

Fourier Coefficients of a Class of Eta Quotients of Weight 18 with Level 12

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Abstract: Williams [16] and later Yao, Xia and Jin[15] discovered explicit formulas for the coefficients of the Fourier series expansions of a class of eta quotients. Williams expressed all coefficients of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$ and Yao, Xia and Jin, following the method of proof of Williams, expressed only even coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$. Here, we will express the even Fourier coefficients of 324 eta quotients in terms of $\sigma_{17}(n), \sigma_{17}(\frac{n}{2}), \sigma_{17}(\frac{n}{3}), \sigma_{17}(\frac{n}{4}), \sigma_{17}(\frac{n}{6})$ and $\sigma_{17}(\frac{n}{12})$.

Keywords: Dedekind Eta Function, Eta Quotients, Fourier Series

1. Introduction

The divisor function $\sigma_i(n)$ is defined for a positive integer i by

$$\sigma_i(n) := \sum_{d \text{ positive integer}, d|n} d^i, \text{ if } n \text{ is a positive integer, and} \quad (1)$$

$$\sigma_i(n) := 0 \text{ if } n \text{ is not a positive integer.}$$

The Dedekind eta function is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (2)$$

Where

$$q := e^{2\pi iz}, z \in H = \{x + iy : y > 0\} \quad (3)$$

and an eta quotient of level n is defined by

$$f(z) := \prod_{m|n} \eta(mz)^{a_m}, n, m \in \mathbb{N}, a_m \in \mathbb{Z}. \quad (4)$$

It is interesting and important to determine explicit formulas of the Fourier coefficients of eta quotients, because they are the building blocks of modular forms of level n and weight k . The book of Köhler [13] (Chapter 3, pg.39) describes such expansions by means of Hecke Theta series and develops algorithms for the determination of suitable eta quotients. One can find more information in [3], [6], [14]. I have determined the Fourier coefficients of the theta series

associated to some quadratic forms, see [7], [8], [9][10], [11] and [12].

It is known that Williams, see [16] discovered explicit formulas for the coefficients of Fourier series expansions of a class of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)}$$

gives the expansion found by Williams.

Then Yao, Xia and Jin [15] expressed the even Fourier coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^{25}(2z)\eta^4(3z)}{\eta^{12}(z)\eta^5(4z)\eta^3(6z)\eta(12z)},$$

where the even coefficients are obtained. After that we find that we can express the even Fourier coefficients of 324 eta quotients in terms of $\sigma_{17}(n), \sigma_{17}(\frac{n}{2}), \sigma_{17}(\frac{n}{3}), \sigma_{17}(\frac{n}{4}), \sigma_{17}(\frac{n}{6})$ and $\sigma_{17}(\frac{n}{12})$, see Table 3. One example is as follows:

$$\eta^8(2z)\eta^{20}(4z)\eta^4(6z)\eta^4(12z).$$

We also see that the odd Fourier coefficients of 650 eta quotients are zero and even coefficients can be expressed by

simple formula.

2. Main Body

Now let

$$\begin{aligned}
 f_1 &= \sum_{n=0}^{\infty} f_1(n) = \frac{\eta^{17}(4z)\eta^{19}(6z)\eta^{13}(12z)}{\eta^{13}(2z)}, \\
 f_2 &= \sum_{n=0}^{\infty} f_2(n) = \frac{\eta^4(4z)\eta^{20}(6z)\eta^{20}(12z)}{\eta^8(2z)}, \\
 f_3 &= \sum_{n=0}^{\infty} f_3(n) = \frac{\eta^{16}(4z)\eta^{20}(6z)\eta^8(12z)}{\eta^8(2z)}, \\
 f_4 &= \sum_{n=0}^{\infty} f_4(n) = \frac{\eta^{18}(4z)\eta^6(6z)\eta^{18}(12z)}{\eta^6(2z)}, \\
 f_5 &= \sum_{n=0}^{\infty} f_5(n) = \frac{\eta^{13}(4z)\eta^{11}(6z)\eta^{17}(12z)}{\eta^5(2z)}, \\
 f_6 &= \sum_{n=0}^{\infty} f_6(n) = \frac{\eta^8(4z)\eta^{16}(6z)\eta^{16}(12z)}{\eta^4(2z)}, \\
 f_7 &= \sum_{n=0}^{\infty} f_7(n) = \frac{\eta^{20}(4z)\eta^{16}(6z)\eta^4(12z)}{\eta^4(2z)}, \\
 f_8 &= \sum_{n=0}^{\infty} f_8(n) = \frac{\eta^6(2z)\eta^{18}(4z)\eta^{18}(6z)}{\eta^6(12z)}, \\
 f_9 &= \sum_{n=0}^{\infty} f_9(n) = \frac{\eta^7(2z)\eta^{13}(4z)\eta^{17}(12z)}{\eta(6z)}, \\
 f_{10} &= \sum_{n=0}^{\infty} f_{10}(n) = \eta^8(2z)\eta^{20}(4z)\eta^4(6z)\eta^4(12z), \\
 f_{11} &= \sum_{n=0}^{\infty} f_{11}(n) = \frac{\eta^{18}(4z)\eta^{18}(6z)\eta^{18}(12z)}{\eta^{18}(2z)}, \\
 f_{12} &= \sum_{n=0}^{\infty} f_{12}(n) = \frac{\eta^{11}(2z)\eta^{17}(4z)\eta^{13}(12z)}{\eta^5(6z)}, \\
 f_{13} &= \sum_{n=0}^{\infty} f_{12}(n) = \frac{\eta^{11}(2z)\eta^{17}(4z)\eta^{19}(6z)}{\eta^{11}(12z)}, \\
 f_{14} &= \sum_{n=0}^{\infty} f_{14}(n) = \frac{\eta^{20}(2z)\eta^{20}(4z)\eta^4(12z)}{\eta^8(6z)}, \\
 f_{15} &= \sum_{n=0}^{\infty} f_{15}(n) = \frac{\eta^{11}(2z)\eta^{19}(6z)\eta^{13}(12z)}{\eta^7(4z)},
 \end{aligned}$$

$$\begin{aligned}
 f_{16} &= \sum_{n=0}^{\infty} f_{16}(n) = \frac{\eta^{20}(2z)\eta^{16}(6z)\eta^{16}(12z)}{\eta^{16}(4z)}, \\
 f_{17} &= \sum_{n=0}^{\infty} f_{17}(n) = \frac{\eta^{17}(2z)\eta^{13}(6z)\eta^{13}(12z)}{\eta^7(4z)}, \\
 f_{18} &= \sum_{n=0}^{\infty} f_{18}(n) = \eta^{17}(2z)\eta^5(4z)\eta^{13}(6z)\eta(12z), \\
 f_{19} &= \sum_{n=0}^{\infty} f_{19}(n) = \frac{\eta^{18}(4z)\eta^{12}(6z)\eta^{18}(12z)}{\eta^{12}(2z)}, \\
 f_{20} &= \sum_{n=0}^{\infty} f_{20}(n) = \frac{\eta^{13}(4z)\eta^{17}(6z)\eta^{17}(12z)}{\eta^{11}(2z)}, \\
 f_{21} &= \sum_{n=0}^{\infty} f_{21}(n) = \frac{\eta^{17}(4z)\eta^{13}(6z)\eta^{13}(12z)}{\eta^7(2z)}, \\
 f_{22} &= \sum_{n=0}^{\infty} f_{22}(n) = \frac{\eta^4(4z)\eta^{14}(6z)\eta^{20}(12z)}{\eta^2(2z)}, \\
 f_{23} &= \sum_{n=0}^{\infty} f_{23}(n) = \frac{\eta^{16}(4z)\eta^{14}(6z)\eta^8(12z)}{\eta^2(2z)}, \\
 f_{24} &= \sum_{n=0}^{\infty} f_{24}(n) = \frac{\eta^{18}(2z)\eta^{18}(6z)\eta^{12}(12z)}{\eta^{12}(4z)}, \\
 f_{25} &= \sum_{n=0}^{\infty} f_{25}(n) = \eta^{18}(4z)\eta^{18}(12z), \\
 f_{26} &= \sum_{n=0}^{\infty} f_{26}(n) = \eta(2z)\eta^{13}(4z)\eta^5(6z)\eta^{17}(12z), \\
 f_{27} &= \sum_{n=0}^{\infty} f_{27}(n) = \frac{\eta^7(2z)\eta^{19}(4z)\eta^{11}(6z)}{\eta(12z)}, \\
 f_{28} &= \sum_{n=0}^{\infty} f_{27}(n) = \frac{\eta^{12}(2z)\eta^{18}(4z)\eta^{18}(12z)}{\eta^{12}(6z)}, \\
 f_{29} &= \sum_{n=0}^{\infty} f_{27}(n) = \frac{\eta^{12}(2z)\eta^{18}(4z)\eta^{12}(6z)}{\eta^6(12z)}, \\
 f_{30} &= \sum_{n=0}^{\infty} f_{30}(n) = \frac{\eta^{13}(2z)\eta^{13}(4z)\eta^{17}(12z)}{\eta^7(6z)}, \\
 f_{31} &= \sum_{n=0}^{\infty} f_{31}(n) = \frac{\eta^{18}(2z)\eta^{12}(4z)\eta^{18}(6z)}{\eta^{12}(12z)}.
 \end{aligned}$$

Now we can state our main Theorem:

Theorem 1. Let b_1, b_2, \dots, b_5 be non-negative integers satisfying

$$b_1 + b_2 + \dots + b_5 \leq 36. \tag{5} \quad a_3 := 3b_1 + 2b_2 + 6b_3 + 4b_4 + 3b_5 - 108, \tag{8}$$

Define the integers $a_1, a_2, a_3, a_4, a_6, a_{12}$ by

$$a_1 := -b_1 + 2b_2 - 2b_3 - 4b_4 - b_5 + 36, \tag{6} \quad a_4 := -2b_1 - b_2 - b_3 - 4b_4 + 2b_5 + 36, \tag{9}$$

$$a_2 := 3b_1 + b_2 + 3b_3 + 10b_4 + b_5 - 90, \tag{7} \quad a_6 := -9b_1 - 7b_2 - 9b_3 - 10b_4 - 7b_5 + 270, \tag{10}$$

$$a_{12} := 6b_1 + 3b_2 + 3b_3 + 4b_4 + 2b_5 - 108. \tag{11}$$

They are functions of q by (3). Now define integers

$$k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}, k_{15}, k_{16}, k_{17}, k_{18}, \\ k_{19}, k_{20}, k_{21}, k_{22}, k_{23}, k_{24}, k_{25}, k_{26}, k_{27}, k_{28}, k_{29}, k_{30}, k_{31}, k_{32}, k_{33}, k_{34}, k_{35}, \text{ and } k_{36}$$

by

$$\frac{1}{2^{b_1+b_5}} x^{b_1} (1-x)^{b_2} (1+x)^{b_3} (1+2x)^{b_4} (2+x)^{b_5} = k_0 + k_1x + k_2x^2 + k_3x + k_4x^4 + k_5x^5 + k_6x^6 + k_7x^7 + k_8x^8 \tag{12}$$

$$+ k_9x^9 + k_{10}x^{10} + k_{11}x^{11} + k_{12}x^{12} + k_{13}x^{13} + k_{14}x^{14} + k_{15}x^{15} \tag{13}$$

$$+ k_{16}x^{16} + k_{17}x^{17} + k_{18}x^{18} + k_{19}x^{19} + k_{20}x^{20} \tag{14}$$

$$+ k_{21}x^{21} + k_{22}x^{22} + k_{23}x^{23} + k_{24}x^{24} + k_{25}x^{25} + k_{26}x^{26} + k_{27}x^{27} \tag{15}$$

$$+ k_{28}x^{28} + k_{29}x^{29} + k_{30}x^{30} + k_{31}x^{31} + k_{32}x^{32} + k_{33}x^{33} + k_{34}x^{34} + k_{35}x^{35} + k_{36}x^{36}. \tag{16}$$

Define the rational numbers

$$c_1, c_2, c_3, c_4, c_6, c_{12}, r_1, r_2, \dots, r_{30}$$

and r_{31} as in www.fatih.edu.tr/~bkendirli/weight18/Table 1. Here $\{f_1, \dots, f_{31}\} \setminus \{f_{11}\} \in S_{18}(\Gamma_0(12))$, $f_{11} \in M_{18}(\Gamma_0(12)) \setminus S_{18}(\Gamma_0(12))$ and

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) = \delta(b_1) + \sum_{n=1}^{\infty} c(n)q^n,$$

where for $n \in \mathbb{N}$,

$$c(n) = -c_1\sigma_{17}(n) - c_2\sigma_{17}\left(\frac{n}{2}\right) - c_3\sigma_{17}\left(\frac{n}{3}\right) - c_4\sigma_{17}\left(\frac{n}{4}\right) - c_6\sigma_{17}\left(\frac{n}{6}\right) - c_{12}\sigma_{17}\left(\frac{n}{12}\right) + r_1f_1(n) + \dots + r_{31}f_{31}(n).$$

In particular,

$$c(2n) = -c_1\sigma_{17}(2n) - c_2\sigma_{17}(n) - c_4\sigma_{17}\left(\frac{n}{2}\right) - (262\ 145c_3 + c_6)\sigma_{15}\left(\frac{n}{3}\right) - (c_{12} - 262\ 144c_3)\sigma_{15}\left(\frac{n}{6}\right) + r_1f_1(2n) + \dots \\ + r_{16}f_{16}(2n),$$

$$c(2n-1) = -c_1\sigma_{15}(2n-1) - c_3\sigma_{15}\left(\frac{2n-1}{3}\right) + r_{17}f_{17}(2n-1) + \dots + r_{31}f_{31}(2n-1),$$

for $n \in \mathbb{N}$.

Proof. It follows from (6-11) that

$$a_{-1} + 2a_{-2} + 3a_{-3} + 4a_{-4} + 6a_{-6} + 12a_{-12} = 24b_{-1}, \tag{17}$$

$$a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 36, \tag{18}$$

$$-\frac{a_1}{6} - \frac{a_2}{3} - \frac{a_3}{6} - \frac{2a_4}{3} - \frac{a_6}{3} - \frac{2a_{12}}{3} = -b_1 - b_5.$$

Now we will use p-k parametrization of Alaca, Alaca and Williams, see [1]:

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{19}$$

where the theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{-\infty}^{\infty} q^{n^2}.$$

etting $x = p$ in (12), and multiplying both sides by k^{18} ,we obtain

$$\frac{k^{18}}{2^{b_1+b_5}} p^{b_1} (1-p)^{b_2} (1+p)^{b_3} (1+2p)^{b_4} (2+p)^{b_5}$$

$$= (k_0 + k_1p + k_2p^2 + k_3p^3 + k_4p^4 + k_5p^5 + k_6p^6 + k_7p^7 + k_8p^8 + k_9p^9 + k_{10}p^{10} + k_{11}p^{11} + k_{12}p^{12} + k_{13}p^{13}$$

$$+ k_{14}p^{14} + k_{15}p^{15} + k_{16}p^{16} + k_{17}p^{17} + k_{18}p^{18} + k_{19}p^{19} + k_{20}p^{20} + k_{21}p^{21} + k_{22}p^{22} + k_{23}p^{23} + k_{24}p^{24}$$

$$+ k_{25}p^{25} + k_{26}p^{26} + k_{27}p^{27} + k_{28}p^{28} + k_{29}p^{29} + k_{30}p^{30} + k_{31}p^{31} + k_{32}p^{32} + k_{33}p^{33} + k_{34}p^{34}$$

$$+ k_{35}p^{35} + k_{36}p^{36})k^{18}.$$

Alaca, Alaca and Williams [2] have established the following representations in terms of p and k:

$$\eta(q) = 2^{-1/6} p^{1/24} (1-p)^{1/2} (1+p)^{1/6} (1+2p)^{1/8} (2+p)^{1/8} k^{1/2}, \tag{20}$$

$$\eta(q^2) = 2^{-1/3} p^{1/12} (1-p)^{1/4} (1+p)^{1/12} (1+2p)^{1/4} (2+p)^{1/4} k^{1/2}, \tag{21}$$

$$\eta(q^3) = 2^{-1/6} p^{1/8} (1-p)^{1/6} (1+p)^{1/2} (1+2p)^{1/24} (2+p)^{1/24} k^{1/2}, \tag{22}$$

$$\eta(q^4) = 2^{-2/3} p^{1/6} (1-p)^{1/8} (1+p)^{1/24} (1+2p)^{1/8} (2+p)^{1/2} k^{1/2}, \tag{23}$$

$$\eta(q^6) = 2^{-1/3} p^{1/4} (1-p)^{1/12} (1+p)^{1/4} (1+2p)^{1/12} (2+p)^{1/12} k^{1/2}, \tag{24}$$

$$\eta(q^{12}) = 2^{-2/3} p^{1/2} (1-p)^{1/24} (1+p)^{1/8} (1+2p)^{1/24} (2+p)^{1/6} k^{1/2}, \tag{25}$$

$$E_6(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$= (1 - 246p - 5532p^2 - 38614p^3 - 135369p^4 - 276084p^5$$

$$- 348024p^6 - 276084p^7 - 135369p^8 - 38614p^9 - 5532p^{10}$$

$$- 246p^{11} + p^{12})k^6,$$

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5$$

$$+ 964p^6 + 124p^7 + p^8)k^4.$$

Therefore, since

$$E_{18}(q) = \frac{5500}{43867} E_6^3(q) + \frac{38367}{43867} E_6(q) E_4^3(q),$$

we have

$$E_{18}(q) = (p^{36} + \frac{775\ 242}{43\ 867} p^{35} - \frac{935\ 300\ 808}{43\ 867} p^{34} - \frac{479\ 718\ 421\ 878}{43\ 867} p^{33}$$

$$- \frac{38\ 618\ 625\ 180\ 573}{43\ 867} p^{32} - \frac{1234\ 640\ 783\ 535\ 120}{43\ 867} p^{31}$$

$$- \frac{21\ 916\ 011\ 691\ 473\ 792}{43\ 867} p^{30} - \frac{252\ 643\ 315\ 041\ 469\ 584}{43\ 867} p^{29}$$

$$- \frac{2065\ 064\ 598\ 427\ 463\ 748}{43\ 867} p^{28} - \frac{12\ 645\ 150\ 195\ 540\ 418\ 600}{43\ 867} p^{27}$$

$$- \frac{60\ 191\ 705\ 453\ 414\ 612\ 832}{43\ 867} p^{26} - \frac{228\ 609\ 396\ 856\ 786\ 345\ 128}{43\ 867} p^{25}$$

$$- \frac{706\ 047\ 862\ 569\ 538\ 312\ 980}{43\ 867} p^{24} - \frac{1798\ 300\ 198\ 147\ 477\ 624\ 560}{43\ 867} p^{23}$$

$$- \frac{3817\ 086\ 982\ 353\ 184\ 063\ 104}{43\ 867} p^{22} - \frac{6804\ 790\ 928\ 276\ 409\ 773\ 424}{43\ 867} p^{21}$$

$$\begin{aligned}
 & - \frac{10\,245\,612\,527\,911\,949\,392\,998}{43\,867} p^{20} - \frac{13\,077\,984\,442\,388\,649\,094\,404}{43\,867} p^{19} \\
 & - \frac{14\,183\,115\,793\,553\,780\,830\,128}{43\,867} p^{18} - \frac{13\,077\,984\,442\,388\,649\,094\,404}{43\,867} p^{17} \\
 & - \frac{10\,245\,612\,527\,911\,949\,392\,998}{43\,867} p^{16} - \frac{6804\,790\,928\,276\,409\,773\,424}{43\,867} p^{15} \\
 & - \frac{3817\,086\,982\,353\,184\,063\,104}{43\,867} p^{14} - \frac{1798\,300\,198\,147\,477\,624\,560}{43\,867} p^{13} \\
 & - \frac{706\,047\,862\,569\,538\,312\,980}{43\,867} p^{12} - \frac{228\,609\,396\,856\,786\,345\,128}{43\,867} p^{11} \\
 & - \frac{60\,191\,705\,453\,414\,612\,832}{43\,867} p^{10} - \frac{12\,645\,150\,195\,540\,418\,600}{43\,867} p^9 \\
 & - \frac{2065\,064\,598\,427\,463\,748}{43\,867} p^8 - \frac{252\,643\,315\,041\,469\,584}{43\,867} p^7 \\
 & - \frac{21\,916\,011\,691\,473\,792}{43\,867} p^6 - \frac{1234\,640\,783\,535\,120}{43\,867} p^5 \\
 & - \frac{38\,618\,625\,180\,573}{43\,867} p^4 - \frac{479\,718\,421\,878}{43\,867} p^3 - \frac{935\,300\,808}{43\,867} p^2 \\
 & + \frac{775\,242}{43\,867} (p + 1)k^{18},
 \end{aligned}$$

$$\begin{aligned}
 E_{18}(q^2) = & (p^{36} + 18p^{35} + \frac{6309\,666}{43\,867} p^{34} + \frac{28\,961\,727}{43\,867} p^{33} \\
 & - \frac{308\,674\,899}{87\,734} p^{32} - \frac{3639\,905\,496}{43\,867} p^{31} - \frac{84\,415\,605\,936}{43\,867} p^{30} \\
 & - \frac{975\,153\,257\,352}{43\,867} p^{29} - \frac{7871\,740\,659\,954}{43\,867} p^{28} \\
 & - \frac{48\,180\,925\,598\,836}{43\,867} p^{27} - \frac{229\,616\,693\,343\,792}{43\,867} p^{26} \\
 & - \frac{872\,230\,167\,761\,940}{43\,867} p^{25} - \frac{2693\,410\,301\,866\,986}{43\,867} p^{24} \\
 & - \frac{6859\,725\,720\,266\,232}{43\,867} p^{23} - \frac{14\,560\,859\,613\,263\,040}{43\,867} p^{22} \\
 & - \frac{25\,958\,442\,940\,644\,024}{43\,867} p^{21} - \frac{39\,084\,193\,930\,940\,595}{43\,867} p^{20} \\
 & - \frac{49\,888\,462\,700\,809\,602}{43\,867} p^{19} - \frac{54\,103\,968\,749\,611\,992}{43\,867} p^{18} \\
 & - \frac{49\,888\,462\,700\,809\,602}{43\,867} p^{17} - \frac{39\,084\,193\,930\,940\,595}{43\,867} p^{16} \\
 & - \frac{25\,958\,442\,940\,644\,024}{43\,867} p^{15} - \frac{14\,560\,859\,613\,263\,040}{43\,867} p^{14} \\
 & - \frac{6859\,725\,720\,266\,232}{43\,867} p^{13} - \frac{2693\,410\,301\,866\,986}{43\,867} p^{12} \\
 & - \frac{872\,230\,167\,761\,940}{43\,867} p^{11} - \frac{229\,616\,693\,343\,792}{43\,867} p^{10} \\
 & - \frac{48\,180\,925\,598\,836}{43\,867} p^9 - \frac{7871\,740\,659\,954}{43\,867} p^8 \\
 & - \frac{975\,153\,257\,352}{43\,867} p^7 - \frac{84\,415\,605\,936}{43\,867} p^6 - \frac{3639\,905\,496}{43\,867} p^5 \\
 & - \frac{308\,674\,899}{87\,734} p^4 + \frac{28\,961\,727}{43\,867} p^3 + \frac{6309\,666}{43\,867} p^2 + 18p + 1)k^{18},
 \end{aligned}$$

$$\begin{aligned}
 E_{18}(q^3) = & (p^{36} + 18p^{35} + 144p^{34} + \frac{29\,080\,230}{43\,867} p^{33} + \frac{81\,862\,371}{43\,867} p^{32} \\
 & + \frac{129\,064\,464}{43\,867} p^{31} - \frac{22\,337\,088}{43\,867} p^{30} - \frac{1193\,934\,960}{43\,867} p^{29} \\
 & - \frac{6473\,096\,388}{43\,867} p^{28} - \frac{28\,970\,785\,048}{43\,867} p^{27} \\
 & - \frac{142\,462\,891\,584}{43\,867} p^{26} - \frac{613\,411\,925\,400}{43\,867} p^{25}
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1922\ 036\ 606\ 100}{43\ 867}p^{24} - \frac{4579\ 376\ 969\ 232}{43\ 867}p^{23} \\
 & -\frac{9461\ 957\ 464\ 512}{43\ 867}p^{22} - \frac{17\ 636\ 769\ 114\ 768}{43\ 867}p^{21} \\
 & -\frac{27\ 308\ 190\ 632\ 166}{43\ 867}p^{20} - \frac{33\ 729\ 332\ 247\ 900}{43\ 867}p^{19} \\
 & -\frac{35\ 484\ 183\ 005\ 664}{43\ 867}p^{18} - \frac{33\ 729\ 332\ 247\ 900}{43\ 867}p^{17} \\
 & -\frac{27\ 308\ 190\ 632\ 166}{43\ 867}p^{16} - \frac{17\ 636\ 769\ 114\ 768}{43\ 867}p^{15} \\
 & -\frac{9461\ 957\ 464\ 512}{43\ 867}p^{14} - \frac{4579\ 376\ 969\ 232}{43\ 867}p^{13} \\
 & -\frac{1922\ 036\ 606\ 100}{43\ 867}p^{12} - \frac{613\ 411\ 925\ 400}{43\ 867}p^{11} \\
 & -\frac{142\ 462\ 891\ 584}{43\ 867}p^{10} - \frac{28\ 970\ 785\ 048}{43\ 867}p^9 \\
 & -\frac{6473\ 096\ 388}{43\ 867}p^8 - \frac{1193\ 934\ 960}{43\ 867}p^7 - \frac{22\ 337\ 088}{43\ 867}p^6 \\
 & + \frac{129\ 064\ 464}{43\ 867}p^5 + \frac{81\ 862\ 371}{43\ 867}p^4 + \frac{29\ 080\ 230}{43\ 867}p^3 \\
 & + 144p^2 + 18p + 1)k^{18},
 \end{aligned}$$

$$\begin{aligned}
 E_{18}(q^4) = & \left(\frac{1}{262\ 144}p^{36} + \frac{401\ 985}{5749\ 735\ 424}p^{35} - \frac{233\ 699\ 517}{2874\ 867\ 712}p^{34} \right. \\
 & + \frac{111\ 942\ 533\ 949}{2874\ 867\ 712}p^{33} - \frac{11\ 657\ 555\ 492\ 571}{5749\ 735\ 424}p^{32} \\
 & + \frac{7703\ 600\ 322\ 195}{6974\ 484\ 891\ 375}p^{31} - \frac{359\ 358\ 464}{53\ 240\ 261\ 186\ 547}p^{30} \\
 & - \frac{22\ 722\ 809\ 742\ 693}{89\ 839\ 616}p^{29} + \frac{359\ 358\ 464}{4607\ 875\ 351\ 917}p^{28} \\
 & + \frac{232\ 858\ 848\ 626\ 717}{179\ 679\ 232}p^{27} - \frac{44\ 919\ 808}{262\ 813\ 037\ 358\ 195}p^{26} \\
 & - \frac{324\ 266\ 361\ 567\ 831}{89\ 839\ 616}p^{25} - \frac{179\ 679\ 232}{112\ 489\ 707\ 780\ 261}p^{24} \\
 & + \frac{215\ 260\ 077\ 592\ 395}{44\ 919\ 808}p^{23} + \frac{44\ 919\ 808}{381\ 947\ 582\ 978\ 163}p^{22} \\
 & - \frac{37\ 710\ 901\ 527\ 003}{5614\ 976}p^{21} - \frac{44\ 919\ 808}{11\ 327\ 286\ 134\ 973}p^{20} \\
 & - \frac{65\ 142\ 446\ 901\ 093}{22\ 459\ 904}p^{19} - \frac{11\ 229\ 952}{107\ 022\ 904\ 612\ 137}p^{18} \\
 & - \frac{42\ 136\ 567\ 354\ 953}{11\ 229\ 952}p^{17} - \frac{22\ 459\ 904}{8257\ 547\ 688\ 291}p^{16} \\
 & - \frac{11\ 229\ 952}{2807\ 488}p^{15} - \frac{3154\ 099\ 233\ 537}{2807\ 488}p^{14} \\
 & - \frac{278\ 753\ 497\ 173}{701\ 872}p^{13} - \frac{501\ 559\ 886\ 151}{2807\ 488}p^{12} \\
 & - \frac{91\ 138\ 556\ 457}{1403\ 744}p^{11} - \frac{811\ 095\ 417}{43\ 867}p^{10} \\
 & - \frac{11\ 577\ 929\ 941}{701\ 872}p^9 - \frac{22\ 511\ 837\ 745}{1403\ 744}p^8 \\
 & - \frac{2462\ 497\ 353}{350\ 936}p^7 + \frac{304\ 619\ 577}{350\ 936}p^6 + \frac{129\ 466\ 656}{43\ 867}p^5 \\
 & + \frac{81\ 919\ 827}{43\ 867}p^4 + 663p^3 + 144p^2 + 18p + 1)k^{18},
 \end{aligned}$$

$$\begin{aligned}
 E_{18}(q^6) = & (p^{36} + 18p^{35} + 144p^{34} + 663p^{33} + \frac{3735}{2}p^{32} + 2952p^{31} \\
 & + \frac{38\ 278\ 992}{43\ 867}p^{30} - \frac{307\ 015\ 416}{43\ 867}p^{29} - \frac{686\ 939\ 634}{43\ 867}p^{28} \\
 & - \frac{515\ 629\ 228}{43\ 867}p^{27} + \frac{438\ 109\ 344}{43\ 867}p^{26} + \frac{1351\ 258\ 740}{43\ 867}p^{25}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1068\,365\,142}{43\,867} p^{24} - \frac{308\,234\,376}{43\,867} p^{23} - \frac{1351\,213\,920}{43\,867} p^{22} \\
 & - \frac{1117\,405\,512}{43\,867} p^{21} - \frac{168\,597\,747}{43\,867} p^{20} + \frac{521\,787\,474}{43\,867} p^{19} \\
 & + \frac{715\,735\,824}{43\,867} p^{18} + \frac{521\,787\,474}{43\,867} p^{17} - \frac{168\,597\,747}{43\,867} p^{16} \\
 & - \frac{1117\,405\,512}{43\,867} p^{15} - \frac{1351\,213\,920}{43\,867} p^{14} - \frac{308\,234\,376}{43\,867} p^{13} \\
 & + \frac{1068\,365\,142}{43\,867} p^{12} + \frac{1351\,258\,740}{43\,867} p^{11} + \frac{438\,109\,344}{43\,867} p^{10} \\
 & - \frac{515\,629\,228}{43\,867} p^9 - \frac{686\,939\,634}{43\,867} p^8 - \frac{307\,015\,416}{43\,867} p^7 \\
 & + \frac{38\,278\,992}{43\,867} p^6 + 2952p^5 + \frac{3735}{2} p^4 + 663p^3 + 144p^2 + 18p + 1)k^{18},
 \end{aligned}$$

$$\begin{aligned}
 E_{18}(q^6) = & \left(\frac{1}{262\,144} p^{36} + \frac{9}{131\,072} p^{35} + \frac{9}{16\,384} p^{34} \right. \\
 & + \frac{2874\,867\,712}{4060\,197} p^{33} + \frac{5749\,735\,424}{586\,713} p^{32} \\
 & + \frac{359\,358\,464}{9258\,939} p^{31} - \frac{359\,358\,464}{201\,932\,793} p^{30} \\
 & - \frac{89\,839\,616}{226\,257\,313} p^{29} - \frac{359\,358\,464}{415\,196\,415} p^{28} \\
 & - \frac{179\,679\,232}{4123\,232\,613} p^{27} + \frac{89\,839\,616}{27\,284\,567\,805} p^{26} \\
 & + \frac{89\,839\,616}{7840\,865\,637} p^{25} + \frac{179\,679\,232}{19\,383\,267\,141} p^{24} \\
 & + \frac{44\,919\,808}{11\,414\,735\,505} p^{23} - \frac{44\,919\,808}{132\,668\,452\,503} p^{22} \\
 & - \frac{5614\,976}{17\,700\,651\,225} p^{21} - \frac{44\,919\,808}{54\,470\,572\,833} p^{20} \\
 & + \frac{22\,459\,904}{157\,667\,087\,871} p^{19} + \frac{5614\,976}{33\,347\,831\,031} p^{18} \\
 & + \frac{11\,229\,952}{61\,501\,278\,789} p^{17} + \frac{22\,459\,904}{82\,956\,848\,247} p^{16} \\
 & - \frac{2807\,488}{4774\,798\,179} p^{15} - \frac{2807\,488}{68\,418\,816\,597} p^{14} \\
 & - \frac{701\,872}{985\,959} p^{13} + \frac{2807\,488}{188\,011} p^{12} \\
 & + \frac{32}{501\,075} p^{11} + \frac{16}{55\,989} p^{10} - \frac{16}{6981} p^9 \\
 & - \frac{32}{501\,075} p^8 - \frac{55\,989}{8} p^7 + \frac{6981}{8} p^6 + 2952p^5 \\
 & \left. + \frac{3735}{2} p^4 + 663p^3 + 144p^2 + 18p + 1)k^{18}. \right.
 \end{aligned}$$

We can also similarly determine f_1, \dots, f_{30} and f_{31} in terms of p and k as in $\{f_1, \dots, f_{31}\} \setminus \{f_{11}\} \in S_{18}(\Gamma_0(12))$, $f_{11} \in M_{18}(\Gamma_0(12)) \setminus S_{18}(\Gamma_0(12))$ by [4]. Now www.fatih.edu.tr/~bkendirli/weight18/Table 3. Obviously, f_1, \dots, f_{31} are functions of , see (3),(19). We see that

$$\begin{aligned}
 & \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) \\
 & = q^{b_1} \prod_{n=1}^{\infty} (1 - q^n)^{a_1} (1 - q^{2n})^{a_2} (1 - q^{3n})^{a_3} (1 - q^{4n})^{a_4} (1 - q^{6n})^{a_6} (1 - q^{12n})^{a_{12}} \\
 & = 2^{-\frac{a_1}{6} - \frac{a_2}{3} - \frac{a_3}{6} - \frac{2a_4}{3} - \frac{a_6}{3} - \frac{2a_{12}}{3}} p^{\frac{a_1}{24} + \frac{a_2}{12} + \frac{a_3}{8} + \frac{a_4}{6} + \frac{a_6}{4} + \frac{a_{12}}{2}} (1 - p)^{\frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{6} + \frac{a_4}{8} + \frac{a_6}{12} + \frac{a_{12}}{24}} \\
 & (1 + p)^{\frac{a_1}{6} + \frac{a_2}{12} + \frac{a_3}{24} + \frac{a_4}{4} + \frac{a_6}{8} + \frac{a_{12}}{8}} (1 + 2p)^{\frac{a_1}{8} + \frac{a_2}{4} + \frac{a_3}{24} + \frac{a_4}{8} + \frac{a_6}{12} + \frac{a_{12}}{24}} (2 + p)^{\frac{a_1}{8} + \frac{a_2}{4} + \frac{a_3}{24} + \frac{a_4}{2} + \frac{a_6}{12} + \frac{a_{12}}{6}} \\
 & k^{\frac{a_1 + a_2 + a_3 + a_4 + a_6 + a_{12}}{2}} = \frac{k^{18}}{2^{b_1 + b_5}} p^{b_1} (1 - p)^{b_2} (1 + p)^{b_3} (1 + 2p)^{b_4} (2 + p)^{b_5}
 \end{aligned}$$

$$\begin{aligned}
 &= k^{18}(k_0 + k_1p + k_2p^2 + k_3p^3 + k_4p^4 + k_5p^5 + k_6p^6 \\
 &+ k_7p^7 + k_8p^8 + k_9p^9 + k_{10}p^{10} + k_{11}p^{11} \\
 &+ k_{12}p^{12} + k_{13}p^{13} + k_{14}p^{14} + k_{15}p^{15} + k_{16}p^{16} \\
 &k_{17}p^{17} + k_{18}p^{18} + k_{19}p^{19} + k_{20}p^{20} + k_{21}p^{21} + k_{22}p^{22} \\
 &+ k_{23}p^{23} + k_{24}p^{24} + k_{25}p^{25} + k_{26}p^{26} + k_{27}p^{27} + k_{28}p^{28} \\
 &+ k_{29}p^{29} + k_{30}p^{30} + k_{31}p^{31} + k_{32}p^{32} + k_{33}p^{33} + k_{34}p^{34} + k_{35}p^{35} + k_{36}p^{36}) \\
 &= \frac{43867c_1}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^n \right) + \frac{43867c_2}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^{2n} \right) \\
 &+ \frac{43867c_3}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^{3n} \right) + \frac{43867c_4}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^{4n} \right) \\
 &+ \frac{43867c_6}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^{6n} \right) + \frac{43867c_{12}}{28728} \left(1 - \frac{28728}{43867} \sum_{n=1}^{\infty} \sigma_{17}(n)q^{12n} \right) \\
 &+ r_1 q^{13} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{17}(1 - q^{6n})^{19}(1 - q^{12n})^{13}}{(1 - q^{2n})^{13}} \\
 &r_2 q^{15} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^4(1 - q^{6n})^{20}(1 - q^{12n})^{20}}{(1 - q^{2n})^8} \\
 &+ r_3 q^{11} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{16}(1 - q^{6n})^{20}(1 - q^{12n})^8}{(1 - q^{2n})^8} \\
 &+ r_4 q^{13} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{18}(1 - q^{6n})^6(1 - q^{12n})^{18}}{(1 - q^{2n})^6} \\
 &+ r_5 q^{13} \prod_{n=1}^{\infty} \frac{((1 - q^{4n})^{13}(1 - q^{6n})^{11}(1 - q^{12n})^{17})}{(1 - q^{2n})^5} \\
 &+ r_6 q^{13} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^8(1 - q^{6n})^{16}(1 - q^{12n})^{16}}{(1 - q^{2n})^4} \\
 &+ r_7 q^9 \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{20}(1 - q^{6n})^{16}(1 - q^{12n})^4}{(1 - q^{2n})^4} \\
 &+ r_8 q^5 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^6(1 - q^{4n})^{18}(1 - q^{6n})^{18}}{(1 - q^{12n})^6} \\
 &+ r_9 q^{11} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^7(1 - q^{4n})^{13}(1 - q^{12n})^{17}}{(1 - q^{6n})} \\
 &+ r_{10} q^7 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^8(1 - q^{4n})^{20}(1 - q^{6n})^4(1 - q^{12n})^4}{(1 - q^{6n})} \\
 &+ r_{11} q^{15} \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{18}(1 - q^{6n})^{18}(1 - q^{12n})^{18}}{(1 - q^{2n})^{18}} \\
 &+ r_{12} q^9 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{11}(1 - q^{4n})^{17}(1 - q^{12n})^{13}}{(1 - q^{6n})^5} \\
 &+ r_{13} q^3 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{11}(1 - q^{4n})^{17}(1 - q^{6n})^{19}}{(1 - q^{12n})^{11}} \\
 &+ r_{14} q^5 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{20}(1 - q^{4n})^{20}(1 - q^{12n})^4}{(1 - q^{6n})^8} \\
 &+ r_{15} q^{11} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{11}(1 - q^{6n})^{19}(1 - q^{12n})^{13}}{(1 - q^{4n})^7}
 \end{aligned}$$

$$\begin{aligned}
& +r_{16}q^{11} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{20}(1-q^{6n})^{16}(1-q^{12n})^{16}}{(1-q^{4n})^{16}} \\
& +r_{17}q^{10} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{17}(1-q^{6n})^{13}(1-q^{12n})^{13}}{(1-q^{4n})^7} \\
& +r_{18}q^6 \prod_{n=1}^{\infty} (1-q^{2n})^{17}(1-q^{4n})^5(1-q^{6n})^{13}(1-q^{12n}) \\
& +r_{19}q^{14} \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{18}(1-q^{6n})^{12}(1-q^{12n})^{18}}{(1-q^{2n})^{12}} \\
& +r_{20}q^{14} \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{13}(1-q^{6n})^{17}(1-q^{12n})^{17}}{(1-q^{2n})^{11}} \\
& +r_{21}q^{12} \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{17}(1-q^{6n})^{13}(1-q^{12n})^{13}}{(1-q^{2n})^7} \\
& +r_{22}q^{14} \prod_{n=1}^{\infty} \frac{(1-q^{4n})^4(1-q^{6n})^{14}(1-q^{12n})^{20}}{(1-q^{2n})^2} \\
& +r_{23}q^{10} \prod_{n=1}^{\infty} \frac{(1-q^{4n})^{16}(1-q^{6n})^{14}(1-q^{12n})^8}{(1-q^{2n})^2} \\
& +r_{24}q^{10} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{18}(1-q^{6n})^{18}(1-q^{12n})^{12}}{(1-q^{4n})^{12}} \\
& +r_{25}q^{12} \prod_{n=1}^{\infty} (1-q^{4n})^{18}(1-q^{12n})^{18} \\
& +r_{26}q^{12} \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{4n})^{13}(1-q^{6n})^5(1-q^{12n})^{17} \\
& +r_{27}q^6 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^7(1-q^{4n})^{19}(1-q^{6n})^{11}}{(1-q^{12n})} \\
& +r_{28}q^{10} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}(1-q^{4n})^{18}(1-q^{12n})^{18}}{(1-q^{6n})^{12}} \\
& +r_{29}q^4 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}(1-q^{4n})^{18}(1-q^{6n})^{12}}{(1-q^{12n})^6} \\
& +r_{30}q^{10} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{13}(1-q^{4n})^{13}(1-q^{12n})^{17}}{(1-q^{6n})^7} \\
& +r_{31}q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{18}(1-q^{4n})^{12}(1-q^{6n})^{18}}{(1-q^{12n})^{12}} \\
& = \delta(b_1) - \sum_{n=1}^{\infty} (c_1\sigma_{17}(n) + c_2\sigma_{17}\left(\frac{n}{2}\right) + c_3\sigma_{17}\left(\frac{n}{3}\right) + c_4\sigma_{17}\left(\frac{n}{4}\right) \\
& + c_6\sigma_{17}\left(\frac{n}{6}\right) + c_{12}\sigma_{17}\left(\frac{n}{12}\right)) + r_1f_1(n) + \dots + r_{31}f_{31}(n),
\end{aligned}$$

where

$$\delta(b_1) = \begin{cases} 0 & \text{if } b_1 \neq 0 \\ 1 & \text{if } b_1 = 0 \end{cases}$$

So

$$c(n) = -(c_1\sigma_{17}(n) + c_2\sigma_{17}\left(\frac{n}{2}\right) + c_3\sigma_{17}\left(\frac{n}{3}\right) + c_4\sigma_{17}\left(\frac{n}{4}\right) + c_6\sigma_{17}\left(\frac{n}{6}\right) + c_{12}\sigma_{17}\left(\frac{n}{12}\right)) + r_1f_1(n) + \dots + r_{31}f_{31}(n).$$

Therefore, for $n=1,2,\dots$,

$$c(2n) = -c_1\sigma_{17}(2n) - c_2\sigma_{17}(n) - c_4\sigma_{17}\left(\frac{n}{2}\right) - (262\ 145c_3 + c_6)\sigma_{17}\left(\frac{n}{3}\right) - (c_{12} - 262\ 144c_3)\sigma_{17}\left(\frac{n}{6}\right) + r_{17}f_{17}(2n) + \dots + r_{31}f_{31}(2n),$$

$$c(2n - 1) = -c_1\sigma_{15}(2n - 1) - c_3\sigma_{15}\left(\frac{2n - 1}{3}\right) + r_1f_1(2n - 1) + \dots + r_{14}f_{14}(2n - 1),$$

since it is easy to see that

$$\sigma_k\left(\frac{2n}{3}\right) = (2^k + 1)\sigma_k\left(\frac{n}{3}\right) - 2^k\sigma_k\left(\frac{n}{6}\right)$$

hence,

$$\sigma_{17}\left(\frac{2n}{3}\right) = 262\ 145\sigma_{17}\left(\frac{n}{3}\right) - 262\ 144\sigma_{17}\left(\frac{n}{6}\right),$$

and, for $n=1,2,\dots$,

$$f_1(2n) = \dots = f_{14}(2n) = 0,$$

$$f_{15}(2n - 1) = \dots = f_{27}(2n - 1) = 0.$$

3. Conclusion

www.fatih.edu.tr/~bkendirli/weight18/Table 4, such that, for $n=1,2,\dots$,

1. We have found 324 eta quotients, see

$$c(2n) = -c_1\sigma_{17}(2n) - c_2\sigma_{17}(n) - c_4\sigma_{17}\left(\frac{n}{2}\right) - (262\ 145c_3 + c_6)\sigma_{17}\left(\frac{n}{3}\right) - (c_{12} - 262\ 144c_3)\sigma_{17}\left(\frac{n}{6}\right)$$

$$c(2n - 1) = -c_1\sigma_{17}(2n - 1) - c_3\sigma_{17}\left(\frac{2n - 1}{3}\right) + r_1f_1(2n - 1) + \dots + r_{14}f_{14}(2n - 1).$$

and 650 eta quotients, such that for $n=1,2,\dots$,

$$c(2n) = -c_1\sigma_{17}(2n) - c_2\sigma_{17}(n) - c_4\sigma_{17}\left(\frac{n}{2}\right) - c_6\sigma_{17}\left(\frac{n}{3}\right) - c_{12}\sigma_{15}\left(\frac{n}{6}\right) + r_{17}f_{17}(2n) + \dots + r_{31}f_{31}(2n),$$

$$c(2n - 1) = 0.$$

2. $S_{18}(\Gamma_0(12))$ is 31 dimensional, $M_{18}(\Gamma_0(12))$ is 37 dimensional, see [5] (Chapter 3, pg.87 and Chapter 5, pg.197),

- $\Delta_{1,18}, \Delta_{1,18}(2z), \Delta_{1,18}(3z), \Delta_{1,18}(4z), \Delta_{1,18}(6z), \Delta_{1,18}(12z),$
- $\Delta_{2,18}, \Delta_{2,18}(2z), \Delta_{2,18}(3z), \Delta_{2,18}(6z),$
- $\Delta_{3,18,1}, \Delta_{3,18,1}(2z), \Delta_{3,18,1}(4z),$
- $\Delta_{3,18,2}(z), \Delta_{3,18,2}(2z), \Delta_{3,18,2}(4z),$
- $\Delta_{3,18,3}(z), \Delta_{3,18,3}(2z), \Delta_{3,18,3}(4z)$ (conjugate of $\Delta_{3,18,2}$ by $x^2 - 594x - 42912$),
- $\Delta_{4,18,1}, \Delta_{4,18,1}(3z),$
- $\Delta_{4,18,2}, \Delta_{4,18,2}(3z)$ (conjugate of $\Delta_{4,18,1}$ by $x^2 + 5880x - 336440304$),
- $\Delta_{6,18,1}, \Delta_{6,18,1}(2z), \Delta_{6,18,2}, \Delta_{6,18,2}(2z), \Delta_{6,18,3}, \Delta_{6,18,3}(2z),$
- $\Delta_{12,18,1}, \Delta_{12,18,2},$

where $\Delta_{1,18}$ is the unique newform in $S_{18}(\Gamma_0(1))$; $\Delta_{2,18}$ is the unique newform in $S_{18}(\Gamma_0(2))$; $\Delta_{3,18,1}, \Delta_{3,18,2}$ and $\Delta_{3,18,3}$ are the unique newforms in $S_{18}(\Gamma_0(3))$, $\Delta_{4,18,1}, \Delta_{4,18,2}$ are the unique newforms in $S_{18}(\Gamma_0(4))$, $\Delta_{6,18,1}, \Delta_{6,18,2}$ and $\Delta_{6,18,3}$ are the unique newforms in $S_{18}(\Gamma_0(6))$, and $\Delta_{12,18,1}, \Delta_{12,18,2}$ are the unique newforms in $S_{18}(\Gamma_0(12))$. By taking t as a root of $x^2 - 594x - 42912$, and s as the root of $x^2 + 5880x - 336440304$, we see f_1, \dots, f_{31} as linear combinations of this basis in [www.fatih.edu.tr/~bkendirli/weight18/Table 3](http://www.fatih.edu.tr/~bkendirli/weight18/Table3).

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