta_\epsilon}{\partial\lambda}$$

$$= \int_\Omega K\left({\delta_\epsilon}^{p-1+\epsilon} + O\left({\theta_\epsilon}^{p-1+\epsilon}+{\delta_\epsilon}^{p-2+\epsilon}\theta_\epsilon\right)\right)v_\epsilon\lambda_\epsilon\frac{\partial P\delta_\epsilon}{\partial\lambda}$$

$$= \int_\Omega K{\delta_\epsilon}^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda} - \int_\Omega K{\delta_\epsilon}^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial\theta_\epsilon}{\partial\lambda} - O\left(\int_\Omega {\delta_\epsilon}^{p-1+\epsilon}|v_\epsilon|\,\theta_\epsilon\right)$$

$$= \int_\Omega K(\delta_\epsilon)^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda}$$

$$- O\left(\frac{\lambda_\epsilon^{\epsilon(n-2)}|v_\epsilon|_{H_0^1}}{(\lambda_\epsilon d_\epsilon^2)^{(n-2)/2}}\left(\int_\Omega {\delta_\epsilon}^{\frac{8n}{n^2-4}}\right)^{(n+2)/2}\right)$$

$$+O\left(\lambda_\epsilon^{\epsilon(n-2)}|v_\epsilon|_{H_0^1}\left(\int_\Omega {\delta_\epsilon}^{2n/(n-2)}\right)^{(n+2)/(2n)}\right)$$

Using (15) we find

$$\int_\Omega K(P\delta_\epsilon)^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial P\delta_\epsilon}{\partial\lambda} = \int_\Omega K{\delta_\epsilon}^{p-1+\epsilon}v_\epsilon\lambda_\epsilon\frac{\partial\delta_\epsilon}{\partial\lambda} + O\left(\frac{\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}|v_\epsilon|_{H_0^1}}{(\lambda_\epsilon d_\epsilon)^{\frac{(n+2)}{2}}}\right) + O\left(\lambda_\epsilon^{\frac{\epsilon(n-2)}{2}}|v_\epsilon|_{H_0^1}\left[\frac{1}{(\lambda_\epsilon d_\epsilon)^{n-2}}(if \quad n<6) + \frac{\log(\lambda_\epsilon d_\epsilon)}{(\lambda_\epsilon d_\epsilon)^4}(if \quad n=6) + \frac{1}{(\lambda_\epsilon d_\epsilon)^{\frac{(n+2)}{2}}}(if \quad n>6)\right]\right) \qquad (24)$$

We also have, using Remark 4

$$\int_\Omega K(\delta_\epsilon)^{p-1+\epsilon} v_\epsilon \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda} = \lambda_\epsilon^{\frac{\epsilon(n-2)}{2}} c_0^\epsilon \int_\Omega K \delta_\epsilon^{p-1} v_\epsilon \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda} + \int_\Omega K\left(\delta_\epsilon^{p-1+\epsilon} - c_0^\epsilon \lambda_\epsilon^{\frac{\epsilon(n-2)}{2}} \delta_\epsilon^{p-1}\right) v_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda} = O\left(\epsilon \int_\Omega K \log(1 + \lambda_\epsilon^2 |x-a_\epsilon|^2)\delta_\epsilon^p |v_\epsilon|\right)$$

whence

$$\int_\Omega K(\delta_\epsilon)^{p-1+\epsilon} v_\epsilon \lambda_\epsilon \frac{\partial \delta_\epsilon}{\partial \lambda} = O\left(\epsilon |v_\epsilon|_{H_0^1}\right) = O(\epsilon). \quad (25)$$

Noticing that in addition $\lambda_\epsilon \frac{\partial P\delta_\epsilon}{\partial \lambda} = O(\delta_\epsilon)$ and

$$\int_\Omega \delta_\epsilon^{p-1+\epsilon} |v_\epsilon|^2 = O\left(\lambda_\epsilon^{\epsilon(n-2)/2} |v_\epsilon|_{H_0^1}^2\right). \quad (26)$$

$$\int_{\delta<|v_\epsilon|} |v_\epsilon|^{p+\epsilon} \delta_\epsilon = O\left(\lambda_\epsilon^{\epsilon(n-2)/2} |v_\epsilon|_{H_0^1}^{p+1}\right). \quad (27)$$

(18), (23), (24),(25), (26), (27) and Lemmas 3 and 5 prove Proposition 6.

We are know able to prove the theorem.

Proof of Theorem 1

Arguing by contradiction, let us suppose that (P_ϵ) has a solution u_ϵ as stated in the theorem. From Proposition 6, we have

$$\alpha_\epsilon c_1 \frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^{n-2}} - \alpha_\epsilon \frac{c_3}{n^2} \frac{\Delta K(a_\epsilon)}{K(a_\epsilon)\lambda_\epsilon^2} + \alpha_\epsilon c_2 \epsilon = o\left(\epsilon + \frac{1}{\lambda_\epsilon^2} + \begin{cases} \frac{1}{(\lambda_\epsilon d_\epsilon)^{n-2}} (if\ n \geq 4) \\ \frac{1}{\lambda_\epsilon d_\epsilon} \quad (if\ n=4) \end{cases}\right) \quad (28)$$

Notice that $H(a_\epsilon,a_\epsilon) \sim d_\epsilon^{n-2}$ if $d_\epsilon \to 0$ as $\epsilon \to 0$ and $H(a_\epsilon,a_\epsilon) \geq c > 0$ as $\epsilon \to 0$ if $d_\epsilon \nrightarrow 0$ as $\epsilon \to 0$.

For $n=3$, it follows from (28) that

$$c_1 \frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon} + c_2 \epsilon = o\left(\epsilon + \frac{1}{\lambda_\epsilon}\right)$$

which is a contradiction.

For $n=4$ it follows from (28) that

$$c_1 \frac{H(a_\epsilon,a_\epsilon)}{\lambda_\epsilon^2} - \frac{c_3}{16} \frac{\Delta K(a_\epsilon)}{K(a_\epsilon)\lambda_\epsilon^2} + c_2 \epsilon = O\left(\epsilon + \frac{1}{\lambda_\epsilon^2} + \frac{1}{(\lambda_\epsilon d_\epsilon)^2}\right)$$

which is a contradiction with assumption (ii) of the theorem.

For $n \geq 5$, it follows from (28) that

$$-\frac{c_3}{n^2} \frac{\Delta K(a_\epsilon)}{K(a_\epsilon)\lambda_\epsilon^2} + c_2 \epsilon = O\left(\epsilon + \frac{1}{\lambda_\epsilon^2}\right)$$

also leads to a contradiction with assumption (iii) .

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