A nonexistence of solutions to a supercritical problem

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Abstract: In this paper, we study the nonlinear elliptic problem involving nearly critical exponent $(P_\epsilon) : -\Delta u = K u^{p+\epsilon} \text{ in } \Omega ; u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega$ where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $K$ is a $C^3$-positive function and $\epsilon$ is a small positive real parameter. We prove that, for $\epsilon$ small, $(P_\epsilon)$ has no positive solutions which blow up at one critical point of the function $K$.

Keywords: Nonlinear Elliptic Equations, Critical Exponent, Variational Problem

1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. We consider the following nonlinear elliptic problem

$$(P_\epsilon) \begin{cases} -\Delta u = K u^{p+\epsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

where $K$ is a $C^3$-positive function, $p + 1 = 2n/n - 2$ is the critical Sobolev exponent and $\epsilon$ is a small positive real parameter.

Problem $(P_\epsilon)$ is in some sense related to the limiting problem (when $\epsilon = 0$) and the interest to it comes from its resemblance to the scalar curvature problem in differential geometry, which consists in finding suitable conditions on a given function $K$ defined on $M$ such that $K$ is the scalar curvature for a metric $\bar{g}$ conformally equivalent to $g$, where $(M, g)$ is a $n$-dimensional Riemannian manifold without boundary.

Note that the limiting problem has been widely studied in various works see for example [1], [2], [7] and [10].

In another view point, it is interesting to study the problem $(P_\epsilon)$ with $\epsilon < 0$ and $\epsilon > 0$ and to understand what happens to the solutions of $(P_\epsilon)$ (if they exist) as $\epsilon \to 0$!

When $\epsilon \in (1 - p, 0)$, the mountain pass lemma proves the existence of solutions of $(P_\epsilon)$ (see [3]). Note that, many works have been devoted to the study of positive solutions of $(P_\epsilon)$ with $\epsilon < 0$. In sharp contrast to this, very little study has been made concerning the sign-changing solutions of $(P_\epsilon)$ with $\epsilon < 0$ and even less for $\epsilon > 0$.

When $\epsilon > 0$, problem $(P_\epsilon)$ becomes more delicate since we lose the Sobolev embedding which is an important difficulty to overcome.

Concerning the supercritical case, $\epsilon > 0$ and $K$ is a constant, it was proved in [4] that $(P_\epsilon)$ has no positive solution which blows up at a single point. This result shows that the situation is different from the subcritical one.

However, del Pino et al [6] gave an existence result for two blow up points, provided that $\Omega$ satisfies some geometrical conditions. In sharp contrast to this, it proved in [5] for the case $K$ is a constant and [8] for the case $K$ is a non constant function that, for $\epsilon$ small, $(P_\epsilon)$ has no sign-changing solutions which blow up at two points.

In this paper, we consider the case $K$ is a non constant function and we look to understand the influence of the function $K$ in the study of the positive solutions of $(P_\epsilon)$ which blows up at a single point.

It is well known that problem $(P_\epsilon)$ has a variational structure. Setting

$$J(u) = \frac{\int_{\Omega} |u|^2}{(\int_{\Omega} K |u|^{p+1+\epsilon})^{2n/2}}, \quad u \in H^1_0(\Omega) \setminus \{0\},$$

the positive critical points of $J$ are solutions to $(P_\epsilon)$, up to a multiplicative constant. $J$ satisfies the Palais-Smale condition in the subcritical case, whereas this condition fails in the critical case. Such a failure is due to the function

$$\delta_{(a, b)}(x) = C_0 \frac{a^{n/2}}{(a + b|x-x_0|^2)^{n/2}}, \quad C_0 = (n(n - 2))^{n/2} a > 0, a \in \mathbb{R}^n \quad (1)$$

which are the only solutions of

$$-\Delta u = u^{n+2} \text{ in } \mathbb{R}^n, \quad u > 0, \text{ with } u \in L^{p+1}(\mathbb{R}^n) \text{ and } \nabla u \in L^2(\mathbb{R}^n)$$

and are also the only minimizers of the
Sobolev inequality on the whole space, that is
\[ S = \|u\|_{L^{2n/(n-2)}(\mathbb{R}^n)}^2, \text{s.t. } \nabla u \in L^2, u \in L^{2n/(n-2)}(\mathbb{R}^n), u \neq 0 \]  

We have the following nonexistence result for \( P_\epsilon \):

**Theorem 1**

Let \( \Omega \) be any smooth bounded domain in \( \mathbb{R}^n \), \( n \geq 3 \)

Assume that \( a_0 \in \Omega \) is a critical point of \( K \) satisfying one of the following conditions:

(i) \( n = 3 \),

(ii) \( n = 4 \) and \( c_1 H(a_0, a_0) - 26K(a_0) > 0 \),

(iii) \( n \geq 5 \) and \(- \Delta K(a_0) > 0 \).

Then the problem \( P_\epsilon \) has no solution \( u_\epsilon \) such that

\[ u_\epsilon = \alpha_\epsilon P\delta_{a_\epsilon, \lambda_\epsilon} + v_\epsilon \quad \text{with} \quad |u_\epsilon|^p \text{ is bounded and} \]

\[ v_\epsilon \to 0 \quad \text{in} \quad H^1_0(\Omega) \quad \alpha_\epsilon \to K(a_\epsilon)^{(2-n)/4}, \quad \lambda_\epsilon \to \Omega, a_\epsilon \to a_0 \]

and \( \lambda_\epsilon d(a_\epsilon, \partial \Omega) \to +\infty \) as \( \epsilon \to 0 \).

2. Preliminary Results

We need to introduce some notations:

\( P\delta_{a, \lambda} \) is defined as the only function in \( H^1_0(\Omega) \) such that

\[ \Delta P\delta_{a, \lambda} = \delta_{a, \lambda} \]

Writing

\[ P\delta_{a, \lambda} = \delta_{a, \lambda} - \theta_{a, \lambda} \]

we have

\[ \Delta \theta_{a, \lambda} = 0 \quad \text{in} \quad \Omega; \quad \theta_{a, \lambda} = \delta_{a, \lambda} \quad \text{on} \quad \partial \Omega \]

We note that projections \( P\delta_{a, \lambda} \) of \( \delta_{a, \lambda} \)'s on \( H^1_0(\Omega) \) are approximate solutions to the limiting problem as \( a_\epsilon \in \Omega \) and \( \lambda_\epsilon d(a_\epsilon, \partial \Omega) \) goes to infinity.

Let \( G \) be the Green's function for the Laplace operator with Dirichlet boundary conditions, that is, for any \( x \in \Omega \):

\[ \{-\Delta G(x, \cdot) = c_n \delta_x \quad \text{in} \quad \Omega \}, \quad G(x, \cdot) = 0 \quad \text{on} \quad \partial \Omega \]

with \( \delta_x \) the Dirac mass at \( x \) and \( c_n = (n-2)|S^{n-1}| \)

We denote by \( H \) the regular part of \( G \), i.e.

\[ H(x_1, x_2) = |x_1 - x_2|^{2-n} - G(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \Omega \times \Omega \]

The maximum principle gives us the uniform estimate

\[ \theta_{a, \lambda}(x) = C_0 \frac{n(x, a, \lambda)}{\lambda} + O\left(\frac{1}{\lambda^{2-n}(d(a, \partial \Omega)^n)}\right) \text{as} \lambda d(a, \partial \Omega) \to +\infty \]

Corresponding estimates hold for the derivatives of \( \theta_{a, \lambda} \) with respect to \( a, \lambda \) and \( x \).

Note that \( H(x, x) = O(d(x, \partial \Omega)^2 \text{-n}) \) as \( d(x, \partial \Omega) \to 0 \) [9]. From [9] we also know that

\[ \int_\Omega |\nabla \theta_{a, \lambda}|^2 = O(\lambda d(a, \partial \Omega)^2 \text{-n}) \text{as} \lambda d(a, \partial \Omega) \to +\infty \]

Next, we recall that for \( u_\epsilon \) satisfying the assumption of the theorem, there is a unique way to choose \( a_\epsilon, \lambda_\epsilon \) and \( v_\epsilon \) such that

\[ u_\epsilon = \alpha_\epsilon P\delta_{a_\epsilon, \lambda_\epsilon} + v_\epsilon \]

with

\[ \begin{cases} 
\alpha_\epsilon \in \mathbb{R}, & \alpha_\epsilon \to K(a_\epsilon)^{(2-n)/4} \\
\lambda_\epsilon \in \mathbb{R}^*, & \lambda_\epsilon d(a_\epsilon, \partial \Omega) \to +\infty \\
v_\epsilon \to 0 & \text{in} \quad H^1_0(\Omega), \quad v_\epsilon \in E_{a_\epsilon, \lambda_\epsilon} 
\end{cases} \]

and for any \( (a, \lambda) \in \Omega \times \mathbb{R}^*, E_{(a, \lambda)} \) denotes the subspace of \( H^1_0(\Omega) \) defined by

\[ E_{(a, \lambda)} = \{ w \in H^1_0(\Omega) \} \quad \text{where} \quad \int_\Omega w^p \to +\infty \quad \text{as} \quad \lambda \to 1 \]

For the proof of this fact, see [1], [9]. In the following, we always assume that \( u_\epsilon \), satisfying the assumption of the theorem, is written as in (8). In order to simplify the notations, we set

\[ \delta_{a_\epsilon, \lambda_\epsilon} = \delta_{a_\epsilon}, \quad P\delta_{a_\epsilon, \lambda_\epsilon} = P\delta_\epsilon \quad \text{and} \quad \theta_{a_\epsilon, \lambda_\epsilon} = \theta_\epsilon \]

Lemma 2

Let \( u_\epsilon \) satisfying the assumption of the theorem 1. Then

(i) \( \int_\Omega |\nabla u_\epsilon|^2 \to S^{n/2} \); \quad (ii) \( \int_\Omega K u_\epsilon^{p+1+\epsilon} \to S^{n/2} \)

as \( \epsilon \to 0 \), \( S, S \) denoting the Sobolev constant defined by (2).

**Proof.**

We have

\[ \int_\Omega |\nabla u_\epsilon|^2 = \int_\Omega |\nabla (\alpha \epsilon P\delta_\epsilon + v_\epsilon)|^2 = \alpha_\epsilon^2 \int_\Omega |\nabla P\delta_\epsilon|^2 + \int_\Omega |v_\epsilon|^2 \text{ since} \quad v_\epsilon \in E_{a_\epsilon, \lambda_\epsilon} \]

From the fact that \( \delta_\epsilon \) satisfies \( -\Delta \delta_\epsilon = \delta_\epsilon^p \text{ in} \mathbb{R}^n \) and is a minimizer for \( S \), we deduce that \( \int_\Omega |\nabla \delta_\epsilon|^2 \to S^{n/2} \)

On the other hand, an explicit computation provides us with

\[ \int_\Omega |\nabla \theta_{a, \lambda}|^2 = \int_\mathbb{R} |\nabla \theta_{a, \lambda}|^2 + O\left(\frac{1}{\lambda d(a, \partial \Omega)^n}\right) \text{as} \lambda d(a, \partial \Omega) \to +\infty \]

Taking account of (6), claim (i) is a consequence of (8). Claim (ii) follows from the fact that \( u_\epsilon \) solves \( P_\epsilon \).

3. Estimating \( v_\epsilon \)

As usual in this type of problems, we first deal with the \( v \)-part of \( u \), in order to show that it is negligible with respect to the concentration phenomenon.
Lemma 3

Let \( u_\varepsilon \) satisfying the assumption of the theorem. \( \lambda_\varepsilon \) occurring in (7) satisfies
\[
\lambda_\varepsilon \rightarrow 1, \text{as} \quad \varepsilon \rightarrow 0.
\]

Proof.

According to Lemma 2, we have
\[
\int_\Omega K u_\varepsilon^{p+1+\varepsilon} = S^{n/2} + o(1) \quad \text{as} \quad \varepsilon \rightarrow 0
\]
and
\[
\int_\Omega K u_\varepsilon^{p+1+\varepsilon} = \int_\Omega K (\alpha_\varepsilon \delta_\varepsilon + v_\varepsilon)^{p+\varepsilon} \alpha_\varepsilon \delta_\varepsilon
\]
\[
+ \int_\Omega K u_\varepsilon^{p+\varepsilon} v_\varepsilon
\]
\[
= \alpha_\varepsilon^{p+1+\varepsilon} \int_\Omega K \delta_\varepsilon^{p+\varepsilon+1}
\]
\[
- \int_\Omega \Delta u_\varepsilon v_\varepsilon
\]
\[
+ O \left( \int_\Omega \delta_\varepsilon^{p+\varepsilon} |v_\varepsilon| + \int_\Omega |v_\varepsilon|^{p+\varepsilon} \delta_\varepsilon \right)
\]
\[
= \alpha_\varepsilon^{p+1+\varepsilon} \int_\Omega K \delta_\varepsilon^{p+\varepsilon+1}
\]
\[
+ O \left( \lambda_\varepsilon^{(n-2)/2} |v_\varepsilon|_{L^{p+1}} + \lambda_\varepsilon^{(n-2)/2} |\nabla v_\varepsilon|_{L^{p+1}} + \lambda_\varepsilon^{(n-2)/2} |v_\varepsilon|_{L^{p+1}} \right)
\]

Thus
\[
\int_\Omega K u_\varepsilon^{p+1+\varepsilon} = \alpha_\varepsilon^{p+1+\varepsilon} \int_\Omega K \delta_\varepsilon^{p+\varepsilon+1} + O \left( \lambda_\varepsilon^{(n-2)/2} + 1 \right)
\]

We observe that
\[
\int_\Omega K \delta_\varepsilon^{p+1+\varepsilon} = \int_\Omega K (\delta_\varepsilon - \Theta_\varepsilon) \delta_\varepsilon^{p+\varepsilon+1} = \int_\Omega K (\delta_\varepsilon - \Theta_\varepsilon) \delta_\varepsilon^{p+\varepsilon+1} + O(\int_\Omega \delta_\varepsilon^{p+\varepsilon+1})
\]
\[
= C_0^{p+\varepsilon+1} K(\alpha_\varepsilon) \int_B \left( \frac{\lambda_\varepsilon}{1 + \lambda_\varepsilon |x-a_\varepsilon|} \right)^{\frac{p+\varepsilon+1}{n}}
\]
\[
+ O \left( \left( \frac{\lambda_\varepsilon}{1 + \lambda_\varepsilon |x-a_\varepsilon|} \right)^{\frac{p+\varepsilon+1}{n}} + \frac{\lambda_\varepsilon^{(n-2)/2}}{(\lambda_\varepsilon d_\varepsilon)^n} \right)
\]

where \( B = B(a_\varepsilon, d_\varepsilon) \). Using Proposition 1 of [9], we obtain
\[
\int_\Omega K \lambda_\varepsilon^{p+1+\varepsilon}
\]
\[
= \lambda_\varepsilon^2 K(\alpha_\varepsilon) \int_\Omega \left( \frac{\lambda_\varepsilon^{p+\varepsilon+1}}{1 + \lambda_\varepsilon |x-a_\varepsilon|} \right)^{\frac{p+\varepsilon+1}{2(n-2)}}
\]
\[
- \frac{C_0^{p+\varepsilon+1}}{(1 + \lambda_\varepsilon |x-a_\varepsilon|)^{\frac{p+\varepsilon+1}{2}}} + O \left( \frac{\lambda_\varepsilon^2}{(\lambda_\varepsilon d_\varepsilon)^n} \right)
\]
\[
= \lambda_\varepsilon^{e(n-2)/2} K(\alpha_\varepsilon) C_0^{p+\varepsilon+1} \int_\Omega \left( 1 + |x|^2 \right)^{\frac{p+\varepsilon+1}{2(n-2)}}
\]
\[
+ O \left( \frac{\lambda_\varepsilon^{(n-2)/2}}{(\lambda_\varepsilon d_\varepsilon)^n} \right)
\]

We note that
\[
C_0^{p+\varepsilon+1} \int_\Omega \left( 1 + |x|^2 \right)^{\frac{p+\varepsilon+1}{2(n-2)}} = C_0^{p+1} \int_\Omega \left( 1 + |x|^2 \right)^{\frac{p+1}{2}} + O(\varepsilon) = S^{n/2} + O(\varepsilon)
\]

Therefore
\[
\int_\Omega K \lambda_\varepsilon^{p+1+\varepsilon} = \lambda_\varepsilon^{e(n-2)/2} K(\alpha_\varepsilon) (S^{n/2} + O(\varepsilon) + o(1))
\]

so (10) and (11) provide us with
\[
\int_\Omega K u_\varepsilon^{p+1+\varepsilon} = \alpha_\varepsilon^{p+1+\varepsilon} \lambda_\varepsilon^{e(n-2)/2} K(\alpha_\varepsilon) (S^{n/2} + o(1)) + o(1)
\]

Combination of (9) and (12) proves the lemma.

Next, we recall the following estimate [10]:

\[
\delta_\varepsilon(x) - C_0^\delta_\varepsilon^{e(n-2)/2} = O \left( \varepsilon \log(1 + \lambda_\varepsilon^2 |x - a_\varepsilon|^2) \right) \quad \text{in} \quad \Omega
\]

We are now able to study the \( v_\varepsilon \) -part of \( u_\varepsilon \).

Lemma 5

Let \( u_\varepsilon \) satisfying the assumption of the theorem. \( v_\varepsilon \) occurring in (7) satisfies
\[
|v_\varepsilon|_{H^1(\Omega)}
\]

\[
\leq C + C \begin{cases} 
\nabla K(a_\varepsilon) \frac{1}{\lambda_\varepsilon} & \text{if } n < 6 \\
\nabla K(a_\varepsilon) \frac{\log(\lambda_\varepsilon d_\varepsilon)}{\lambda_\varepsilon} & \text{if } n = 6 \\
\n\nabla K(a_\varepsilon) \frac{1}{\lambda_\varepsilon} & \text{if } n > 6
\end{cases}
\]

where \( B = B(a_\varepsilon, d_\varepsilon) \).
with $C$ independent of $\epsilon$.

**Proof.**

Multiplying $(P_2)$ by $v e$ and integrating on $\Omega$, we obtain

$$0 = \int_{\Omega} \nabla u e \cdot \nabla v e - \int_{\Omega} K u^{p+\epsilon} v e$$

Thus

$$0 = \int_{\Omega} |\nabla v e|^2 - \int_{\Omega} \left[ (a e P \delta e)^{p+\epsilon} + (p + \epsilon)(a e P \delta e)^{p-1+\epsilon} v e + O(\delta e^{p-2+\epsilon} v e^2 \chi_{|v e| < \delta e} + |v e|^{p+\epsilon}) \right] v e.$$

Using the assumption that $|u e|^\epsilon$ is bounded, we find

$$0 = Q_e(v e, v e) - f_e(v e) + O \left( |v e|_{H^2_0}^\min(2,p+1) + |v e|_{H^2_0}^{p+1} \right)$$

(13)

with

$$Q_e(v, v) = |v e|_{H^2_0}^2 - (p + \epsilon) \int_{\Omega} K(a e P \delta e)^{p-1+\epsilon} v e^2$$

and

$$f_e(v) = \int_{\Omega} K(a e P \delta e)^{p+\epsilon} v e.$$

We observe that

$$Q_e(v, v) = |v e|_{H^2_0}^2 - p \int_{\Omega} K(a e P \delta e)^{p-1+\epsilon} v e^2$$

$$+ O \left( |v e|_{H^2_0}^2 \right)$$

$$= |v e|_{H^2_0}^2 - p a e^{p-1+\epsilon} K(a e) \int_{\Omega} (\delta e^{p-1+\epsilon}$$

$$+ O(\delta e^{p-1+\epsilon} \theta e)) v e^2 + o \left( |v e|_{H^2_0}^2 \right)$$

$$= |v e|_{H^2_0}^2 - p a e^{p-1+\epsilon} K(a e) C_0 \delta e^{(n-2)/2} \int_{\Omega} \delta e^{p-1} v e^2$$

$$+ O \left( \int_{\Omega} (\delta e^{p-1+\epsilon}$$

$$- C_0 \delta e^{(n-2)/2} \delta e^{p-1} |v e|^2) \right) + o \left( |v e|_{H^2_0}^2 \right)$$

Using Remark 4, we find

$$Q_0(v, v) = Q_0(v, v) + o \left( |v|_{H^2_0}^2 \right)$$

$$Q_0(v, v) = |v|_{H^2_0}^2 - \int_{\Omega} \delta e^{p-1} v e^2.$$  

According to [1], $Q_0$ is coercive, that is, there exists some constant $c > 0$ independent of $\epsilon$, for $\Omega$ small enough, such that

$$Q_0(v, v) \geq c |v|_{H^2_0}^2 \quad \forall v \in E(a e, \lambda e).$$

We also observe that

$$f_e(v) = a e^{p+\epsilon} \int_{\Omega} K(\delta e^{p+\epsilon} + O(\delta e^{p-1+\epsilon} \theta e)) v e$$

$$= a e^{p+\epsilon} \left[ C_0 \delta e^{(n-2)/2} \int_{\Omega} \delta e^{p-1} v e$$

$$+ O \left( \int_{\Omega} K(a e) \theta e v e \right) \delta e^{p-1} v e \right]$$

The last equality follows from Remark 4. Therefore we can write,

$$f_e(v) = C_0 \delta e^{(n-2)/2} \int_{\Omega} \delta e^{p-1} v e$$

$$+ O \left( \int_{\Omega} K(a e) \theta e v e \right) \delta e^{p-1} v e$$

$$+ o \left( |v|_{H^2_0}^2 \right)$$

(14)

We notice that

$$\int_{\Omega} \delta e^{2n/(n-2)} = O \left( \frac{1}{(\lambda e d e)^n} \right)$$

and

$$\left( \int_{\Omega} \delta e^{\frac{8n}{n-2}} \right)^{(n+2)/2n} \leq C \left\{ \begin{array}{cl}
\frac{d e^{(n-6)/2}}{\lambda e^{2}} & \text{if } n > 6 \\
\log(\lambda e d e) \frac{1}{\lambda e^{2}} & \text{if } n = 6 \\
\frac{1}{\lambda e^{(n-2)/2}} & \text{if } n < 6
\end{array} \right.$$  

Using (5), we obtain

$$|f_e(v)| \leq C |v|_{H^2_0}$$

$$+ C \left\{ \begin{array}{cl}
\frac{K(a e)}{\lambda e} + \frac{1}{(\lambda e d e)^{n-2}} & \text{if } n > 6 \\
\frac{K(a e)}{\lambda e} + \log(\lambda e d e) \frac{1}{(\lambda e d e)^{n-2}} & \text{if } n = 6 \\
\frac{1}{(\lambda e d e)^{(n+2)/2}} & \text{if } n > 6
\end{array} \right.$$  

(15)

Combining (13), (14) and (15), we obtain the desired estimate.
4. Proof of Theorem

Let us start by proving the following crucial result:

**Proposition 6**

Let \( u_e \) satisfying the assumption of the theorem. Then,

\[
\left| \alpha_e c_1 \frac{H(a_e, a_e)}{\lambda_e^2} - \alpha_e c_2 \frac{\delta K(a_e)}{\lambda_e^2} + \alpha_e c_3 \frac{d_e}{\lambda_e^2} \right| \leq c \left( \epsilon^2 + \frac{1}{\lambda_e^2} + \log(\lambda_e d_e) \right) (n \geq 4) \leq \left( \frac{1}{\lambda_e^2} \right) (n = 3) \]

(16)

where \( \alpha_e, \lambda_e \) and \( d_e = (a_e, d) \) are given in (7) and \( c_1, c_2, c_3 \) are positive constants defined by

\[
c_1 = c_0 \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^{n+2}}
\]

\[
c_2 = \frac{n-2}{2} c_0 \int_{\mathbb{R}^n} \log(1 + |x|^2) \frac{|x|^2-1}{(1 + |x|^2)^{n+2}} \ dx
\]

and

\[
c_3 = c_0 \int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^2} \ dx.
\]

**Proof.**

Multiplying \((P_d)\) by \( \lambda_e \frac{\delta P_{\delta_e}}{\delta} \) and integrating on \( \Omega \), we obtain

\[
0 = -\int_{\Omega} \Delta u_e \frac{\partial P_{\delta_e}}{\partial \lambda} - \int_{\Omega} K u_e^{p+e} \lambda_e \frac{\partial P_{\delta_e}}{\partial \lambda} = \int_{\Omega} \nabla (\alpha_e P_{\delta_e} + v_e) \nabla \left( \lambda_e \frac{\partial P_{\delta_e}}{\partial \lambda} \right) - \int_{\Omega} K (\alpha_e P_{\delta_e})
\]

\[
+ v_e^{p+e} \lambda_e \frac{\partial P_{\delta_e}}{\partial \lambda}
\]

\[
= \alpha_e \int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda} - \int_{\Omega} K \left( \alpha_e P_{\delta_e} \right)^{p+e} + (p+e) (\alpha_e P_{\delta_e})^{p-1+e} v_e
\]

\[
+ O(\delta_e^{p-2+e} |v_e|^2)
\]

\[
+ |v_e^{p+e}| \lambda_e \frac{\partial P_{\delta_e}}{\partial \lambda}.
\]

(17)

We estimate each term of the right hand side in (17). First, we have

\[
\int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda} = \int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda} - \int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda}
\]

whence

\[
\int_{\Omega} \delta_e \frac{\partial P_{\delta_e}}{\partial \lambda} = \int_{\Omega} \delta_e \frac{\partial P_{\delta_e}}{\partial \lambda} - \int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda}
\]

\[
+ \int_{\Omega} \frac{\partial P_{\delta_e}}{\partial \lambda}.
\]

(18)
and we have to estimate each term of the right hand side of (18). Using the fact that
\[ \lambda_e \frac{\partial \delta_e}{\partial \lambda} = \frac{n-2}{2} \left( 1 - \frac{2}{1 + \lambda_e^2 |x - a_e|^2} \right) \delta_e, \]
we derive that
\[ \int_B K \delta_e p+\epsilon \lambda_e \frac{\partial \delta_e}{\partial \lambda} = \frac{n-2}{2} \lambda_e \frac{\partial \epsilon}{\partial \lambda} \int_B (1 + |x|^2)^{(n-2)/2} \left( 1 - \frac{2}{1 + \lambda_e^2 |x - a_e|^2} \right) \delta_e \]
\[ = \frac{n-2}{2} \lambda_e \frac{\epsilon}{\lambda_e} K(a_e) c_0^{p+\epsilon} \left( \frac{1}{1 + \lambda_e^2 |x - a_e|^2} \right) dx + O \left( \lambda_e^{\epsilon(n-2)/2} (\lambda_e d_e)^n \right) \]
\[ = \lambda_e \frac{\epsilon}{\lambda_e} \left( c_2 \lambda_e (a_e) - c_3 \frac{\lambda_e^2 K(a_e)}{\lambda_e^2} + O \left( \epsilon^2 + \frac{1}{\lambda_e^2} \right) \right) + O \left( \lambda_e^{\epsilon(n-2)/2} (\lambda_e d_e)^n \right) \]
\[ = \lambda_e \left( c_2 \lambda_e (a_e) - c_3 \frac{\lambda_e^2 K(a_e)}{\lambda_e^2} + O \left( \epsilon^2 + \frac{1}{\lambda_e^2} \right) \right) + O \left( \lambda_e^{\epsilon(n-2)/2} (\lambda_e d_e)^n \right) \quad (20) \]

For the other terms in (19), we write
\[ \int_B K \delta_e p+\epsilon \lambda_e \frac{\partial \theta_e}{\partial \lambda} = K(a_e) \lambda_e \frac{\partial \theta_e}{\partial \lambda} (a_e) \int_B \delta_e p+\epsilon \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} |x - a_e|^2 \right) \]
\[ = \frac{n-2}{2} \lambda_e \frac{\epsilon}{\lambda_e} \left( K(a_e) c_0^{p+\epsilon} \left( \frac{1}{1 + \lambda_e^2 |x - a_e|^2} \right) \lambda_e \frac{\epsilon}{\lambda_e} \right) (p+\epsilon) \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} (\lambda_e d_e)^n \right) \]
\[ = \frac{n-2}{2} c_1 \lambda_e (a_e) - \frac{\lambda_e^{\epsilon(n-2)/2}}{\lambda_e} \}
\[ = \frac{n-2}{2} c_1 \lambda_e (a_e) - \frac{\lambda_e^{\epsilon(n-2)/2}}{\lambda_e} \}
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \quad (21) \]

and
\[ \int_B K \delta_e p+\epsilon \theta_e \lambda_e \frac{\partial \delta_e}{\partial \lambda} = \theta_e(a_e) K(a_e) \int_B \delta_e p+\epsilon \lambda_e \frac{\partial \delta_e}{\partial \lambda} \]
\[ + O \left( \epsilon^2 + \frac{1}{\lambda_e^2} \right) \]
\[ = \theta_e(a_e) K(a_e) \int_B \lambda_e \theta_e(a_e) \left( \frac{1}{1 + \lambda_e^2 |x - a_e|^2} \right) \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \]
\[ = \theta_e(a_e) K(a_e) c_0^{p+\epsilon} \left( \frac{1}{1 + \lambda_e^2 |x - a_e|^2} \right) \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \quad (22) \]

Using (15) we find
\[ \int_B K \delta_e p+\epsilon \theta_e \lambda_e \frac{\partial \delta_e}{\partial \lambda} = \int_B K \delta_e p+\epsilon \theta_e \lambda_e \frac{\partial \delta_e}{\partial \lambda} + \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \quad (23) \]

We also have, using Remark 4

\[ \int_B K(P \delta_e) p+\epsilon \lambda_e \frac{\partial P \delta_e}{\partial \lambda} = \int_B K(P \delta_e) p+\epsilon \lambda_e \frac{\partial P \delta_e}{\partial \lambda} + \]
\[ + O \left( \lambda_e^{\epsilon(n-2)/2} \log(\lambda_e d_e) \right) \quad (24) \]
\[
\int_{\Omega} K(\delta) \delta^p v_\epsilon \lambda \frac{\partial \delta}{\partial \lambda} = \lambda^p \frac{c_0^p}{\epsilon} \int_{\Omega} K \delta \delta^p v_\epsilon \lambda \frac{\partial \delta}{\partial \lambda} + \int_{\Omega} K \left( \frac{\epsilon}{\lambda} \delta^p \right) v_\epsilon \lambda \frac{\partial \delta}{\partial \lambda} - \frac{c_0^p \lambda^p}{\epsilon^p} v_\epsilon \lambda \frac{\partial \delta}{\partial \lambda} = O \left( \epsilon \int_{\Omega} K \log(1) \right)
+ \lambda^2 |x - a| \delta^p v_\epsilon \right)
\]
whence
\[
\int_{\Omega} K(\delta) \lambda \frac{\partial \delta}{\partial \lambda} = 0 \left( \epsilon \int_{\Omega} |v_\epsilon|^2 \right) = O(\epsilon). \quad (25)
\]

Noticing that in addition \( \lambda \frac{\partial \delta}{\partial \lambda} = O(\delta) \) and
\[
\int_{\delta} \delta \delta^p |v_\epsilon|^2 = O \left( \epsilon^{n-2} |v_\epsilon|^2 \right). \quad (26)
\]
\[
\int_{\delta \leq |v_\epsilon|} |v_\epsilon|^p \delta^p = O \left( \epsilon^{n-2} |v_\epsilon|^p \right). \quad (27)
\]

(18), (23), (24), (25), (26), (27) and Lemmas 3 and 5 prove Proposition 6.

We are now able to prove the theorem.

**Proof of Theorem 1**

Arguing by contradiction, let us suppose that \( (P_\epsilon) \) has a solution \( u_\epsilon \) as stated in the theorem. From Proposition 6, we have
\[
\alpha \epsilon \frac{H(a_\epsilon, a_\epsilon)}{\lambda^2} - \alpha \epsilon \frac{c_3 \lambda K(a_\epsilon)}{\pi^2} \frac{\lambda^2}{\lambda^2 a_\epsilon} + \alpha \epsilon \epsilon \epsilon = o \left( \epsilon \frac{1}{\lambda^2} \right)
\]
\[
\left\{ \begin{array}{l}
\frac{1}{\lambda^2 a_\epsilon} \quad (\text{if } n \geq 4) \\
\frac{1}{\lambda^2 a_\epsilon} \quad (\text{if } n = 4)
\end{array} \right. \quad (28)
\]

Notice that \( H(a_\epsilon, a_\epsilon) \sim a_\epsilon^{n-2} \) if \( a_\epsilon \to 0 \) as \( \epsilon \to 0 \) and \( H(a_\epsilon, a_\epsilon) \geq c_0 > 0 \) as \( \epsilon \to 0 \) if \( a_\epsilon \to 0 \) as \( \epsilon \to 0 \).

For \( n = 3 \), it follows from (28) that
\[
c_1 \frac{H(a_\epsilon, a_\epsilon)}{\lambda^2} + c_2 \epsilon = o \left( \epsilon \frac{1}{\lambda^2} \right)
\]
which is a contradiction.

For \( n = 4 \) it follows from (28) that
\[
c_1 \frac{H(a_\epsilon, a_\epsilon)}{\lambda^2} + c_2 \epsilon \left( \epsilon \frac{1}{\lambda^2} \right)
\]
which is a contradiction with assumption (ii) of the theorem.

For \( n \geq 5 \), it follows from (28) that
\[
c_1 \frac{H(a_\epsilon, a_\epsilon)}{\lambda^2} + c_2 \epsilon = o \left( \epsilon \frac{1}{\lambda^2} \right)
\]
also leads to a contradiction with assumption (iii).

**References**


