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The Mean Field Forward Backward Stochastic Differential Equations and Stochastic Partial Differential Equations

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Abstract: Since 1990 Pardoux and Peng, proposed the theory of backward stochastic differential equation Backward stochastic differential equation and is backward stochastic differential equations (short for FBSDE) theory has been widely research (see El Karoui, Peng and Cauenez, Ma and Yong, etc.) Generally, a backward stochastic differential equation is a type Ito stochastic differential equation and a coupling Pardoux - Peng and backward stochastic differential equation. Antonelli, Ma, Protter and Yong is backward stochastic differential equation for a series of research, and apply to the financial. One of the research direction is put forward by Hu and Peng first. Peng and Wu Peng and Shi made a further research, and Yong to a more detailed discussion of this method, by introducing the concept of the bridge, systematically studied the FBSDE continuity method. Because such a system can be applied to random Feynman - Kac of partial differential equations of research, And a double optimal control problem of stochastic control systems, we will be working in Peng and Shi further in-depth study on the basis of this category are backward stochastic differential equation. In this paper, we are considering various constraint conditions with backward stochastic differential equation.

Keywords: FBSDE, Mean-Field Forward Backward, Stochastic Differential Equations, Stochastic Partial Differential Equations

1. Introduction

In order to study the stochastic partial differential equations are non local (SPDEs):

$$\begin{cases} u(t,x) = \mathrm{E}[\Phi(y^{0,x_0}(T),x)] + \int_t^T [\aleph u(s,x) + \hat{F}(s,y^{0,x_0}(s),x)]ds + \int_t^T p\hat{G}(s,y^{0,x_0}(s),x)\overleftarrow{d}B(s), \\ \nabla u(t,x)\mu(t,x) = q\hat{G}(t,y^{0,x_0}(t),x),\ p+q=1, q\neq 0, p,q\in\Re, \\ v(t,x) = \nabla u(t,x)\hat{g}(t,y^{0,x_0}(t),x), \forall (t,x)\in[0,T]\times\Re^n, \end{cases} \tag{1}$$

Where

$$u:\Re_+\times\Re^n\to\Re^m,\quad \aleph u = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_m \end{pmatrix},$$

$$Lu_k(t,x) := \sum_{i=1}^n \hat{f}_i(t,y^{0,x_0}(t),x)\frac{\partial u_k}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^n \mathrm{E}(\hat{g}\hat{g}^T)_{ij}(t,y^{0,x_0}(t),x)\frac{\partial^2 u_k}{\partial x_i \partial x_j}(t,x) - \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 u_k}{\partial x_i\partial x_j}(t,x)(\mu(t,x)\mu(t,x)^T)_{ij}, k=1,\cdots,m.$$

and

$$\hat{\phi}(s,y^{0,x_0}(s),x)=\mathrm{E}[\phi(s,y^{0,x_0}(s),x,u(s,y^{0,x_0}(s)),u(s,x),\mu(s,y^{0,x_0}(s)),\mu(s,x),v(s,y^{0,x_0}(s)),v(s,x))],$$

In this section we study the mean field forward backward stochastic differential equations (MF-FBDSDE)

$$\begin{cases} y(t)=x+\int_0^t \mathrm{E}'f(s,\xi(s))ds+\int_0^t \mathrm{E}'g(s,\xi(s))\overrightarrow{d}W(s)-\int_0^t z(s)\overleftarrow{d}B(s),\\ Y(t)=\mathrm{E}'\Phi(y'(T),y(T))-\int_t^T E'F(s,\xi(s)ds-\int_t^T E'G(s,\xi(s))\overleftarrow{d}B(s)-\int_t^T Z(s)\overrightarrow{d}W(s),\end{cases}$$

where

$$\mathrm{E}'l(s,\xi(s))=\mathrm{E}'l(s,y(s),Y(s),z(s),Z(s),y'(s),Y'(s),z'(s),Z'(s))$$
$$=\int_\Omega l(s,\omega,\omega',y(s,w),Y(s,w),z(s,w),Z(s,w),y(s,w'),Y(s,w'),z(s,w'),Z(s,w'))P(d\omega'),l=f,g,F,G,$$

and

$$\mathrm{E}'\Phi(y(T),y'(T))=\int_\Omega \Phi(\omega,\omega',y(T,\omega),y(T,\omega'))P(d\omega'). \quad (2)$$

Equation (2) to Carmona and Delarue's results are extended to the stochastic case, the Peng and Shi results are extended to the case of the average field. Under certain monotonicity conditions, through the continuity of MF-FBDSDE (2) method to get the existence and uniqueness of the solution. Finally, the application of WF-FBDSDE (2), are non stochastic partial differential equation (1) represents the local solution of the probability.

2. Problem Presentation

Hypothesis $(\Omega^2,F^2,P^2)=(\Omega\times\Omega,F\otimes F,P\otimes P)$ is a complete space with its own product space here, for any $t\in[0,T], F_t^2=F_t\otimes F_t, F_t\otimes F_t$ *is* $F_t\times F$ completion. Arbitrary definition of $\xi=\xi(\omega)$ in the Ω can be extended to Ω^2 natural space, namely $\xi'(\omega,\omega')=\xi(\omega),(\omega,\omega')\in\Omega^2$. $H=R^n$, and so on, we define

$$L^1(\Omega^2,F^2,\mathrm{P}^2;H)=\left\{\xi\middle|\xi:\Omega^2\to H,\text{is } F^2\textit{-measurable} \text{ and } \textit{meeting}\,\mathrm{E}^2|\xi|\equiv\int_{\Omega^2}|\xi(\omega',\omega)|\mathrm{P}(d\omega)<\infty\right\}.$$

for any $\eta\in L^1(\Omega^2,F^2,\mathrm{P}^2;H)$,

note $\mathrm{E}'\eta(\omega,\cdot)=\int_\Omega \eta(\omega,\omega')\mathrm{P}(d\omega'),\mathrm{E}^*\eta(\cdot,\omega)=\int_\Omega \eta(\omega',\omega)\mathrm{P}(d\omega').$

if $\eta_1(\omega,\omega')=\eta_1(\omega'),\eta_2(\omega,\omega')=\eta_2(\omega)$, *then*

$$\mathrm{E}'\eta_1=\int_\Omega \eta_1(\omega')\mathrm{P}(d\omega')=\mathrm{E}\eta_1,\mathrm{E}^*\eta_2=\int_\Omega \eta_2(\omega)\mathrm{P}(d\omega)=\mathrm{E}\eta_2.$$

When ω,ω' at the same time,

In order to distinguish between ω and ω', we use mark E' and E*.

From now on, when we talk about MF-BDSVIE , the mapping of Γ^f and Γ^g are defined by the E' operator. Obviously, Γ^f is a non local means the $\Gamma^f(s,Y(s),Z(s))$ value in the $\Gamma^f(s,w,Y(s,w),Z(s,w))$ depends on the whole set

$$\{Y(s,w'),Z(s,w')|\omega'\in\Omega\}$$

Not only

$$(Y(s,w),Z(s,w))$$

Introduction of mark

$$U=\begin{pmatrix}y\\Y\\z\\Z\end{pmatrix},U'=\begin{pmatrix}y'\\Y'\\z'\\Z'\end{pmatrix},\xi=\begin{pmatrix}U\\U'\end{pmatrix},A(t,\xi)=\begin{pmatrix}-F\\f\\-G\\g\end{pmatrix}(t,\xi).$$

considering MF-FBDSDEs(1), *where*

$$F:\Omega\times[0,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^n,$$
$$f:\Omega\times[0,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^n,$$
$$G:\Omega\times[0,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^{n\times l},$$

$$g:\Omega\times[0,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^{n\times d},$$
$$\Phi:\Omega\times\Re^n\times\Re^n\to\Re^n.$$

Definition.

$F_t - measurable\ process\ (y,Y,z,Z)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$ is called the solution of MF-FBDSDEs if (y,Y,z,Z) is meeting (2)

We assume that for any (H1) $\xi\in R^{n+n+n\times l+n\times d+n+n+n\times l+n\times d}$, $A(\cdot,\xi)$ is $F_t - measurable\ process$ on the $[0,T]$, it is meeting $A(\cdot,0)\in M^2\left(0,T;R^{n+n+n\times l+n\times d+n+n+n\times l+n\times d}\right)$

(H2) $A(t,\xi)$ *and* $\Phi(y)$ is meeting Lipschitz condition: there exist contact $k>0, 0<\lambda<\frac{1}{2}$, making

$$\left|f(t,\xi)-f(t,\bar\xi)\right|^2\le k(|\hat y|^2+|\hat Y|^2+|\hat z|^2+|\hat Z|^2+|\hat y'|^2+|\hat Y'|^2+|\hat z'|^2+|\hat Z'|^2),$$
$$\left|F(t,\xi)-F(t,\bar\xi)\right|^2\le k(|\hat y|^2+|\hat Y|^2+|\hat z|^2+|\hat Z|^2+|\hat y'|^2+|\hat Y'|^2+|\hat z'|^2+|\hat Z'|^2),$$
$$\left|g(t,\xi)-g(t,\bar\xi)\right|^2\le k(|\hat y|^2+|\hat Y|^2+|\hat Z|^2+|\hat y'|^2+|\hat Y'|^2+|\hat Z'|^2)+\lambda(|\hat z|^2+|\hat z'|^2),$$
$$\left|G(t,\xi)-G(t,\bar\xi)\right|^2\le k(|\hat y|^2+|\hat Y|^2+|\hat z|^2+|\hat y'|^2+|\hat Y'|^2+|\hat z'|^2)+\lambda(|\hat Z|^2+|\hat Z'|^2),$$

(H3) $A(t,\xi)$ *and* $\Phi(y)$ are meeting the following monotonicity condition, that is to say, for constant $\mu>0,\beta>0$, making

$$\mathrm{E}'\langle A(t,\xi)-A(t,\bar\xi),U-\bar U\rangle\le-\mu\left|U-\bar U\right|^2,$$
$$\forall U=(y,Y,z,Z)^T,\bar U=(\bar y,\bar Y,\bar z,\bar Z)^T,U'=(y',Y',z',Z')^T,$$
$$\bar U'=(\bar y',\bar Y',\bar z',\bar Z')^T\in\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d},\forall t\in[0,T].$$
$$\mathrm{E}'\langle\Phi(y,y')-\Phi(\bar y,\bar y'),y-\bar y\rangle\ge\beta\left|y-\bar y\right|^2,\forall y,\bar y\in\Re^n.$$

Where

$$\mu>0,\beta>0$$

3. The Existence and Uniqueness of Solutions for MF-FBDSDEs

In order to hypothesis (H1) -(H3), prove the equation (3.68) the existence and uniqueness of the solution, we need the following lemma. The following lemma is discussed in class. $\alpha\in[0,1]$ is a priori parameter estimation of MF-FBDSDEs:

$$\begin{cases} dy(t)=[\alpha\mathrm{E}'f(t,\xi(t))+f_0(t)]dt-z(t)\overleftarrow{d}B(t)+[\alpha\mathrm{E}'g(t,\xi(t))+g_0(t)]\vec{d}W(t),\\ dY(t)=[\alpha\mathrm{E}'F(t,\xi(t))-(1-\alpha)\mu y(t)+F_0(t)]dt+Z(t)\vec{d}W(t)+[\alpha\mathrm{E}'G(t,\xi(t))-(1-\alpha)\mu z(t)+G_0(t)]\overleftarrow{d}B(t),\\ y_0=x,Y(T)=\alpha\mathrm{E}'\Phi(y'(T),y(T))+(1-\alpha)y(T)+\phi, \end{cases} \tag{3}$$

Where $\xi=(y,Y,z,Z,y',Y',z',Z')$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$, $\varphi\in L^2\left(\Omega,F_T,P,R^n\right)$ are arbitrary given vector-valued random variables.

When α = 1, the existence of solution of equation (3.69) means that the existence of solution of equation (3.68) exist. When α = 0, according to the [111] of the MF - BDSDE results about the existence and uniqueness of the solution, that the equation (3.69) is the only solution can be got.

The following lemma is the key in the continuous method, it provides for a fixed $\alpha=\alpha_0\in[0,1)$, if the equation (3.69) is the only solution, there exist a α_0 has nothing to do with the normal number δ_0 , to make the equation (3.69) is the only solution also for $\alpha\in[\alpha_0,\alpha_0+\delta_0]$.

Lemma 2. Supposing that (H1)-(H3), if the equation (3.69) is the only solution for certain $\alpha=\alpha_0\in[0,1)$, $\psi\in L^2\left(\Omega,F_T,P,R^n\right),\varphi\in L^2\left(\Omega,F_T,P,R^m\right)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$,

There exist a α_0 has nothing to do with the normal number δ_0 , to make the equation (3.69) is the only solution also in the $M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$ for $\alpha\in[\alpha_0,\alpha_0+\delta_0]$, $\psi\in L^2\left(\Omega,F_T,P,R^n\right),\varphi\in L^2\left(\Omega,F_T,P,R^m\right)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$.

Proof

Supposing that

$$U=(y,Y,z,Z),U=(\tilde y,Y,\tilde z,Z),$$
$$U=(\bar y,Y,\bar z,Z),U=(\tilde y,Y,\tilde z,Z),$$
$$\xi=(y',Y',z',Z',y,Y,z,Z),\xi=(\tilde y',Y',\tilde z',Z',\tilde y,Y,\tilde z,Z),$$
$$\bar\xi=(\bar y',\bar Y',\bar z',\bar Z',\bar y,\bar Y,\bar z,\bar Z),\xi=(\bar{\tilde y}',\bar Y',\bar{\tilde z}',\bar Z',\bar{\tilde y},\bar Y,\bar{\tilde z},\bar Z),$$
$$\xi=\xi-\xi,\xi=\xi-\xi,$$
$$U=(\hat y,Y,\hat z,Z)=(y-\tilde y,Y-Y,z-\tilde z,Z-Z),$$
$$\bar U=(\hat{\bar y},\bar Y,\hat{\bar z},\bar Z)=(\bar y-\bar{\tilde y},\bar Y-Y,\bar z-\bar{\tilde z},\bar Z-Z).$$

When $\alpha=\alpha_0$, if the equation (3.69) is the only solution for $x\in R^n$, $\left(F_0,f_0,G_0,g_0\right)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$, $\varphi\in L^2\left(\Omega,F_{T,}P,R^n\right)$, there exist only $U=\left(y,Y,z,Z\right)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$ meeting the following equations for each $\bar{U}=\left(\bar{y},\bar{Y},\bar{z},\bar{Z}\right)\in M^2\left(0,T;R^{n+n+n\times l+n\times d}\right)$.

$$\begin{cases} dy(t)=\left[\alpha_0 E'f(t,\xi(t))+\delta E'f(t,\bar{\xi}(t))+f_0(t)\right]dt-z(t)\vec{d}B(t)+\left[\alpha_0 E'g(t,\xi(t))+\delta E'g(t,\bar{\xi}(t))+g_0(t)\right]\vec{d}W(t), \\ dY(t)=\left[\alpha_0 E'F(t,\xi(t))-(1-\alpha_0)\mu y(t)+\delta(E'F(t,\bar{\xi}(t))+\mu\bar{y}(t))+F_0(t)\right]dt \\ +[\alpha_0 E'G(t,\xi(t))-(1-\alpha_0)\mu z(t)+\delta(E'G(t,\bar{\xi}(t))+\mu\bar{z}(t))+G_0(t)]\vec{d}B(t)+Z(t)\vec{d}W(t), \\ y_0=x, \\ Y(T)=\alpha E'\Phi(y'(T),y(T))+(1-\alpha_0)y(T)+\delta(E'\Phi(\bar{y}'(T),\bar{y}(T))-\bar{y}(T))+\phi, \end{cases}$$

Where $\delta\in(0,1)$ is independent of α_0, Our aim is to prove the following mapping

$$U=I_{\alpha_0+\delta}(\bar{U}):M^2(0,T;\Re^{n+n+n\times l+n\times d})\to M^2(0,T;\Re^{n+n+n\times l+n\times d})$$

It can be compressed for small enough $\delta>1$

Supposing that $\bar{\bar{U}}=(\bar{\bar{y}},\bar{\bar{Y}},\bar{\bar{z}},\bar{\bar{Z}})\in M^2(0,T;R^{n+n+n\times l+n\times d})$,

$$\tilde{U}=(\tilde{y},\tilde{Y},\tilde{z},\tilde{Z})=I_{\alpha_0+\delta}(\bar{\bar{U}}).$$

On the $[0,T]$, The *Itô* formula was applied with $\langle\hat{y},Y\rangle$,we can obtain

$$E\left\langle\hat{y}(T),\alpha_0\hat{\Phi}(y(T))+(1-\alpha_0)\hat{y}(T)\right\rangle+(1-\alpha_0)\mu E\int_0^T(|\hat{y}(t)|^2+|\hat{z}(t)|^2)dt-E\int_0^T E'\left\langle\alpha_0\left(A(t,\xi(t))-A\left(t,\tilde{\xi}(t)\right),\hat{U}(t)\right\rangle dt$$
$$=E\left\langle\hat{y}(T),\delta\hat{\bar{y}}(T)\right\rangle-E\left\langle\hat{y}(T),\delta\hat{\Phi}(\bar{y}(T))\right\rangle+\delta E\int_0^T\left(\left\langle\hat{Y}(t),\hat{f}(t,\bar{\xi}(t))\right\rangle+\left\langle\hat{y}(t),\hat{F}(t,\bar{\xi}(t))\right\rangle+\left\langle\hat{Z}(t),\hat{g}(t,\bar{\xi}(t))\right\rangle+\left\langle\hat{z}(t),\hat{G}(t,\bar{\xi}(t))\right\rangle\right)dt$$
$$+\delta\mu E\int_0^T\left(\left\langle\hat{y}(t),\hat{\bar{y}}(t)\right\rangle+\left\langle\hat{z}(t),\hat{\bar{z}}(t)\right\rangle\right)dt$$

Where

$$\begin{aligned} \hat{f}\left(t,\bar{\xi}(t)\right)&=E'f\left(t,\bar{\xi}(t)\right)-E'f\left(t,\bar{\xi}(t)\right), \\ \hat{g}\left(t,\bar{\xi}(t)\right)&=E'g\left(t,\bar{\xi}(t)\right)-E'g\left(t,\bar{\xi}(t)\right), \\ \hat{F}\left(t,\bar{\xi}(t)\right)&=E'F\left(t,\bar{\xi}(t)\right)-E'F\left(t,\bar{\xi}(t)\right), \\ \bar{G}\left(t,\bar{\xi}(t)\right)&=E'G\left(t,\bar{\xi}(t)\right)-E'G\left(t,\xi(t)\right), \\ \Phi(\bar{y}(T))&=E'\Phi(\bar{y}'(T),\bar{y}(T))-E'\Phi(\bar{\bar{y}}'(T),\bar{\bar{y}}(T)), \\ \Phi(y(T))&=E'\Phi(y'(T),y(T))-E'\Phi(\tilde{y}'(T),\tilde{y}(T)). \end{aligned}$$

according to the H1-H3,

$$(1-\alpha_0+\alpha_0\beta)E|\hat{y}(T)|^2+\mu E\int_0^T\left(|\hat{y}(t)|^2+|\hat{z}(t)|^2\right)dt\le\delta CE\int_0^T\left(\left|\hat{U}(t)\right|^2+\left|\hat{\bar{U}}(t)\right|^2\right)dt+\delta C\left(E|\hat{y}(T)|^2+E\left|\hat{\bar{y}}(T)\right|^2\right),$$

Where constant C>0, thereafter, it would be appropriate constants. It can be progressive different and only depends on the Lipschitz constant. On the other hand, we apply estimation technology to $(Y,W)=(Y-Y,W-W)$. We apply *Itô* formula to $|Y(t)|^2$ on the $[0,T]$.we can get the following,

$$
\begin{aligned}
&\mathrm{E}\left|\widehat{Y}(t)\right|+\mathrm{E}\int_t^T\left|\widehat{Z}(s)\right|^2 ds=\mathrm{E}\left|\alpha_0\widehat{\Phi}(y(T))+(1-\alpha_0)\widehat{y}(T)+\delta\left(\widehat{\Phi}(\overline{y}(T))-\widehat{\overline{y}}(T)\right)\right|^2\\
&-2\mathrm{E}\int_t^T\left\langle\widehat{Y}(s),\alpha_0\widehat{F}(s,\xi(s))-(1-\alpha_0)\mu\widehat{y}(s)+\delta\left(\widehat{F}(s,\overline{\xi}(s))+\mu\widehat{\overline{y}}(s)\right)\right\rangle ds\\
&+\mathrm{E}\int_t^T\left|\alpha_0\widehat{G}(s,\xi(s))-(1-\alpha_0)\mu\widehat{z}(s)+\delta\left(\widehat{G}(s,\overline{\xi}(s))+\mu\widehat{\overline{z}}(s)\right)\right|^2 ds,
\end{aligned}
$$

Where

$$
\begin{aligned}
&F\left(t,\xi(t)\right)=\mathrm{E}'F\left(t,\xi(t)\right)-\mathrm{E}'F\left(t,\xi(t)\right),\\
&\overline{F}\left(t,\overline{\xi}(t)\right)=\mathrm{E}'F\left(t,\overline{\xi}(t)\right)-\mathrm{E}'F\left(t,\xi(t)\right),\\
&G\left(t,\xi(t)\right)=\mathrm{E}'G\left(t,\xi(t)\right)-\mathrm{E}'G\left(t,\xi(t)\right),\\
&\overline{G}\left(t,\overline{\xi}(t)\right)=\mathrm{E}'G\left(t,\overline{\xi}(t)\right)-\mathrm{E}'G\left(t,\xi(t)\right),\\
&\Phi\left(y(T)\right)=\mathrm{E}'\Phi\left(y'(T),y(T)\right)-\mathrm{E}'\Phi\left(\tilde{y}'(T)\tilde{y}(T)\right),\\
&\Phi\left(\overline{y}(T)\right)=\mathrm{E}'\Phi\left(\overline{y}'(T),\overline{y}(T)\right)-\mathrm{E}'\Phi\left(\overline{\tilde{y}}'(T)\overline{\tilde{y}}(T)\right).
\end{aligned}
$$

Form the (H.3), we can get the following

$$
\begin{aligned}
&\mathrm{E}\left|\widehat{Y}(t)\right|^2+\mathrm{E}\int_t^T\left|\widehat{Z}(s)\right|^2 ds\le 4\mathrm{E}\left[\alpha_0^2\left|\widehat{\Phi}\left(y(T)\right)\right|^2+(1-\alpha_0)^2\left|\widehat{y}(T)\right|^2+\delta^2\left|\widehat{\Phi}(\overline{y}(T))\right|^2+\delta^2\left|\widehat{\overline{y}}(T)\right|^2\right]\\
&+2\mathrm{E}\int_t^T\left|\widehat{Y}(s)\right|\left(\alpha_0\left|\widehat{F}\left(s,\xi(s)\right)\right|+(1-\alpha_0)\mu\left|\widehat{y}(s)\right|+\delta\left|\widehat{F}\left(s,\overline{\xi}(s)\right)\right|+\delta\mu\left|\widehat{\overline{y}}(s)\right|\right)ds\\
&+\mathrm{E}\int_t^T\left[\frac{1+2\lambda}{4\lambda}\alpha_0^2\left|\widehat{G}\left(s,\xi(s)\right)\right|^2\right]ds+3\mathrm{E}\int_t^T\left[\frac{1+2\lambda}{1-2\lambda}\left((1-\alpha_0)^2\mu^2\left|\widehat{z}(s)\right|^2+\delta^2\left|\widehat{G}\left(s,\overline{\xi}(s)\right)\right|^2+\delta^2\mu^2\left|\widehat{\overline{z}}(s)\right|^2\right)\right]ds\\
&\le C\mathrm{E}\left|\widehat{y}(T)\right|^2+\delta C\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2+\mathrm{E}\int_t^T\left[\left(\frac{8k}{1-2\lambda}\left|\widehat{Y}(s)\right|^2+\frac{1-2\lambda}{8k}\left|\widehat{F}\left(s,\xi(s)\right)\right|^2\right)+(1-\alpha_0)\mu\left(\left|\widehat{Y}(s)\right|^2+\left|\widehat{y}(s)\right|^2\right)\right]ds\\
&+\delta\mathrm{E}\int_t^T\left[\left(\left|\widehat{Y}(s)\right|^2+\left|\widehat{F}\left(s,\overline{\xi}(s)\right)\right|^2\right)+\mu\left(\left|\widehat{Y}(s)\right|^2+\left|\widehat{\overline{y}}(s)\right|^2\right)\right]ds+\mathrm{E}\int_t^T\left[\frac{1+2\lambda}{4\lambda}\left|\widehat{G}\left(s,\xi(s)\right)\right|^2\right]ds\\
&+3\mathrm{E}\int_t^T\left[\frac{1+2\lambda}{1-2\lambda}(1-\alpha_0)^2\mu^2\left|\widehat{z}(s)\right|^2+\delta^2\left|\widehat{G}\left(s,\overline{\xi}(s)\right)\right|^2+\delta^2\mu^2\left|\widehat{\overline{z}}(s)\right|^2\right]ds\\
&\le C\mathrm{E}\left|\widehat{y}(T)\right|^2+\delta C\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2+C\mathrm{E}\int_t^T\left|\widehat{Y}(s)\right|^2 ds+\delta C\mathrm{E}\int_t^T\left|\widehat{\overline{\xi}}(s)\right|^2 ds+C\mathrm{E}\int_t^T\left(\left|\widehat{y}(s)\right|^2+\left|\widehat{z}(s)\right|^2\right)ds+\frac{3+2\lambda}{4}\mathrm{E}\int_t^T\left|\widehat{Z}(s)\right|^2 ds.
\end{aligned}
$$

Thus

$$
\mathrm{E}\left|\widehat{Y}(t)\right|^2+\frac{1-2\lambda}{4}\mathrm{E}\int_t^T\left|\widehat{Z}(s)\right|^2 ds\le C\mathrm{E}\int_t^T\left|\widehat{Y}(s)\right|^2 ds+C\left(\mathrm{E}\left|\widehat{y}(T)\right|^2+\delta\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2\right)+C\mathrm{E}\int_t^T\left(\left|\widehat{y}(s)\right|^2+\left|\widehat{z}(s)\right|^2+\delta\left|\widehat{\overline{U}}(s)\right|^2\right)ds.
$$

Form Gronwall inequality, we can obtain the following

$$
\mathrm{E}\left|\widehat{Y}(t)\right|^2+\mathrm{E}\int_t^T\left|\widehat{Z}(s)\right|^2 ds\le C\left(\mathrm{E}\left|\widehat{y}(T)\right|^2+\delta\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2\right)+C\mathrm{E}\int_0^T\left(\left|\widehat{y}(t)\right|^2+\left|\widehat{z}(t)\right|^2+\delta\left|\widehat{\overline{U}}(t)\right|^2\right)dt
$$

so

$$
\mathrm{E}\int_0^T\left(\left|\widehat{Y}(t)\right|^2+\left|\widehat{Z}(t)\right|^2\right)dt\le C\left(\mathrm{E}\left|\widehat{y}(T)\right|^2+\delta\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2\right)+C\mathrm{E}\int_0^T\left(\left|\widehat{y}(t)\right|^2+\left|\widehat{z}(t)\right|^2+\delta\left|\widehat{\overline{U}}(t)\right|^2\right)dt.(3.71)
$$

Combined with the above estimates (3.70) and (3.71), The constant of the full C>0, the following is available.

$$
\mathrm{E}\int_0^T\left|\widehat{U}(t)\right|^2 dt+\mathrm{E}\left|\widehat{y}(T)\right|^2\le\delta C\left(\mathrm{E}\int_0^T\left|\widehat{U}(t)\right|^2 dt+\mathrm{E}\left|\widehat{y}(T)\right|^2+\mathrm{E}\int_0^T\left|\widehat{\overline{U}}(t)\right|^2 dt+\mathrm{E}\left|\widehat{\overline{y}}(T)\right|^2\right)
$$

$\delta_0=\dfrac{1}{3C}$ It is easy to see, for each fixed $\delta\in[0,\ \delta_0]$, the mapping is compressed, that is to say,

$$\mathrm{E}\int_0^T \left|\widehat{U}(t)\right|^2 dt + \mathrm{E}\left|\widehat{y}(T)\right|^2 \le \frac{1}{2}\left(\mathrm{E}\int_0^T \left|\widehat{\widehat{U}}(t)\right|^2 dt + \mathrm{E}\left|\widehat{\widehat{y}}(T)\right|^2\right).$$

$$\mathrm{E}\left\langle \widehat{y}(T), \widehat{\Phi}(y(T))\right\rangle = \mathrm{E}\int_0^T \mathrm{E}'\left\langle A(t,U(t)) - A(t,\overline{U}(t)), \widehat{U}(t)\right\rangle dt.$$

From that, we could know

$$\mu\mathrm{E}\int_0^T \mathrm{E}'\left\langle A(t,U(t)) - A(t,\overline{U}(t)), \widehat{U}(t)\right\rangle dt \le 0$$

Thus the existence of the fixed point of mapping, conclusion is proved.

Existence and uniqueness theorem of solutions of MF-FBDSDE is given below.

Theorem 3. We suppose the (H1)-(H3) are tenable, thus MF-FBDSDE exist uniqueness solutions in the

$$M^2\left(0,T;R^{n+n+n\times l+n\times d}\right).$$

Proof

(Uniqueness) We take $U=(y,Y,z,Z)$ and $\overline{U}=(\overline{y},\overline{Y},\overline{z},\overline{Z})$ as too solutions of the equation (3.68). We continue to use the mark in 3.5.2 lemma. Application formula. $It\hat{o}$ to $\left\langle \widehat{y},\widehat{Y}\right\rangle$ on [0，T]

We can get,

$$\mathrm{E}\left\langle \widehat{y}(T), \widehat{\Phi}(y(T))\right\rangle = \mathrm{E}\int_0^T \mathrm{E}'\left\langle A(t,U(t)) - A(t,\overline{U}(t)), \widehat{U}(t)\right\rangle dt.$$

According to assumption (H.3), we can know

$$\mu\mathrm{E}\int_0^T \mathrm{E}'\left\langle A(t,U(t)) - A(t,\overline{U}(t)), \widehat{U}(t)\right\rangle dt \le 0.$$

So $U \equiv U'$, Uniqueness is proved.

(Existence) when $\alpha=0$,The equation(3.69)has only one solution in $M^2(0,T;\Re^{n+n+n\times l+n\times d})$

According to Lemma3.5.2, we can know

There exists positive $\delta_0=\delta_0(k,\lambda,\mu,\beta)$ so to any

$$\delta\in[0，\delta_0]，x\in\Re^n, \phi\in L^2(\Omega,F_T,P;\Re^n),$$
$$(F_0,f_0,G_0,g_0)\in M^2(0,T;\Re^{n+n+n\times l+n\times d}),$$

When $\alpha=\delta$, The equation(3.69)has only one solution. Because δ_0 only relies on (k,λ,μ,β) , Repeat the above process many times, Leading to $1\le N\delta_0<1+\delta_0$ .Particularly, when $\alpha=1$,take $(F_0,f_0,G_0,g_0)\equiv 0$, $\phi\equiv 0$, The equation(3.69)has only one solution in $M^2(0,T;\Re^{n+n+n\times l+n\times d})$

Conclusion is proved.

Note: hypothesis (H.3) could be replaced by the following

$$(\mathrm{H}3.5.3)'\begin{cases}\mathrm{E}'\left\langle A(t,\xi)-A(t,\overline{\xi}),U-\overline{U}\right\rangle \ge \mu\left|U-\overline{U}\right|^2,\\ \forall U=(y,Y,z,Z)^T,\overline{U}=(\overline{y},\overline{Y},\overline{z},\overline{Z})^T,U'=(y',Y',z',Z')^T,\\ \overline{U}'=(\overline{y}',\overline{Y}',\overline{z}',\overline{Z}')^T\in\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d},\forall t\in[0,T].\\ \mathrm{E}'\left\langle \Phi(y,y')-\Phi(\overline{y},\overline{y}'),y-\overline{y}\right\rangle \le -\beta\left|y-\overline{y}\right|^2,\forall y,\overline{y}\in\Re^n,\end{cases}$$

Where μ and β both are positive constant.

Theorem 5. We suppose the (H1), (H2) and (H3)' are tenable, thus MF-FBDSDE exist uniqueness solutions in the

$$M^2\left(0,T;R^{n+n+n\times l+n\times d}\right).$$

4. Probabilistic Representations of Non Local SPDEs Solutions

Using the above MF-FBDSDEs, discuss the non probabilistic local SPDE solutions. For any $x\in R^n$, consider the following WF-FBDSDE:

$$\begin{cases}dy(s)=\mathrm{E}'f(s,\xi(s))ds+\mathrm{E}'g(s,\xi(s))\bar{d}W(s)-z(s)\bar{d}B(s),\\ dY(s)=\mathrm{E}'F(s,\xi(s))ds+\int_t^T\mathrm{E}'G(s,\xi(s))\bar{d}B(s)-Z(s)\bar{d}W(s),\\ y(t)=x,Y(T)=\mathrm{E}'\Phi\left(y'(T),y(T)\right),\end{cases}$$

$$\begin{aligned}&where\xi(s)=\left(y(s),Y(s),z(s),Z(s),y'(s),Y'(s),z'(s),Z'(s)\right),\\ &F:[t,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^n,\\ &f:[t,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^n,\\ &G:[t,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^{n\times l},\\ &g:[t,T]\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\times\Re^n\times\Re^n\times\Re^{n\times l}\times\Re^{n\times d}\to\Re^{n\times d},\\ &\Phi:\Re^n\times\Re^n\to\Re^n.\end{aligned}$$

We suppose that $\left(F,f,Y,\Psi,\Phi\right)$ of MF-FBDSDE is deterministic, there exist uniqueness solution

$\left(y(s),Y(s),z(s),Z(s)\right),s\in[t,T]$ of MF-FBDSDE. We suppose that

$$u(t,x):=Y^{t,x}(t),v(t,x):=O^{t,x}(t)$$

According the uniqueness solution of MF-FBDSDE, we know that for any $t\le s\le T$, we could obtain the following,

$$Y^{t,x}(s)=Y^{t,y^{t,x}(s)}(s)=u(s,y^{t,x}(s)).$$

In order to mark is simple, for $\varphi=F,f,G,g,$ we suppose that

$$\hat{\varphi}(s,y^{0,x_0}(s),x)=\mathrm{E}[\varphi(s,y^{0,x_0}(s),x,u(s,y^{0,x_0}(s)).u(s,x),\mu(s,y^{0,x_0}(s)),$$
$$\mu(s,x),v(s,y^{0,x_0}(s)),v(s,x)].$$

Notice the marks in the second quarter, we know that

$$\hat{\phi}(s,y^{0,x_0}(s),x)=\mathrm{E}'[\phi(\omega',s,y^{0,x_0}(\omega',s),x,u(s,y^{0,x_0}(\omega',s)),u(s,x),\mu(s,y^{0,x_0}(\omega',s)),\mu(s,x),v(s,y^{0,x_0}(\omega',s)),v(s,x))].$$

If there exist A is the following second-order quasilinear nonlocal SPDE:

$$\begin{cases} u(t,x)=\mathrm{E}[\Phi(y^{0,x_0}(T),x)]+\int_t^T[\aleph u(s,x)+\hat{F}(s,y^{0,x_0}(s),x)]ds+\int_t^T p\hat{G}(s,y^{0,x_0}(s),x)\overleftarrow{d}B(s)\\ \nabla u(t,x)\mu(t,x)=q\hat{G}(t,y^{0,x_0}(t),x),p+q=1,q\neq 0,p,q\in\Re,\\ v(t,x)=\nabla u(t,x)\hat{g}(t,y^{0,x_0}(t),x),\forall(t,x)\in[0,T]\times\Re^n,\end{cases}$$

$$where\ \ u:\Re_+\times\Re^n\to\Re^m,\aleph u=\begin{pmatrix}Lu_1\\ \vdots\\ Lu_m\end{pmatrix}$$

$$Lu_k(t,x):=\sum_{i=1}^n\hat{h}_i(t,z^{0,x_0}(t),x)\frac{\partial u_k}{\partial x_i}(t,x)+\frac{1}{2}\sum_{i,j=1}^n\mathrm{E}(\hat{g}\hat{g}^T)_{ij}(t,z^{0,x_0}(t),x)\frac{\partial^2u_k}{\partial x_i\partial x_j}(t,x)$$
$$-\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2u_k}{\partial x_i\partial x_j}(t,x)(\mu(t,x)\mu(t,x)^T)_{ij},k=1,\cdots,m.$$

We can obtain the following:

Theorem3.5.6.Suppose （F,f,G,g,Φ）is deterministic in MF-FBDSDE(3.72).

There is only one solution in MF-FBDSDE(3.72).F,f,G and g are three order continuous differentiable. Φ is two order continuous differentiable.

If （u,v）is the solution of nonlocal SPDE(3.74), so (3.73) is right. (Y,Z) is only determined by equation (3.72)

Proof We only need to prove $\left\{u\left(s,y^{t,x}(s)\right),\hat{g}\left(s,y^{0,x_0}(s),x\right)\nabla u\left(s,y^{t,x}(s)\right);0\le s\le t\right\}$ is the solution of MF-BDSDE. Thus we suppose $t=t_0<t_1<t_2<....<t_n=T$, we can obtain the following

$$u(t_i,z_{t_i}^{t,x})-u(t_{i+1},z_{t_{i+1}}^{t,x})=u(t_i,z_{t_i}^{t,x})-u(t_i,z_{t_{i+1}}^{t,x})+u(t_i,z_{t_{i+1}}^{t,x})-u(t_{i+1},z_{t_{i+1}}^{t,x})$$
$$=-\int_{t_i}^{t_{i+1}}\aleph u(t_i,z^{t,x}(s))ds+\int_{t_i}^{t_{i+1}}\nabla u(t_i,z^{t,x}(s))z(s)\overleftarrow{d}B(s)+\int_{t_i}^{t_{i+1}}\hat{g}(t_i,z^{0,x_0}(s),z^{t,x}(s))\nabla u(s,z^{t,x}(s))\vec{d}W(s)$$
$$+\int_{t_i}^{t_{i+1}}\left[\aleph u(s,z_{t_{i+1}}^{t,x})+\hat{H}(s,z^{0,x_0}(s),z_{t_{i+1}}^{t,x})\right]ds+\int_{t_i}^{t_{i+1}}p\hat{G}(s,z^{0,x_0}(s),z_{t_{i+1}}^{t,x})\overleftarrow{d}B(s),$$

Here we used to meet the Ito formula and U condition equation. Finally, the cell length tends to 0. We can obtain the following

$$u(t,z(t))-u(T,z(T))=\int_t^T\hat{H}(s,z^{0,x_0}(s),z^{t,x}(s))ds+\int_t^T\hat{G}(s,z^{0,x_0}(s),z^{t,x}(s))\overleftarrow{d}B(s)+\int_t^T\hat{g}(s,z^{0,x_0}(s),z^{t,x}(s))\nabla u(s,z^{t,x}(s))\vec{d}W(s).$$

It is easy to verify $Y^{t,x}(s):=u(s,y^{t,x}(s)),Z^{t,x}(s):=\hat{g}(s,y^{0,x_0}(s),x)\nabla u(s,y^{t,x}(s))$ have the same solution with MF-BDSDE.

Note 3.5.7. (I) in non local SPDE, When p=0, the non local SPDE (3.73) degradation as follows

$$\begin{cases} u(t,x)=\mathrm{E}[\Phi(z^{0,x_0}(T),x)]+\int_t^T[\aleph u(s,x)+\hat{H}(s,z^{0,x_0}(s),x)]ds, \\ \nabla u(t,x)\mu(t,x)=\hat{G}(t,z^{0,x_0}(t),x), \\ v(t,x)=\nabla u(t,x)\hat{g}(t,z^{0,x_0}(t),x),\forall(t,x)\in[0,T]\times\Re^n. \end{cases}$$

II) (3.73) can be called non local SPDE (3.74) Feynman-Kac formula.

III) (3.73) to PDE with algebraic equations to the mean field.

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