On the symmetry classes of tensors associated with certain frobenius groups

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Abstract: In this paper, we study the symmetry classes of tensors associated with some Frobenius groups of order pq , where $q|p-1$ as a subgroups of the full symmetric group on p letters. We calculate the dimension of the symmetry classes of tensor associated with some Frobenius groups and some irreducible complex characters and we obtain two useful corollary with an example.

Keywords: Symmetry Classes of Tensors, Frobenius Groups

1. Introduction

Let $n \geq 2$ and $m \geq 2$ be positive integer numbers. Denote by S_n the full symmetric group on $\{1, 2, \dots, n\}$. Let V be a unitary complex vector space of dimension m over the complex numbers field. Let $\otimes^n V$ be the n -th tensor power of V , and write

$$v^{\otimes} := v_1 \otimes v_2 \otimes \dots \otimes v_n$$

for the decomposable tensor product of the indicated vectors.

For $\sigma \in S_n$, there is a unique linear operator $P(\sigma^{-1})$ on $\otimes^n V$ which has the effect

$$P(\sigma^{-1})(v_1 \otimes v_2 \otimes \dots \otimes v_n) :=$$

$$v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)},$$

for all $v_1, v_2, \dots, v_n \in V$.

Let G be a subgroup of S_n and χ be an irreducible complex character of G and $I(G)$ be the set of all the irreducible complex characters of G . We define $T(G, \chi)$ as a linear operator on $\otimes^n V$ with the following definition

$$T(G, \chi) := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma).$$

With respect to the induced inner product in $\otimes^n V$, $T(G, \chi)$ is an orthogonal projection onto its range $V_{\chi}^n(G)$,

which is called the symmetry class of tensors associated with G and χ , and the dimension of $V_{\chi}^n(G)$ is (see [3], [7])

$$\dim V_{\chi}^n(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) m^{c(\sigma)}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ , (see [6]). It follows from the orthogonality relations for characters that $\{T(G, \chi) \mid \chi \in I(G)\}$ is a set of annihilating idempotents which sum to the identity. With respect to the induced inner product in $\otimes^n V$, and the orthogonal relations for characters we have:

$$\otimes^n V = \sum_{\chi \in I(G)} V_{\chi}^n(G)$$

which is an orthogonal direct sum. Let Γ_m^n be the set of all sequences

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad 1 \leq \alpha_i \leq m, \quad i = 1, 2, \dots, n.$$

Then the group G acts on Γ_m^n by

$$\sigma \cdot \alpha := \alpha \circ \sigma^{-1} = (\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \dots, \alpha_{\sigma^{-1}(n)})$$

which is a composition of two functions σ^{-1} and α .

Let Δ be a system of distinct representatives of the orbits of

G acting on Γ_m^n and define:

$$\bar{\Delta} = \left\{ \alpha \in \Delta \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\},$$

Which $G_\alpha := \{\sigma \in G \mid \sigma \cdot \alpha = \alpha\}$ is the stabilizer of α . Let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis of V and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, the image of $e_\alpha^\otimes = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}$ under $T(G, \chi)$ is denoted by

$$e_\alpha^\chi = e_{\alpha_1} * e_{\alpha_2} * \dots * e_{\alpha_n}.$$

For $\gamma \in \bar{\Delta}$, $V_\gamma^\chi = \langle e_{\sigma\gamma}^\chi \mid \sigma \in G \rangle$ is called the orbital subspace of $V_\chi^n(G)$. In [3], prove that

$$\dim V_\gamma^\chi = \frac{\chi(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma).$$

Definition 1.1.

An orthogonal basis of the form $\{e_\gamma^\chi : \gamma \in B\}$, where B is a subset of Γ_m^n , is called an orthogonal basis of decomposable n -symmetrized tensor for $V_\chi^n(G)$, in this case we say that $V_\chi^n(G)$ has an O-basis.

$$V_\chi^n(G) = \sum_{\gamma \in \Delta} V_\gamma^\chi$$

Since $V_\chi^n(G)$ has an O-basis if

and only if V_γ^χ has an O-basis. Several papers are devoted in investigation of the existence of an O-basis for $V_\chi^n(G)$ for example [9, 1, 2, 4]. Some parts of this paper was presented as a talk in the 43rd Annual Iranian Mathematics Conference [8]. In this paper we study the symmetry classes of tensors associated with some Frobenius groups of order pq , where $q \mid p-1$. We calculate the dimension of the symmetry classes of tensor associated with some groups and we obtain two usefull corollary with an example.

2. The Group $F_{p,q}$

Let G be a group and let T be a faithful transitive G -set with each nontrivial element $g \in G$ having at most one fixed point. If no such g has a fixed point, then T is a regular G -set; if some g does have a fixed point, then G is called a Frobenius group.

Definition 2.1. [5]

If p is a prime and $q \mid p-1$, then we write $F_{p,q}$ for the group of order pq with presentation:

$$F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u \rangle$$

where u is an element of order q in the product group the integer numbers modulo p , i.e., Z_p^* .

The groups $F_{p,q}$ belong to a wider class of Frobenius

groups. This group is of order pq and its elements are of the form

$$F_{p,q} = \{a^i b^j : 0 \leq i \leq p-1, 0 \leq j \leq q-1\}.$$

It is not hard to see that $F_{p,q}$ has $\frac{p-1}{q} + q$ conjugacy classes which are

$$\{1\}, \{a^{v_i s} \mid s \in S\}, \quad i = 1, 2, \dots, \frac{p-1}{q},$$

$$\{a^h b^l \mid h = 0, 1, \dots, p-1\}, \quad l = 1, 2, \dots, q-1,$$

where S is the subgroup of Z_p^* generated by u , an element of order q in Z_p^* and $\left\{v_1, v_2, \dots, v_{\frac{p-1}{q}}\right\}$ is a transversal set of S in Z_p^* , and the irreducible complex characters table of $F_{p,q}$ is:

σ	1	a^{v_i}	b^l
χ_k	1	1	ω^{lk}
ϕ_j	q	$\sum_{s \in S} \omega^{p \frac{q}{s} v_i v_j}$	0

where $\omega = \exp\left(\frac{2\pi i}{q}\right)$, $0 \leq k \leq q-1, 1 \leq l \leq q-1$

$$1 \leq i, j \leq \frac{p-1}{q}.$$

From the above table we see that $F_{p,q}$ has q linear characters $\chi_k, 0 \leq k \leq q-1$, and $\frac{p-1}{q}$ non-linear

$$\phi_j, 1 \leq j \leq \frac{p-1}{q} \text{ of degree } q.$$

Now we will embed this group in a suitable full symmetric group. We can consider

$$(1 \ 2 \ 3 \ \dots \ p)$$

is a cyclic permutation in S_p , it can be verified that the mapping

$$a \mapsto (1 \ 2 \ 3 \ \dots \ p),$$

since $b^{-1}ab = a^u \mapsto (1 \ 1+u \ 1+2u \ \dots \ 1+(p-1)u)$, so b can be map to the following permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 1 & 1+u & 1+2u & \cdots & 1+(p-1)u \end{pmatrix}$$

where for $k=1,2,\dots,p-1$,

$$\overline{1+ku} \equiv 1+ku \pmod{p}, 1 \leq \overline{1+ku} \leq p,$$

in S_p . Note that

$$\{1, \overline{1+u}, \overline{1+2u}, \dots, \overline{1+(p-1)u}\} = \{1, 2, \dots, p\}.$$

So $F_{p,q}$ can be considered as a permutation subgroup in S_p , i.e., $F_{p,q} \leq S_p$.

Theorem 2.2.

Let p be a prime number and $G = F_{p,p-1}$. We have only one nonlinear irreducible character ϕ , where $\phi(1) = p-1$, $\phi(a) = -1$ and $\phi(b^l) = 0$, $l=1,2,\dots,p-2$. Let V be an m -dimensional inner product space, then the dimension of the symmetry classes of tensor associated with G and ϕ is

$$\dim V_{\phi}^p(G) = \frac{(p-1)}{p} (m^p - m).$$

Proof. We know

$$\dim V_{\phi}^p(F_{p,p-1}) = \frac{\phi(1)}{|G|} \sum_{\sigma \in F_{p,p-1}} \phi(\sigma) m^{c(\sigma)}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ . Therefore:

$$\begin{aligned} \dim V_{\phi}^p(F_{p,p-1}) &= \frac{p-1}{p(p-1)} \left[\phi(1)m^p + \sum_{1 \neq \sigma \in \langle (12 \dots p) \rangle} -1 \times m^1 \right] \\ &= \frac{1}{p} [(p-1)m^p - (p-1)m] \\ &= \frac{(p-1)}{p} (m^p - m). \end{aligned}$$

The proof is complete.

Corollary 2.3. (little Fermat theorem)

Let p be a prime and m be a positive integer number, then $m^p \equiv m \pmod{p}$.

Proof. By using the above Theorem 2.2, since

$\frac{(p-1)}{p} (m^p - m)$ is an integer number and the greatest common divided of p and $(p-1)$ is 1, so p must divide $m^p - m$, then $m^p \equiv m \pmod{p}$.

In general, the mapping from $F_{p,q}$ to S_p is not known, therefore we can't calculate the dimension of the symmetry classes of tensor associated with $F_{p,q}$ and linear irreducible character, in the next example we calculate these for $F_{5,4}$.

Example 2.4.

If $(1\ 2\ 3\ 4\ 5)$ and $(2\ 3\ 5\ 4)$ are cyclic permutations of order 5 and 4, respectively in S_5 , then it can be verified that the mapping

$$\begin{aligned} a &\mapsto (1\ 2\ 3\ 4\ 5), \\ b &\mapsto (2\ 3\ 5\ 4), \end{aligned}$$

embeds $F_{5,4}$ in S_5 . Now considering $F_{5,4}$ as a subgroup of S_5 . By use the character table of $F_{5,4}$, we find the dimensions of the symmetry classes of tensors associated with this group as follows:

$$\dim V_{\chi_0}^5 = \frac{1}{20} [m^5 + 5m^3 + 10m^2 + 4m]$$

$$\dim V_{\chi_1}^5 = \frac{1}{20} [m^5 - 5m^3 + 4m]$$

$$\dim V_{\chi_2}^5 = \frac{1}{20} [m^5 + 5m^3 - 10m^2 + 4m]$$

$$\dim V_{\chi_3}^5 = \frac{1}{20} [m^5 - 5m^3 + 4m]$$

$$\dim V_{\phi}^5 = \frac{4}{5} [m^5 - m]$$

Corollary 2.5.

Let m be a positive integer number, then the number 20 divided the numbers $[m^5 + 5m^3 + 10m^2 + 4m]$, $[m^5 + 5m^3 - 10m^2 + 4m]$ and $[m^5 - 5m^3 + 4m]$.

Proof. By using Example 2.3., since the dimension of a vector space is an integer number, so the number 20 must divide the numbers $[m^5 + 5m^3 + 10m^2 + 4m]$, $[m^5 + 5m^3 - 10m^2 + 4m]$ and $[m^5 - 5m^3 + 4m]$ and the proof is complete.

In continue we try to calculate the dimension of $V_{\phi_j}^p(F_{p,q})$. By use conjugacy classes of $F_{p,q} \leq S_p$, we

have $c(1)=p$ and $c(a^{v_i^s}) = 1, i=1,\dots, \frac{p}{q-1}, s \in S$.

If we consider $\omega = \exp\left(\frac{2\pi i}{q}\right)$, we have:

$$\dim V_{\phi_j}^p(F_{p,q}) = \frac{\phi_j(1)}{pq} \left[qm^p + \sum_{i=1}^{\frac{p-1}{q}} \sum_{s \in S} \omega^{\frac{p}{q} s v_i v_j} m^1 \right]$$

On the other hand, we know the set

$$\left\{ S v_i : i = 1, \dots, \frac{p-1}{q} \right\}$$

and for $j = 1, \dots, \frac{p}{q-1}$ the set

$$\left\{ S v_i v_j : i = 1, \dots, \frac{p-1}{q} \right\}$$

are partitions of Z_p^* , therefore

$$\begin{aligned} \sum_{i=1}^{\frac{p-1}{q}} \sum_{s \in S} \omega^{\frac{p}{q} s v_i v_j} &= \sum_{r \in Z_p^*} \omega^{\frac{p}{q} r} = \left(\omega^{\frac{p}{q}} + \omega^{\frac{2p}{q}} + \dots + \omega^{\frac{(p-1)p}{q}} \right) \\ &= \frac{1 - \omega^{\frac{p^2}{q}}}{1 - \omega^{\frac{p}{q}}} - 1. \end{aligned}$$

Therefore

$$\dim V_{\phi_j}^p(F_{p,q}) = \frac{1}{p} \left[qm^p + \frac{1 - \omega^{\frac{p^2}{q}}}{1 - \omega^{\frac{p}{q}}} m - m \right].$$

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