Existence of Time Periodic Solutions of New Classes of Nonlinear Problems

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Abstract: We study the existence of one or more weak periodic solutions of nonlinear evolution PDEs in a cylinder of \( \mathbb{R}^{N+1} \) with conditions on lateral surface by using the results connected to a general evolution variational equation depending on a parameter.

Keywords: Time Periodic, Evolution PDEs, Nonstationarity, Weak Periodic Solutions

1. Introduction

Let \( 0 < T < \infty \). Let \( \Omega \subseteq \mathbb{R}^N \) be an open, bounded and connected set, with boundary \( \partial \Omega \in C^{0,1} \).

For \( N=1 \), the condition “\( \partial \Omega \in C^{0,1} \)” means that \( \Omega \) is a bounded open interval.

Let us set
\[
Q = \Omega \times [0,T] \quad \text{and} \quad \Sigma = \partial \Omega \times [0,T];
\]
\[P_T(\Omega \times \mathbb{R}) = \text{the class of the real functions } v(x,t) \text{ defined a.e. in } \Omega \times \mathbb{R}, \text{measurable and T-periodic with respect to t;}
\]
\[v \in P_T(\Omega \times \mathbb{R}) \text{ and } t \in [0,T] \quad \overset{\text{def}}{=} \quad t = v(\cdot,t).
\]

We denote by \( F \) the linear map \( v \in \Omega \times \mathbb{R} \rightarrow \overset{\text{def}}{=} v \).

Let \( 1 < p_1 < \infty, 1 < p_2 < \infty \) and \( V \) be a closed subspace of \( W^{n,n}(\Omega) \) \( (n = 1,2,...) \) such that \( C^0(\Omega) \subseteq V \). We do not exclude \( V = W^{n,n}(\Omega) \).

Let \( \| \| \) be a norm equivalent to the one of \( W^{n,n}(\Omega) \) on \( V \).

Set \( D^\alpha = \frac{\partial^{\left| \alpha \right|}}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}} \), we consider the normed spaces:
\[
W^{n,n}_T = \left\{ v \in P_T(\Omega \times \mathbb{R}) \cap L^n(Q) : \text{as } 0 < \left| \alpha \right| \leq n \text{ there exists the weak derivative } \right. \\
D^\alpha v \text{ with } D^\alpha v \in L^n(Q), v(\cdot,t) \in V \text{ for a.e. } t \in \mathbb{R} \right\}, \left\| v \right\|_n = \left\{ \left( \int_0^T \left( \int_\Omega \left| v(\cdot,t) \right|^n dt \right) \right)^{\frac{1}{n}} \right\} \forall v \in W^{n,n}_T;
\]
\[
W_T = \left\{ v \in W^{n,n}_T : \text{there exists the weak derivative } \frac{\partial v}{\partial t} \text{ with } \frac{\partial v}{\partial t} \in L^n(Q) \right\}.
\]
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\[ \|w\|_{W} = \|w\|_{W_{0}} + \left( \int_{Q} \left| \frac{\partial w}{\partial t} \right|^{p} \, dx \, dt \right)^{\frac{1}{p}} \quad \forall w \in W_{r}; \]  

where \( \tilde{W}_{0} = L^{p_{r}}(0,T;V) \), \( \|w\|_{\tilde{W}_{0}} = \left( \int_{0}^{T} \|w(t)\|_{L^{p_{r}}(Q)}^{p_{r}} \, dt \right)^{\frac{1}{p_{r}}} \) \( \forall w \in \tilde{W}_{0} \);

\[ \tilde{W} = \{ w \in \tilde{W}_{0} : w' \in L^{p}(0,T;L^{p_{r}}(\Omega)) \} , \|w\|_{\tilde{W}} = \|w\|_{\tilde{W}_{0}} + \left( \int_{0}^{T} \|w'(t)\|_{L^{p_{r}}(Q)}^{p_{r}} \, dt \right)^{\frac{1}{p_{r}}} \quad \forall w \in \tilde{W}. \]

**Remark 1.1.** When \( p_{1} = p_{2} = p \) we assume

\[ \int_{0}^{T} \left| \frac{\partial w}{\partial t} \right|^{p} \, dx \, dt \quad \forall w \in W_{r} \text{ and } \|w\|_{W} = \left( \int_{0}^{T} \|w(t)\|_{L^{p}(Q)}^{p} \, dt + \int_{0}^{T} \|w'(t)\|_{L^{p_{r}}(Q)}^{p_{r}} \, dt \right)^{\frac{1}{p}} \quad \forall w \in \tilde{W}. \]

We recall ([16], Chap.23) \( \tilde{W}_{0} \) and \( \tilde{W} \) are reflexive and separable Banach spaces. It is not difficult to prove that \( W_{0} \) and \( W_{r} \) are Banach spaces; the restriction of \( F \) to \( W_{0} \) resp\( W_{r} \) is a norm-preserving linear transformation into \( \tilde{W}_{0} \) resp\( \tilde{W} \).

Consequently \( W_{0} \) and \( W_{r} \) are reflexive and separable spaces.

**Remark 1.2.** These conclusions hold even if \( V = L^{p_{0}}(\Omega) \). In this case \( W_{r}^{0} = P_{r}(\Omega \times R) \cap L^{p_{r}}(Q) \) and

\[ \int_{0}^{T} \left| \frac{\partial w}{\partial t} \right|^{p} \, dx \, dt \quad \forall w \in \tilde{W}. \]

Let us denote by \( \langle \cdot, \cdot \rangle \) the duality between \( W_{r}^{*} \) (dual space of \( W_{r} \)) and \( W_{r} \), and by "\( \partial \)" Fréchet differential operator. Let \( A^{\sim} = 0, \ D_{j}^{-} = 0 \) \( (j=1,...,m; \ m \geq 1) \) and \( B \) be real functionals defined in \( W_{r} \) satisfying the conditions

\[ \begin{cases} \text{A is weakly lower semicontinuous in} \ W_{r} \text{ and} \ C^{1} \left( W_{r} \right), \\ (i_{1}) \text{B is weakly continuous in} \ W_{r} \text{ and} \ C^{1} \left( W_{r} \right), \\ \exists p > 1 : A(\{v\}) = r^{p}A(\{v\}) \text{ and} \ B(\{v\}) = r^{p}B(\{v\}) \forall r > 0 \text{ and} \forall v \in W_{r}; \\ (i_{2}) \text{D}_{j} \text{ is weakly continuous in} \ W_{r} \text{ and} \ C^{1} \left( W_{r} \right), \exists q_{j} > 1 : \text{D}_{j} \text{ (} \{v\} \text{) = r}^{q_{j}} \text{D}_{j} \text{ (} \{v\} \text{) \forall r > 0 \text{ and} \forall v \in W_{r}, q_{j} < ... < q_{m} \text{ if} m > 1.} \end{cases} \]

Let us consider the following problem.

**Problem \( P^{T} \).** Find \( u \in W_{r} \setminus \{0\} \) such that

\[ \langle \partial A(\{u\}), v \rangle = \lambda \langle \partial B(\{u\}), v \rangle + \sum_{j=1}^{m} \langle \partial D_{j}(\{u\}), v \rangle \quad \forall v \in W_{r}, \]

where \( \lambda \) is a real parameter.

**Problem \( P^{T} \)** is a particular case of Problem(P) studied in ([10], [12]) by using the Lagrange multipliers and the “algebraic” approach which is based on the fibering method [14]. In ([11], [12]) many applications of the results connected to Problem (P) related to nonlinear elliptic systems are present.

In section 2 we considered convenient about Problem \( P^{T} \) to state existence theorems (Theorems 2.1-2.3) included in the results of ([4], [7] [8], [10], [12]) whose validity, by the way, depends on \( \lambda \). Furthermore we added Propositions 2.1 and 2.2 useful in some concrete cases in order to establish the nonstationarity of the found solutions.

In section 3 we study some evolution PDEs in the cylinder \( Q \), with also nonlocal nonlinearities and with different conditions on...
About these problems, whose variational formulation is included in Problem \( TP \), we find the existence of one or more weak periodic solutions, giving also some sufficient conditions to their nonstationarity.

It is known that the search for periodic solutions of nonlinear problems has attracted the attention of many researchers. In particular Pohozaev in ([13], [15]) introduced “the separation variables method” for nonlinear equations in which it is possible to find weak periodic solutions in the form

\[
 u(x, t) = u_1(x) u_2(t).
\]

As far as we know, the methods developed in literature are not applicable in this article.

2. Existence Theorems Preliminary Results

As \( \lambda \in \mathbb{R}, v \in W_r \), \( r \geq 0, j \in \{1, ..., m\} \) and \( \{j_1, ..., j_s\} \subseteq \{1, ..., m\}\{j_i < ... < j_s\} \) if \( s > 1 \) we set:

\[
 H_\lambda(v) = A(v) - \lambda B(v), E(v) = H_\lambda(v) - \sum_{j=1}^{m} D_j(v), E^r(v) = E(rv) = r^p H_\lambda(v) - \sum_{j=1}^{m} r^p D_j(v),
\]

\[
 \Psi(r,v) = \frac{\partial E^r}{\partial r}(r,v). S_\lambda = \{v \in W_r : H_\lambda(v) = 1\}, V_\lambda^- = \{v \in W_r : H_\lambda(v) < 0\},
\]

\[
 S(D_j) = \{v \in W_r : D_j(v) = -1\}, V^+(D_\lambda, ..., D_\lambda) = \{v \in W_r : D_j(v) + ... + D_\lambda(v) > 0\}.
\]

We find the solvability of Problem \( TP \) in the cases based on the following assumptions

\[
 (i_{21}) \exists c(\lambda) > 0 : \|v\|^p \leq c(\lambda) H_\lambda(v) \forall v \in W_r;
\]

\[
 (i_{22}) \exists c(\lambda) > 0 : \|v\|^p \leq c(\lambda) H_\lambda(v) \forall v \in V^+(D_j);
\]

\[
 (i_{23}) \exists c(\lambda) > 0 : \|v\|^p \leq c(\lambda) H_\lambda(v) \forall v \in V^+(D_m);
\]

\[
 (i_{24}) \exists m_i \in \{1, ..., m\} : V^- \cap S(D_m) \text{ is nonempty and bounded in } W_r.
\]

At first we consider the cases in which one of the assumptions \((i_{21}) - (i_{24})\) is present:

\[
 (c_1) m = 1, q_i = p, V^+(D_1) = \emptyset, (i_{21}) \text{ holds};
\]

\[
 (c_2) m > 1, q_i < p, V^+(D_j) = \emptyset, D_j \leq 0 \forall j \geq 2, (i_{22}) \text{ holds};
\]

\[
 (c_3) m > 1, q_m > p, V^+(D_m) = \emptyset, D_j \leq 0 \forall j \leq m - 1, (i_{23}) \text{ holds};
\]

\[
 (c_4) m > 1, \exists m_i \in \{2, ..., m\} : q_m < p, D_j \geq 0 \forall j \leq m_i \text{ and } D_j \leq 0 \forall j > m_i \text{ if } m_i < m, (i_{24}) \text{ holds};
\]

\[
 (c_5) m > 1, \exists m_i \in \{1, ..., m - 1\} : q_m > p, D_j \geq 0 \forall j \geq m_i \text{ and } D_j \leq 0 \forall j < m_i \text{ if } m_i > 1, (i_{24}) \text{ holds};
\]

\[
 (c_6) m > 1, q_m < p, D_j \geq 0 \forall j < m, D_m \text{ changes sign, } V^+(D_1, ..., D_{m-1}) = W_r \setminus \{0\}, (i_{24}) \text{ holds};
\]

\[
 (c_7) m > 1, q_i > p, D_i \text{ changes sign, } D_j \geq 0 \forall j > 1, V^+(D_2, ..., D_m) = W_r \setminus \{0\}, (i_{24}) \text{ holds}.
\]

Let us introduce the open set \( \mathcal{A} \) of the space \( W_r \):

\[
 \mathcal{A} = V^+(D_1) \text{ in } (c_1), \mathcal{A} = V^+(D_m) \text{ in } (c_4), \mathcal{A} = V^+(D_1, ..., D_m) \text{ in } (c_5),
\]

\[
 \mathcal{A} = V^+(D_m, ..., D_m) \text{ in } (c_6), \mathcal{A} = W_r \setminus \{0\} \text{ in } (c_7)
\]

\[
 \mathcal{A} = V^+(D_1, ..., D_m) \text{ in } (c_4), \mathcal{A} = W_r \setminus \{0\} \text{ in } (c_7)
\]

Theorem 2.1 ([10], Section 2). Under assumptions \((i_{21}) \text{ and } (i_{24})\), in case \( (c_1) \) we have:
\[ \exists v_0 \in S_{\mathcal{A}} \cap \mathcal{A} : D_{1}(v_0) = \sup \{ D_1(v) : v \in S_{\mathcal{A}} \cap \mathcal{A} \} ; \]

with \( r_0 = \left( q, p^{-1} D_{1}(v_0) \right)^{\frac{1}{p-n}} \), \( u_0 = r_0 v_0 \) is solution of Problem \( \left( P^r \right) \).

When \( W_r \) is a vector lattice, if \( H_{\lambda}(v) = H_{\lambda}(\|v\|) \) and \( D_{j}(v) = D_{j}(\|v\|) \quad \forall v \in W_r \), then \( r_0 v_0 \)

\[ \left[ \text{resp. } r_0 v_0 \text{ and } - r_0 v_0 \right] \text{ is solution [resp. are solutions] of Problem } \left( P^r \right) . \]

Consequently we can assume \( v_0 \geq 0 \) i.e. \( u_0 \geq 0 \).

When \( m \geq 1 \), for any \( v \in \mathcal{A} \) the equation \( \Psi(r,v) = 0 \) has only one positive root \( r(v) \) and we have \( \frac{\partial \Psi}{\partial r}(r(v),v) = 0 \).

Evidently the functionals \( r(v) \) and \( \tilde{E}(v) = \tilde{E}(r(v),v) = (r(v))^T H_{\lambda}(v) - \sum_{j=1}^{m} (r(v))^j D_{j}(v) \)

belong to \( C^{1}(\mathcal{A}) \).

\[ (2.1) \]

**Theorem 2.2** (\([10], \text{Section 2;[12], Section 2}\). Under assumptions \( (i_{1}) \) and \( (i_{2}) \), in cases \( (c_{1}) \) - \( (c_{7}) \) we have:

\[ \exists v_0 \in S_{\mathcal{A}} \cap \mathcal{A} : \tilde{E}(v_0) = \inf \{ \tilde{E}(v) : v \in S_{\mathcal{A}} \cap \mathcal{A} \} ; \]

with \( r_0 = r(v_0) \), \( u_0 = r_0 v_0 \) is solution of Problem \( \left( P^r \right) \).

When \( W_r \) is a vector lattice, if \( H_{\lambda}(v) = H_{\lambda}(\|v\|) \) and \( D_{j}(v) = D_{j}(\|v\|) \quad \forall v \in W_r \) and \( \forall j \in \{1, \ldots, m\} \), then \( r_0 v_0 \)

\[ \left[ \text{resp. } r_0 v_0 \text{ and } - r_0 v_0 \right] \text{ are solutions of Problem } \left( P^r \right) . \]

Consequently we can assume \( v_0 \geq 0 \) i.e. \( u_0 \geq 0 \).

**Proposition 2.1.** In cases \( (c_{1}) \) - \( (c_{7}) \) let \( u_0, v_0, v_0 \) be as in Theorems 2.1 and 2.2. If \( \tilde{v} \in W_{r} \) is such that \( < \partial H_{\lambda}(v_0), \tilde{v} > \neq 0 \), then

\[ \sum_{j=1}^{m} \left[ < \partial D_{j}(u_0), u_0 > - p \left( < \partial H_{\lambda}(v_0), \tilde{v} > \right)^{-1} < \partial D_{j}(u_0), r_0 \tilde{v} > \right] = 0 \quad \text{as } m \geq 1. \]

\[ (2.2) \]

**Proof.** Let us set \( f(s, \tau) = H_{\lambda}(sv_0 + \tau \tilde{v}) \forall s > 0 \) and \( \forall \tau \in \mathbb{R} \). We note that \( f \in C^{1}\left( [0, +\infty) \times \mathbb{R} \right) \), \( \frac{df}{ds}(s, \tau) = < \partial H_{\lambda}(sv_0 + \tau \tilde{v}), v_0 > \) and \( \frac{df}{d\tau}(s, \tau) = < \partial H_{\lambda}(sv_0 + \tau \tilde{v}), \tilde{v} > \forall (s, \tau) \in [0, +\infty) \times \mathbb{R} \). Since \( f(1,0) = 1 \) and \( \frac{df}{d\tau}(1,0) = < \partial H_{\lambda}(v_0), \tilde{v} > \neq 0 \), there exist \( \delta \in [0,1] \) and only one function \( \tau(s) \in C^{1}\left( [1 - \delta, 1 + \delta] \right) \) such that \( \tau(1) = 0 \) and

\[ f(s, \tau(s)) = 1 \quad \forall s \in [1 - \delta, 1 + \delta] ; \quad \text{Moreover } \tau'(1) = - < \partial H_{\lambda}(v_0), v_0 > < \partial H_{\lambda}(v_0), \tilde{v} >^{-1} = - pH_{\lambda}(v_0) \]

\[ < \partial H_{\lambda}(v_0), \tilde{v} >^{-1} = - p < \partial H_{\lambda}(v_0), \tilde{v} >^{-1} \].

Let \( \delta_0 \in [0,\delta] \) such that \( v(s) = sv_0 + \tau(s) \tilde{v} \in \mathcal{A} \quad \forall s \in [1 - \delta_0, 1 + \delta_0] \); then

\[ v(s) \in S_{\mathcal{A}} \cap \mathcal{A} \quad \forall s \in [1 - \delta_0, 1 + \delta_0] . \]

\[ (2.3) \]

In \( (c_{1}) \) \( (2.3) \Rightarrow D_{1}(v(s)) \leq D_{1}(v_0) = D_{1}(v(1)) \quad \forall s \in [1 - \delta_0, 1 + \delta_0] \). Then \( \left[ \frac{d}{ds} D_{1}(v(s)) \right]_{s=1} = 0 \), from which \( (2.2) \) since

\[ \left[ \frac{d}{ds} D_{1}(v(s)) \right]_{s=1} = < \partial D_{1}(v_0), v_0 > - p < \partial H_{\lambda}(v_0), \tilde{v} >^{-1} < \partial D_{1}(v_0), \tilde{v} >= \]
\[ r_{0}^{-n} \left[ < \partial D_1(u_0), u_0 > - p \left( < \partial H_1(v_0), \tilde{v} > \right)^{-1} < \partial D_1(u_0), r_0 \tilde{v} > \right] \]

In \((c_2) - (c_4)\) and (2.1) \(\Rightarrow\) \(\hat{E}(v(s)) \in C^1 [1 - \delta_0, 1 + \delta_0] \), \((2.3) \Rightarrow \hat{E}(v(1)) = \tilde{E}(v) \leq \hat{E}(v(s)) \forall s \in [1 - \delta_0, 1 + \delta_0]. \)

Consequently \(\left[ \frac{d}{ds} \hat{E}(v(s)) \right]_{s=1} = 0\), i.e. (2.2). In fact, since

\[
\frac{d}{ds} \hat{E}(v(s)) = 0, \quad \text{for any } v \in V_\lambda \cap S \{D_1 \}, \text{ where } \hat{E}(v) = \inf \{ E(v) : v \in V_\lambda \cap S \{D_1 \} \},
\]

we have

\[
\left[ \frac{d}{ds} \hat{E}(v(s)) \right]_{s=1} = - \sum_{j=1}^{m} \left[ q^j \left( \partial D_j(v_0), v_0 + \tau(s) \tilde{v} > \right)^{\alpha j} \right]_{s=1} = 0.
\]

Let us pass to the cases in which \((i_2)\) is present:

\((c_4)\) \(m = 1, q_1 \neq p, (i_2a) \text{ holds with } m_1 = 1\);

\((c_5)\) \(m > 1, (i_2a) \text{ holds, either } p < q_1 \text{ or } q_m < p, D_j \leq 0 \text{ as } j \neq m_1\).

In \((c_6)\) for any \(v \in V_\lambda \cap S \{D_m \}\) the equation \(\Psi(r, v) = 0\) has only one positive root \(r(v)\) with \(\frac{\partial \Psi}{\partial r}(r(v), v) \neq 0\). Let us set \(\hat{E}(v) = \hat{E}(r(v), v) \forall v \in V_\lambda \cap S \{D_m \}\).

**Theorem 2.3** ([10], Section 4). Let \((i_1)\) and \((i_2)\) hold. In case \((c_6)\)

\[
\forall v \in V_\lambda \cap S \{D_1 \}, \quad \hat{E}(v) = \inf \{ E(v) : v \in V_\lambda \cap S \{D_1 \} \}
\]

with \(\bar{r} = \left( -p q_1^{-1} H_1(v) \right)^{1/\eta - p}, \underline{v} = \bar{r} \bar{v} \text{ is solution of Problem } \left( P^\tau \right) \).

In case \((c_5)\)

\[
\forall v \in V_\lambda \cap S \{D_m \}, \quad \hat{E}(v) = \inf \{ E(v) : v \in V_\lambda \cap S \{D_m \} \}
\]

with \(\bar{r} = r(v), \underline{v} = \bar{r} \bar{v} \text{ is solution of Problem } \left( P^\tau \right) \).

When \(W_\tau\) is a vector lattice, if \(H_1(v) = H_1(\bar{v})\) and \(D_j(v) = D_j(\bar{v}) \forall v \in W_\tau\) and \(\forall j \in \{1, \ldots, m\}\), then in \((c_6)\) and \((c_5)\)

\(\bar{v} \in \underline{v} \) and \(\tilde{v} \in \overline{v} \) are solutions of Problem \( \left( P^\tau \right) \). Consequently we can assume \(v \geq 0\) i.e. \(\underline{v} \geq 0\).

**Proposition 2.2.** Let \(w, \underline{v}, \overline{v}\) be as in Theor. 2.3. In case \((c_6)\) if \(\tilde{v} \in W_\tau\) is such that \(< \partial D_1(\bar{v}), v > \neq 0\), then
< \partial H_A (u), \bar{u} > + q_1 < \partial D_1 (\bar{u}), \bar{v} > ^{-1} < \partial H_A (u), \bar{v} > = 0. \tag{2.4}

In case \( \psi \in W_r \) is such that \(< \partial D_m (\bar{u}), \bar{v} > \neq 0 \), then

\[
\sum_{j=m} < \partial D_j (u), \bar{u} > + q_m < \partial D_m (\bar{u}), \bar{v} > ^{-1} < \partial D_j (u), \bar{v} > = 0. \tag{2.5}
\]

Proof. As \( j_0 \in \{1, m \} \), let us consider the function of \( C^1 ([0, +\infty) \times \mathbb{R}) \) \( f(s, \tau) = D_{\tau_0} (s \bar{v} + \tau \bar{v}) \). Since \( f(1, 0) = D_{\tau_0} (\bar{v}) = -1 \) and \( \frac{\partial f}{\partial \tau} (1, 0) = -< \partial D_{\tau_0} (\bar{v}), \bar{v} > \neq 0 \), there exist \( \delta \in [0, 1] \) and only one function \( \tau (s) \in C^1 ([1 - \delta, 1 + \delta]) \) such that \( f(s, \tau (s)) = -1 \ \forall s \in [1 - \delta, 1 + \delta] \), \( \tau (1) = 0 \) and we have

\[
\tau' (1) = -< \partial D_{\tau_0} (\bar{v}), \bar{v} > \left( < \partial D_{\tau_0} (\bar{v}), \bar{v} > \right) ^{-1} = -q_{\tau_0} < \partial D_{\tau_0} (\bar{v}), \bar{v} > ^{-1} = q_{\tau_0} < \partial D_{\tau_0} (\bar{v}), \bar{v} > ^{-1}.
\]

In \( \psi \), where \( j_0 = 1 \), with \( \delta \in [0, 1] \), such that \( v(s) = s \bar{v} + \tau (s) \bar{v} \in V_c \ \forall s \in [1 - \delta, 1 + \delta] \), we have

\[
H_{\lambda} (v(s)) \geq H_{\lambda} (v(1)) \ \forall s \in [1 - \delta, 1 + \delta];
\]

then

\[
0 = \left. \frac{d}{ds} H_{\lambda} (v(s)) \right|_{s=1} = < \partial H_{\lambda} (\bar{v}), \bar{u} > + q_1 < \partial D_1 (\bar{v}), \bar{v} > ^{-1} < \partial H_{\lambda} (\bar{v}), \bar{v} > = \tau^- < \partial H_{\lambda} (\bar{v}), \bar{u} > + q_1 < \partial D_1 (\bar{v}), \bar{v} > ^{-1} < \partial H_{\lambda} (\bar{v}), \bar{v} >
\]

from which (2.4) follows.

In \( \psi \), where \( j_0 = m \), we note that relations \( \bar{v} (x, \bar{u}) = 0 \) and \( \frac{\partial \bar{v}}{\partial v} (x, \bar{u}) = 0 \) imply that there exist an open ball \( B' \) centered in \( \bar{v} \) included in \( V_c \) and only one functional \( \tau^* (v) \in C^1 (B') \) such that \( \bar{v} (x, \bar{v}) = 0 \) for any \( v \in B' \). Since \( \tau^* (v) = r (v) \forall v \in B' \cap S \), set \( \delta \in [0, 1] \) such that \( v(s) = s \bar{v} + \tau (s) \bar{v} \in B' \ \forall s \in [1 - \delta, 1 + \delta] \), we have

\[
r^* (v(s)) = r (v(s)) \ \forall s \in [1 - \delta, 1 + \delta].
\]

Then functional

\[
\bar{E} (v(s)) = r (v(s)) ^{-1} H_{\lambda} (v(s)) - \sum_{j=1} ^m r (v(s)) ^{-1} D_j (v(s))
\]

belongs to \( C^1 ([1 - \delta, 1 + \delta]) \).

Taking into account that

\[
D_m (v(s)) = -1 \ \text{and} \ p \left( r (v(s)) \right) ^{-1} H_{\lambda} (v(s)) - \sum_{j=1} ^m q_j \left( r (v(s)) \right) ^{-1} D_j (v(s)) = 0 \ \forall s \in [1 - \delta, 1 + \delta],
\]

we have
\[
\begin{align*}
\left[ \frac{d}{ds} \tilde{E}(v(s)) \right]_{s=1} &= r_p \left[ < \partial H_{\lambda}(\varphi), \varphi > + q_{m_i} \left( < \partial D_{j}(\varphi), \partial \varphi > \right)^{-1} < \partial H_{\lambda}(\varphi), \partial \varphi > \right] - \\
&\sum_{j=m_i}^{n} \left[ < \partial D_{j}(\varphi), \partial \varphi > + q_{m_i} \left( < \partial D_{m_i}(\varphi), \partial \varphi > \right)^{-1} < \partial D_{j}(\varphi), \partial \varphi > \right] - \\
&\left[ < \partial H_{\lambda}(\varphi), \partial \varphi > + q_{m_i} \left( < \partial D_{m_i}(\varphi), \partial \varphi > \right)^{-1} < \partial H_{\lambda}(\varphi), \partial \varphi > \right] - \\
&\sum_{j=m_i}^{n} \left[ < \partial D_{j}(\varphi), \partial \varphi > + q_{m_i} \left( < \partial D_{m_i}(\varphi), \partial \varphi > \right)^{-1} < \partial D_{j}(\varphi), \partial \varphi > \right]
\end{align*}
\]

from which (2.5) since
\[
\tilde{E}(v(s)) \geq \tilde{E}(v) = \tilde{E}(v(1)) \forall s \in [1 - \delta_0, 1 + \delta_0].
\]

3. Some Applications

In this section we suppose \( N \geq 2 \) and set

\[ \nu = (\nu_1, \ldots, \nu_N) = \text{the outward orthogonal unitary vector to } \partial \Omega; \]
\[ P_T(R) = \text{the class of the real functions defined a.e. in } R, \text{ measurable and } T\text{-periodic}; \]
\[ \forall \varphi \in P_T(R) \cap C^\infty(R), \text{ support of the restriction of } \varphi \text{ to } [0,T]. \]

We warn that the weak continuity in \( W_\nu \) of the functionals \( B \) and \( D_j \) present in the applications can be easily proved by using embedding Sobolev theorems [1], a compactness lemma ([6], Theor. 5.1 page 58) and the isomorphism \( F \) (section 1).

We add the following clarification:

\[ \left[ \begin{array}{l}
\text{We suppose } 0 < t_0 - \varepsilon_0 < T \text{ resp. } t_0 + \varepsilon_0 < T, \\
\text{resp } (0, \varepsilon_0) \cup (T - \varepsilon_0, T) \text{ and } \omega = 1 \text{ in } [t_0 - \varepsilon, t_0 + \varepsilon]
\end{array} \right. \]

\[ \text{Application 3.1 (connected to Theor.2.1). Let us assume in the definition (1.1) of } W_\nu \text{ for } n = 1 \text{ and } V = W_0^{1,\nu}(\Omega), \text{ then}
\]
\[
\|v\| = \left( \int_0^T \left| \frac{\partial v}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}} + \left( \int_0^T |\nabla v|^p dx dt \right)^{\frac{1}{p}} \forall v \in W_\nu,
\]

and let us set as any \( v \in W_\nu \)

\[
A(v) = p^{-\frac{1}{n}} \left( \int_0^T \left| \frac{\partial v}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}} + \left( \int_0^T a(x,t) |\nabla v|^p dx dt \right)^{\frac{1}{p}},
\]
\[
B(v) = p^{-\frac{1}{n}} \left( \int_0^T b(x,t) |v|^p dx dt \right)^{\frac{1}{p}}, \quad D_j(v) = q_j^{-\frac{1}{n}} \left( \int_0^T d_j(x,t) v^p dx dt \right)^{\frac{1}{p}},
\]

where
1 < p ≤ p, 1 < q ≤ q, p = a, b ∈ \( \mathbb{P}_r (\Omega \times \mathbb{R}) \cap L^q (Q) \) \setminus \{0\}, a (x, t) ≥ a_0 and 0 ≤ b (x, t) ≤ b_0
dot

ea.e. in Q (a_o, b_o = \text{const.} > 0), d_j ∈ \( \mathbb{P}_r (\Omega \times \mathbb{R}) \cap L^q (Q) \) \setminus \{0\} \( p_j = p^j / (p_j - 1) \).

(3.1)

Problem \( P^\tau \) becomes:

Find \( u ∈ W_\tau \setminus \{0\} \) such that

\[
\int_{\Omega} \left( \int_{t_0}^{T} \frac{\partial u}{\partial t} \, dt \right) \, dx = \int_{\Omega} \left( \int_{t_0}^{T} \frac{\partial u}{\partial t} \, dt \right) \, dx + \int_{\Omega} a (x, t) \left| \nabla u \right|^p \, dx dt = \int_{\Omega} a (x, t) \left| \nabla u \right|^p \, dx dt + \int_{\Omega} b (x, t) \, dx dt + \int_{\Omega} d_j (x, t) \, dx dt \forall \in W_\tau.
\]

(3.2)

Each solution \( u \) of (3.2) is for definition a weak solution of the problem:

\[
- \int_{\Omega} \frac{\partial u}{\partial t} \, dx dt - \int_{\Omega} \frac{\partial u}{\partial t} \, dx dt = \int_{\Omega} a (x, t) \left| \nabla u \right|^p \, dx dt = \int_{\Omega} a (x, t) \left| \nabla u \right|^p \, dx dt + \int_{\Omega} b (x, t) \, dx dt + \int_{\Omega} d_j (x, t) \, dx dt \forall \in W_\tau.
\]

(3.3)

Evidently

\[
V^+ (D_i) = \varnothing.
\]

Let \( \lambda^* and z^* \) be the first eigenvalue and the first eigenfunction of the problem:

\[
z ∈ W^{1,p}_0 (\Omega) : \quad - a_0 \text{div} \left( \left| \nabla z \right|^{p-2} \nabla z \right) = \theta b_0 \left| z \right|^{p-2} z \quad \text{in } \Omega.
\]

We remember that [3] \( z^* > 0 \) in \( \Omega \) and

\[
\lambda^* = \left( \int_{\Omega} \left| \nabla z^* \right|^p \, dx \right)^{1/p} \leq \left( \int_{\Omega} \left| \nabla z \right|^p \, dx \right)^{1/p} \left( \int_{\Omega} \left| z \right|^p \, dx \right)^{-1} \forall z ∈ W^{1,p}_0 (\Omega) \setminus \{0\}.
\]

(3.5)

(3.5) implies that

\[
b_0 \int_{\Omega} v (x, t) \, dx ≤ \left( \lambda^* \right)^{1/p} a_0 \int_{\Omega} \left| \nabla v (x, t) \right|^p \, dx \quad \text{a.e. in } [0, T] \forall v ∈ W_\tau.
\]

then as \( 0 < \lambda < \left( \lambda^* \right)^{1/p} \)
\[
\left( a_0 \int_q |\nabla v|^p \, dx \right)^\frac{1}{p} - \lambda \left( b_0 \int_q |v|^q \, dx \right)^\frac{1}{q} \geq \left( 1 - \lambda^\frac{1}{\lambda} \right)^{\frac{1}{\lambda}} \left( a_0 \int_q |\nabla v|^p \, dx \right)^\frac{1}{p} \forall v \in W_r
\]
(3.6)

from which

\[
H_\lambda (v) = A(v) - \lambda B(v) \geq p^{-1} \min \left\{ 1, a_0^\frac{1}{\lambda} \left( 1 - \lambda^\frac{1}{\lambda} \right)^{\frac{1}{\lambda}} \right\} 2^{-p} \| v \|_{p^*} \forall v \in W_r.
\]

Then, since

\[
as \lambda \leq 0 \quad \| v \|_{p^*} \leq p 2^p H_\lambda (v) \quad \forall v \in W_r,
\]

we have

\[
(\tilde{z}_\lambda), \text{ in particular } (i_{2\lambda}), \text{ holds if } \lambda \in \left[ -\infty, \lambda^* \right]. \quad (3.7)
\]

Relations (3.4),(3.7) and Theor.2.1 let us to state the following proposition.

**Proposition 3.1.** Under conditions (3.1), with \( \lambda \) as in (3.7) problem (3.3) has at least two weak solutions \( u_0 \) and \( -u_0 \) (\( u_0 = r_0 \), \( r_0 = \text{ const.} > 0 \), \( v_0 \in S_0 \cap V^+ (D_1) \)).

**Remark 3.1.** If \( d_1 \geq 0 \), we have \( D_1 (v) \leq D_1 (|v|) \forall v \in W_r \). Then, since \( H_\lambda (v) = H_\lambda (|v|) \forall v \in W_r \), it results in \( u_0 \geq 0 \).

Additionally, when \( a(x,t) \equiv a_0 \) and \( b(x,t) \equiv b_0 \) in \( Q \), since \( z' \in V^+ (D_1) \) and \( H_\lambda (z') \leq 0 \) as \( \lambda \geq \left( \lambda^* \right)^{\frac{1}{p^*}} \), \( (i_{2\lambda}) \) holds if and only if \( \lambda \in \left[ -\infty, \lambda^* \right] \).

**Proposition 3.2.** Let \( p_2 \leq N p_1 / (N - p_1) \) if \( N > p_1 \). If there exist a measurable set \( I \subseteq [0,T] \) with \( |I| > 0 \), a limit point \( t_0 \) of \( I \) and \( g \in L^p (\Omega) \left( \int_{\Omega} p_0 = \min \left\{ p_1', p_2' \right\}, p_1' = p_1 / (p_1 - 1) \right) \) such that \( \lim_{t \to t_0} d_t (x,t) = 0 \) a.e. in \( \Omega \) and \( |d_t (x,t)| \leq g(x) \) a.e. in \( \Omega \times I \), then \( u_0 \) is nonstationary.

**Proof.** The additional assumption on \( p_2 \) implies that

\[
W^{1,n}_0 (\Omega) \subseteq L^p (\Omega) \text{ with continuous embedding.} \quad (3.8)
\]

Reasoning by contradiction, let \( \frac{\partial u_0}{\partial t} \equiv 0 \) in \( Q \). Set \( \omega (t) = \int_I d_t (x,t) u_0 \, dx \) a.e. in \([0,T] \), a Lebesgue theorem assures that

\[
\lim_{t \to t_0} \omega (t) = 0 . \quad (3.9)
\]

Let \( \varphi \in P_\tau (\mathbb{R}) \cap C^\infty (\mathbb{R}) \text{ with } \varphi \geq 0 \text{ and } \text{supp} \varphi \subset [0,T] \). Since by (3.8) \( \varphi u_0 \in W_r \), from (3.2) with \( a = u_0 \) and \( v = \varphi u_0 \) we get

\[
\left( \int_q a(x,t)|\nabla u_0|^p \, dx \right)^\frac{1}{p} - \lambda \left( \int_q b(x,t)|u_0|^q \, dx \right)^\frac{1}{q} \varphi \, d\lambda - \lambda \left( \int_q b(x,t)|u_0|^q \, dx \right)^\frac{1}{q} \varphi \, d\lambda = \int_q \varphi d(x,t)|u_0|^p \, dx \]

Let us add that

\( \text{the left side of (3.10) is } \geq (T)^{-1} \delta \int_{0}^{r} \varphi dt \) \hfill (3.11)

where

\[
\delta = \left( \int_{Q} a_{0} |\nabla u_{0}|^{p} dxdt \right)^{\frac{1}{p}} > 0 \quad \text{if } \lambda \leq 0,
\]

\[
\delta = \left( \int_{Q} a_{0} |\nabla u_{0}|^{p} dxdt \right)^{\frac{1}{p}} - \lambda \left( \int_{Q} b_{0} |u_{0}|^{q} dxdt \right)^{\frac{1}{q}} > 0 \quad \text{(from (3.6)) if } 0 < \lambda < \lambda_{0}^{\frac{1}{q}}.
\]

Since \( u_{0} \in V^{+}(D) \Rightarrow \int_{0}^{r} \omega dt \neq 0 \), from (3.10),(3.11) we get

\[
\omega(t) \geq \text{const.} > 0 \quad \text{if } \int_{0}^{r} \omega dt > 0, \quad \omega(t) \leq \text{const.} < 0 \quad \text{if } \int_{0}^{r} \omega dt < 0 \quad \text{a.e. in } [0,T].
\] \hfill (3.12)

Then (3.12) contradicts (3.9). \( \square \)

**Application 3.2** (connected to Theor.2.1 and Theor.2.3 (case \((c_{8})\))). Let us assume in the definition (1.1) of \( W_{T} \) \( p_{1} = p_{2} = p, n = 1 \) and \( V = W^{1,p}(\Omega) \), then

\[
\|v\| = \left( \int_{Q} \left| \frac{\partial v}{\partial t} \right|^{p} dxdt + \int_{Q} |v|^{p} dxdt + \int_{Q} |\nabla v|^{p} dxdt \right)^{\frac{1}{p}},
\]

and let us set as any \( v \in W_{T} \nabla \)

\[
A(v) = p^{-1} \left( \int_{Q} \left| \frac{\partial v}{\partial t} \right|^{p} dxdt + \int_{Q} a(x,t) |\nabla v|^{p} dxdt \right),
\]

\[
B(v) = p^{-1} \int_{Q} b(x,t) |v|^{p} dxdt, \quad D_{i}(v) = q_{i}^{-1} \int_{Q} d_{i}(x,t) |v|^{q_{i}} dxdt,
\]

where

\[
1 < q_{i} < p; \ a, b, d_{i} \in \left(P_{T} (\Omega \times R) \cap L^{\infty}(Q) \right) \setminus \{0\}, \ a(x,t) \geq a_{0} \text{ and } 0 < b(x,t) \leq b_{0}
\]

\[
a.e. \text{in } Q \left(a_{0}, b_{0} = \text{const.} > 0 \right).
\] \hfill (3.13)

Problem \( \left(P^{T} \right) \) becomes:

Find \( u \in W_{T} \setminus \{0\} \) such that
\[
\int_Q \left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \ dt + \int_Q a(x,t) |\nabla u|^{p-2} \nabla u \ dt = \lambda \int_Q b(x,t) u \ dt + \int_Q d_i(x,t) |u|^{p-2} u \ dt \quad \forall v \in W^*_T. \tag{3.14}
\]

Each solution \( u \) of (3.14) is for definition a weak solution of the problem:

\[
- \frac{\partial}{\partial t} \left( \left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \right) - \text{div} \left( a(x,t) |\nabla u|^{p-2} \nabla u \right) = \lambda b(x,t) u + d_i(x,t) |u|^{p-2} u \quad \text{in } Q,
\]

\[
a(x,t) |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma, \ u(x,0) = u(x,T) \text{ and } \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \text{ on } \Omega. \tag{3.15}
\]

Let us introduce the conditions

\[
d^+_i = \max \{d_i, 0\} \quad \forall \in Q, \tag{3.16}
\]

\[
\int_Q d_i(x,t) \ dt < 0. \tag{3.17}
\]

Evidently

\[
(3.16) \Rightarrow V^+(D_i) = \emptyset, (3.17) \Rightarrow V^- \cap S(D_i) = \emptyset \quad \forall \lambda > 0.
\]

Proposition 3.3. Under conditions (3.13) (with \( p > 1 \) and not necessarily \( q \)), (3.16) holds if \( \lambda < 0 \).

Proof. Let \( \lambda < 0 \). Reasoning by contradiction, as any \( k \in \mathbb{N} \) there exists \( v_k \in W_T \) such that

\[
H_i(v_k) < k^{-1} \|v_k\|_r.
\]

Then with \( w_k = \|v_k\|^{-1} v_k \) we have

\[
\int_Q |\frac{\partial w}{\partial t}|^{p} \ dt + \int_Q a(x,t) |\nabla w_k|^{p} \ dt - \lambda \int_Q b(x,t) |w_k|^{p} \ dt \leq p k^{-1};
\]

moreover there exists \( w \in W_T \) such that (within a subsequence) \( w_k \to w \) weakly in \( W_T \).

Consequently

\[
\int_Q |\frac{\partial w}{\partial t}|^{p} \ dt = \lim_{k \to \infty} \int_Q |\frac{\partial w_k}{\partial t}|^{p} \ dt = 0, \int_Q a(x,t) |\nabla w_k|^{p} \ dt = \lim_{k \to \infty} \int_Q a(x,t) |\nabla w_k|^{p} \ dt = 0,
\]

\[
\int_Q b(x,t) |w_k|^{p} \ dt = \lim_{k \to \infty} \int_Q b(x,t) |w_k|^{p} \ dt = 0,
\]

from which \( w_k \to 0 \) strongly in \( W_T \) and the contradiction \( 1 = \lim_{k \to \infty} \|v_k\| = 0 \). \( \square \)

Proposition 3.4. Under conditions (3.13), (3.16) and (3.17), there exists \( \delta^*_i > 0 \) satisfying the condition

\[
\forall \lambda \in \left[0, \delta^*_i \right) \exists c(\lambda) > 0 : \int_Q |\frac{\partial v}{\partial t}|^{p} \ dt + a_i \int_Q |\nabla v|^{p} \ dt - \lambda b \int_Q |v|^{p} \ dt \geq c(\lambda) \|v\| \quad \forall v \in V^+(D_i).
\]

Consequently \( (i_{2s}) \) holds if \( \lambda \in \left[0, \delta^*_i \right] \).

Proof. Reasoning by contradiction, as any \( k \in \mathbb{N} \) there exist \( v_k \in V^+(D_i) \) and \( \lambda_k \in [0, k^{-1}] \) such that
\[
\int \left[ \frac{\partial v}{\partial t} \right]^p dxdt + a \int \left[ \nabla v \right]^p dxdt < \lambda b \int \left[ v \right]^p dxdt + k^{-1} \| v \|^p .
\]

Then with \( w_k = \| v_k \|^{-1} v_k \) there exists \( w \in W_T \) such that (within a subsequence)

\[
w_k \to w \text{ weakly in } W_T ,
\]

\[
\int \frac{\partial w}{\partial t}^p dxdt = \lim_{k \to +\infty} \int \frac{\partial w_k}{\partial t}^p dxdt = 0, \quad \int \left[ \nabla w \right]^p dxdt = \lim_{k \to +\infty} \int \left[ \nabla w_k \right]^p dxdt = 0, \]

\[
\int d_i (x, t) \left[ w \right]^p dxdt = \lim_{k \to +\infty} \int d_i (x, t) \left[ w_k \right]^p dxdt \geq 0,
\]

from which since (3.17) we get \( w \equiv 0 \) in \( Q \). Then \( w_k \to 0 \) strongly in \( W_T \) and the contradiction \( 1 = \lim_{k \to +\infty} \| w_k \| = 0. \)

**Proposition 3.5.** Under conditions (3.13) and (3.17), there exists \( \delta^*_k > 0 \) such that (3.17) holds if \( \lambda \in [0, \delta^*_k] \).

**Proof.** Reasoning by contradiction, as any \( k \in \mathbb{N} \) there exist \( \lambda_k \in [0, k^{-1}] \) and \( \left( v_{k, h} \right)_{h \in \mathbb{N}} \subseteq W_T \) such that

\[
D_i \left( v_{k, h} \right) = -1 ,
\]

\[
H_{\lambda_k} \left( v_{k, h} \right) < 0 ,
\]

\[
\lim_{k \to +\infty} \left\| v_{k, h} \right\|^p = +\infty .
\]

Relation (3.20) implies there exists \( \left( h_k \right)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) strictly increasing such that

\[
\lim_{k \to +\infty} \left\| v_{k, h_k} \right\|^p = +\infty .
\]

Set \( w_k = \| v_{k, h_k} \|^{-1} v_{k, h_k} \), from (3.18), (3.19) we get

\[
\int d_i (x, t) \left[ w_k \right]^p dxdt = -q_i \left\| v_{k, h_k} \right\|^p ,
\]

\[
\int \frac{\partial w_k}{\partial t}^p dxdt + \int a (x, t) \left[ \nabla w_k \right]^p dxdt < \lambda_k \int b (x, t) \left[ w_k \right]^p dxdt .
\]

Let \( w \in W_T \) such that (within a subsequence) \( w_k \to w \) weakly in \( W_T \).

From (3.21), (3.22) we get

\[
\int d_i (x, t) \left[ w \right]^p dxdt = \lim_{k \to +\infty} \int d_i (x, t) \left[ w_k \right]^p dxdt = 0, \quad \int \frac{\partial w}{\partial t}^p dxdt = \lim_{k \to +\infty} \int \frac{\partial w_k}{\partial t}^p dxdt = 0 ,
\]

\[
\int a (x, t) \left[ \nabla w \right]^p dxdt = \lim_{k \to +\infty} \int a (x, t) \left[ \nabla w_k \right]^p dxdt = 0 ,
\]

from which since (3.17) \( w_k \to 0 \) strongly in \( W_T \) and the contradiction \( 1 = \lim_{k \to +\infty} \| w_k \| = 0. \)
Propositions 3.3-3.5, Theorems 2.1 and 2.3 (case c) allow the following proposition.

Proposition 3.6. Under assumptions (3.13) we have:

when (3.16) and (3.17) hold, with \( \lambda \in (\infty, \delta_i^c) \) problem (3.15) has at least two weak solutions \( u_0 \geq 0 \) and \( -u_0 \cdot (u_0 = r_0v_0, r_0 = \text{const}. > 0, v_0 \in S_0 \cap V^+(D_i)) \);

when (3.17) holds, with \( \lambda \in \left[0, \delta_i^c\right] \) problem (3.15) has at least two weak solutions \( u \geq 0 \) and \(-u\) \( u = r_vv \cdot \), \( r = \text{const}. > 0, v \in V \cap S(D_i)) \).

Consequently, when (3.16) and (3.17) hold, with \( \lambda \in \left[0, \min\{\delta_i^c, \delta_i^c\}\right] \) problem (3.15) has at least four different weak solutions.

Proposition 3.7. Let (3.13), (3.16) and (3.17) hold. Let \( \lambda \in \left(\infty, \delta_i^c\right) \) \( \lambda \). If there exists a measurable set \( \{0, 1\} \subseteq \Omega \times I \) such that \( I \Delta I = 0 \) a.e. in \( \Omega \times I \), then \( u_0 \) is nonstationary.

Proof. Reasoning by contradiction let \( \lambda \in \left(\infty, \delta_i^c\right) \) \( \lambda \). With \( \omega(t) = \int_{I} d(t)(u_0)^{\kappa} \) a.e. in \( [0, T] \) we have

\[
\omega(t) \leq 0 \text{ a.e. in } I. \tag{3.23}
\]

Let \( \varphi \in \mathcal{P}_r \left(\mathbb{R}\right) \cap C^\infty \left(\mathbb{R}\right) \) with \( \varphi \geq 0 \) and \( \sup \varphi \subseteq [0, T] \). Setting in (3.14) \( u = u_0 \) and \( v = \varphi u_0 \in W_r \), we have

\[
\int_{Q} a(x, t) \nabla u_0 \varphi \, dx \, dt - \lambda \int_{Q} b(x, t)(u_0)^{\kappa} \varphi \, dx \, dt = \int_{0}^{T} \omega \varphi \, dt
\]

from which

\[
(T)^{-1} \delta \int_{0}^{T} \varphi \, dt \leq \int_{0}^{T} \omega \varphi \, dt,
\]

where

\[
\delta = a_0 \int_{Q} \nabla u_0 \varphi \, dx \, dt > 0 \text{ (since } u_0 \in V^+(D_i)) \text{ if } \lambda < 0, \delta = a_0 \int_{Q} \nabla u_0 \varphi \, dx \, dt - \lambda b_0 \int_{Q} (u_0)^{\kappa} \, dx \, dt > 0 \text{(since Prop.3.4) if } \lambda \in \left[0, \delta_i^c\right].
\]

Then \( \omega(t) \geq (T)^{-1} \delta \) a.e.in \([0, T]\), and this contradicts (3.23). \( \square \)

Proposition 3.8. Let (3.13) and (3.17) hold. Let \( \lambda \in \left[0, \delta_i^c\right] \). Let one of the following conditions holds:

There exists a measurable set \( I \subseteq [0, T] \) with \( |I| > 0 \) such that \( d_1(x, t) \geq 0 \) a.e. in \( \Omega \times I \); \( \tag{3.24} \)

There exists a measurable set \( I \subseteq [0, T] \) with \( |I| > 0 \), \( c_0 > 0 \) and a limit point \( t_0 \) of \( I \) such that \( b(x, t) \geq c_0 \) a.e. in \( \Omega \times I \),

\[
\lim_{t \to t_0} d_1(x, t) = 0 \text{ a.e. in } \Omega; \tag{3.25}
\]

There exist \( t_0 \in [0, T] \) and \( \varepsilon_0 > 0 \) [resp.there exists \( \varepsilon_0 > 0 \) as in (c+), \( \eta > 0, h > 1 \) and \( \theta \in \mathcal{P}_r \left(\mathbb{R}\right) \cap W^1 \left([0, T]\right) \), with \( \theta(t) > 0 \) \( \forall t \in \left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right] \setminus \{t_0\} \) \[ resp.\forall t \in \left[0, \varepsilon_0\right] \cup \left[T - \varepsilon_0, T\right] \] and \( \theta(t_0) = 0 \) [resp.\( \theta(0) = 0 \) ] , such that
Let \( u \) be as in \( (c) \). Since \( \omega_\varepsilon \in W_\varepsilon \), we have
\[
< \partial D_1(u), \omega_\varepsilon > = \int_Q d_j(x,t)(u)^{\eta-1} \omega_\varepsilon \quad dxt < 0 ,
\]
\[
< \partial D_1(u), \omega_\varepsilon > \geq -\eta \int_Q (u)^{\eta-1} \omega_\varepsilon \quad dxdt = \eta \int_0^T \int_\Omega \theta^{\eta+1} \omega_\varepsilon \quad dxdt.
\]
Then
\[
< \partial H_\lambda(u), u > + q_1 < \partial D_1(u), \omega_\varepsilon >^{-1} < \partial H_\lambda(u), \varepsilon \omega_\varepsilon > =
\]
\[
pH_\lambda(u) - \lambda q_1 \left( < \partial D_1(u), \omega_\varepsilon >^{-1} \int_Q b(x,t)(u)^{\eta-1} \omega_\varepsilon \quad dxdt \right) \geq
\]
\[
pH_\lambda(u) + \lambda q_1 \left[ \eta \left( \int_0^T \int_\Omega \theta^{\eta+1} \omega_\varepsilon \quad dxdt \right) \int_0^T \int_\Omega \theta^{\eta-1} \quad dx \right].
\] (3.28)
Since as $\varepsilon \to 0^+$

$$
\theta_{h^{-1}}(t_{0}) \to \theta_{h^{-1}}(t_{0}) = 0 \text{ resp. } \theta_{h^{-1}}(t_{h}) \to \theta_{h^{-1}}(0) = 0 \text{ and } \theta_{h^{-1}}(T) \to \theta_{h^{-1}}(T) = 0,
$$

from (3.28) we get that it is possible to choose $\varepsilon$ such that (3.27) holds with $\tilde{v} = \theta_{w_{\lambda}}$. □

**Application 3.3** (connected to Theor.2.2 (case (c2)) and Theor.2.3 (case (c9) with $m_{1}=1$)). We premise some clarifications. Let

$$
\frac{2N}{N+2} \leq p \leq \frac{2N}{N-2} \text{ if } N > 2 \text{ , } 1 < p < \infty \text{ if } N = 2.
$$

We note

$$
(3.29) \Rightarrow W_{0}^{1}(\Omega) \subseteq L^{p'}(\Omega) \cap L^{p}(\Omega) \text{ with continuous embedding } \left(p' = p/\left(p-1\right)\right). \tag{3.30}
$$

Let $X = W_{2}^{p}(\Omega) \cap W_{0}^{1}(\Omega)$. We note the norm on $X$

$$
\|\cdot\|_{p} = \left(\int_{\Omega} |\Delta z|^{p} \, dx\right)^{1/p}
$$

is equivalent to the natural one on $X.$ It is also equivalent to the norm of $W^{2,p}(\Omega)$ on $X$. In fact, there exist $c_{1}, c_{2}, c_{3} > 0$ such that for any $z \in X$

$$
\|z\|_{W^{2,p}(\Omega)} \leq c_{1}\left\|\Delta z\|_{L^{p}(\Omega)} + \|\nabla z\|_{L^{p}(\Omega)}\right\| \text{ (2), Theor.8.2 page 444),}
$$

$$
\|\nabla z\|_{L^{p}(\Omega)} \leq c_{3}\|\Delta z\|_{L^{p}(\Omega)} \text{ (from (3.30)).}
$$

Therefore $X$ is a closed subspace of $W^{2,p}(\Omega)$. It is easy to verify that

$$
\exists z^{*} \in X: \int_{\Omega} |\nabla z^{*}|^{p} \, dx = 1 \text{ and } \int_{\Omega} |\Delta z|^{p} \, dx = \inf \left\{ \int_{\Omega} |\Delta z|^{p} \, dx : z \in X \text{ and } \int_{\Omega} |\nabla z|^{p} \, dx = 1 \right\};
$$

then, set $\lambda^{*} = \int_{\Omega} |\Delta z^{*}|^{p} \, dx > 0$, we have

$$
\int_{\Omega} |\Delta z|^{p} \, dx \geq \lambda^{*} \int_{\Omega} |\nabla z|^{p} \, dx \quad \forall z \in X, \tag{3.31}
$$

$$
\int_{\Omega} |\Delta z|^{p} \, dx - \lambda \int_{\Omega} |\nabla z|^{p} \, dx \geq \left(1 - \lambda(\lambda^{*})^{-1}\right) \int_{\Omega} |\Delta z|^{p} \, dx \quad \forall \lambda \in [0, \lambda^{*}] \text{ and } \forall z \in X. \tag{3.32}
$$

Let us assume in the definition (1.1) of $W_{r}^{1}$ $p_{1} = p_{2} = p$, $n = 2$ and $V = X$, then

$$
\|v\| = \left(\int_{Q} |\frac{\partial v}{\partial t}|^{p} \, dxdt + \int_{Q} |\Delta v|^{p} \, dxdt\right)^{1/p} \quad \forall v \in W_{r}^{1},
$$

and let us set for any $v \in W_{r}^{1}$

$$
A(v) = p^{-1} \|v\|^{p}, \quad B(v) = p^{-1} \int_{Q} |\nabla v|^{p} \, dxdt,
$$
\[ D_j(v) = q_j^{-1} \int_q d_j(x,t) |\nabla u_j|^p\, dx dt; \quad D_j^*(v) = -q_j^{-1} \int_q d_j(x,t) |v_j|^p\, dx dt \quad \text{as } j = 2,\ldots,m, \]

where

\[ 1 < q_1 < \ldots < q_m < p; \quad d_j \in \left( \left\{ P_r \left( \Omega \times \mathbb{R} \right) \right\} \cap L^p(Q) \right) \setminus \{0\} \quad \text{as } j = 1,\ldots,m, \]

\[ d_j \geq 0 \text{ a.e. in } Q \text{ as } j = 2,\ldots,m. \quad (3.33) \]

Problem \( (P^*_T) \) becomes:

Find \( u \in W_T \setminus \{0\} \) such that

\[ \int_Q \left[ \frac{\partial u}{\partial t} \right]^{p-2} \frac{\partial u}{\partial t} dx dt + \int_Q |u|^{p-2} u \Delta v dx dt = \lambda \int_Q |\nabla u|^{p-2} \nabla u \nabla v dx dt + \int_Q d_j(x,t) |\nabla u_j|^{p-2} \nabla u_j \nabla v dx dt - \sum_{j=2}^m d_j(x,t) |v_j|^{p-2} v_j dx dt \quad \forall v \in W_T. \quad (3.34) \]

Each solution \( u \) of (3.34) is for definition a weak solution of the problem:

\[ -\frac{\partial}{\partial t} \left[ \frac{\partial u}{\partial t} \right]^{p-2} \frac{\partial u}{\partial t} + \Delta \left[ |u|^{p-2} u \right] = -\lambda \text{div} \left[ |\nabla u|^{p-2} \nabla u \right] - \text{div} \left[ d_j(x,t) |\nabla u_j|^{p-2} \nabla u_j \right] - \sum_{j=2}^m d_j(x,t) |v_j|^{p-2} v_j \quad \text{in } Q, \]

\[ u = 0 \text{ and } |u|^{p-2} u = 0 \quad \text{on } \Sigma, \]

\[ u(x,0) = u(x,T) \text{ and } \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \quad \text{on } \Omega. \quad (3.35) \]

Let us introduce the conditions

There exist a compact set \( K \subseteq \left[ 0, T \right] \) with \( |K| > 0 \) and an open

set \( \Omega^+ \subseteq \Omega \) such that \( d_j(x,t) > 0 \text{ a.e. in } \Omega^+ x K \),

\[ \int_0^T d_j(x,t) dt < 0 \text{ a.e. in } \Omega. \quad (3.37) \]

We note

\[ (3.36) \Rightarrow V^+ \left( D_j \right) = \emptyset, (3.37) \Rightarrow \frac{\partial v}{\partial t} < 0 \text{ in } Q \quad \forall v \in V^+ \left( D_j \right); \]

besides

\[ (3.37) \Rightarrow V^- \cap S \left( D_j \right) = \emptyset \quad \forall \lambda > \lambda^*, \]

since as \( v = D_j(z^-)^{1/\lambda} \) we have \( D_j(v) = -1 \text{ and } H_j(v) < 0. \)

Taking into account from (3.32)

\[ p^{-1} \int_Q \left[ \frac{\partial v}{\partial t} \right]^{p} dx dt + \int_Q |\nabla v|^{p} dx dt - \lambda \int_Q |\nabla v|^{p} dx dt \geq p^{-1} \int_Q \left[ \frac{\partial v}{\partial t} \right]^{p} dx dt + \left( 1 - \lambda \left( \lambda^* \right)^{-1} \right) \int_Q |\nabla v|^{p} dx dt \]
∀λ ∈ [0,λ*] and ∀v ∈ W_r,

we have

\((i_2)\), in particular \((i_2)\), holds if λ ∈ [−∞,λ*].

**Proposition 3.9.** Under conditions (3.36) and (3.37), there exists \(δ^*_1 > 0\) such that

\((i_2)\) holds if \(λ ∈ [λ^*,λ^* + δ^*_1]\).

**Proof.** Reasoning by contradiction, for any \(k ∈ N\) there exist \(v_k ∈ V^+(D_1)\) and \(λ_k ∈ [λ^*,λ^* + k^{-1}]\) such that

\[\left\|v_k\right\| - λ_k \int_q \left\|\nabla v_k\right\|^p dxdt < pk^{-1} \left\|v_0\right\|.\]

Then with \(w_k = \left\|v_k\right\|^{-1} v_k \) and \(w ∈ W_r\) such that (within a subsequence) \(w_k → w\) weakly in \(W_r\), from the relations

\[\int_q \left|\frac{\partial w}{\partial t}\right|^p dxdt + \int_q \left|\Delta w \right|^p dxdt - λ_k \int_q \left|\nabla w_k\right|^p dxdt < pk^{-1} \int_q d_1(x,t) \left|\nabla w_k\right|^p dxdt > 0,\]

passing to limit as \(k → +∞\) we get

\[\int_q \left|\frac{\partial w}{\partial t}\right|^p dxdt + \int_q \left|\Delta w \right|^p dxdt - λ^* \int_q \left|\nabla w\right|^p dxdt ≤ 0,\]

(3.38)

\[\int_q d_1(x,t) \left|\nabla w\right|^p dxdt ≥ 0.\]

(3.39)

Since from (3.31) \(\int_q \left|\Delta w \right|^p dxdt ≥ λ^* \int_q \left|\nabla w\right|^p dxdt,\) according to (3.38) we get \(\frac{\partial w}{\partial t} ≡ 0\) in \(Q\). Consequently

\(\left(3.37\right) \text{ and } \left(3.39\right) → \left|\nabla w\right| ≡ 0 \text{ in } Q, \text{ that is } w ≡ 0 \text{ in } Q,\)

from which the contradiction \(1 = \lim_{k → +∞} \left\|w_k\right\| ≤ \lim_{k → +∞} \left\{λ_k \int_q \left|\nabla w_k\right|^p dxdt + pk^{-1}\right\} = 0.\)

**Proposition 3.10.** Under conditions (3.37), there exists \(δ^*_2 > 0\) such that

\((i_3)\) holds with \(m_1 = 1\) if \(λ ∈ [λ^*,λ^* + δ^*_2]\).

**Proof.** Reasoning by contradiction, as in Prop.3.5 there exist \(λ_k \) with \(λ_k ∈ [λ^*,λ^* + k^{-1}]\), and \(v_{k,λ_k} \subseteq W_r\) such that

\[H_{λ_k} \left(v_{k,λ_k}\right) < 0, D_1 \left(v_{k,λ_k}\right) = -1, \lim_{k → +∞} \left\|v_{k,λ_k}\right\| = +∞.\]

Then, set \(w_k = \left\|v_{k,λ_k}\right\|^{-1} v_{k,λ_k},\) we have

\[\int_q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt + \int_q \left|\Delta w_k \right|^p dxdt < λ_k \int_q \left|\nabla w_k\right|^p dxdt,\]

(3.40)
\[
\int_Q d_1(x,t) \left| \nabla w_k \right|^p \, dx dt = -q_k \left\| y_{q_k} \right\|^q \to 0 \text{ as } k \to +\infty.
\] (3.41)

Let \( w \in W_t \) such that (within a subsequence) \( w_k \to w \) weakly in \( W_t \). Relations (3.40), (3.41) imply

\[
\int_Q \left| \frac{\partial w}{\partial t} \right|^p \, dx dt + \int_Q \left| \Delta w \right|^p \, dx dt - \lambda^* \int_Q \left| \nabla w \right|^p \, dx dt \leq 0 \quad \int_Q d_1(x,t) \left| \nabla w \right|^p \, dx dt = 0,
\]

from which, taking into account (3.37), we deduce that \( w \equiv 0 \) in \( Q \) and the contradiction

\[
1 = \lim_{k \to +\infty} \left\| w_k \right\| \leq \lim_{k \to +\infty} \lambda_k \int_Q \left| \nabla w_k \right|^p \, dx dt = 0. \quad \square
\]

How established far allows the following result.

**Proposition 3.11** (Theor. 2.2 (case \((c_2)\)); Theor. 2.3 (case \((c_3)\) with \( m_1 = 1 \))). Under assumptions (3.29) and (3.33) we have:

- when (3.36) and (3.37) hold, with \( \lambda \in [-\infty, \lambda^* + \delta^*_1] \) problem (3.35) has at least two nonstationary weak solutions \( u_0 \) and \(-u_0\) (\( u_0 = r_0 v_0 \), \( r_0 = \text{const.} \) > 0, \( v_0 \in S_1 \cap V^+(D_1) \));
- when (3.37) holds, with \( \lambda \in [\lambda^*, \lambda^* + \delta^*_2] \) problem (3.35) has at least two weak solutions \( u \) and \(-u\) (\( u = r v \), \( r = \text{const.} \) > 0, \( v \in V^*_1 \cap S(D_1) \)).

Consequently, when (3.36) and (3.37) hold, with \( \lambda \in [\lambda^*, \lambda^* + \min \{\delta^*_1, \delta^*_2\}] \) problem (3.35) has at least four different weak solutions.

**Proposition 3.12.** Let \( \lambda \in [\lambda^*, \lambda^* + \delta^*_2] \). If there exist a measurable set \( I \subseteq [0,T] \) with \( \left| I \right| > 0 \) and a limit point \( t_0 \) of \( I \) such that

\[
\lim_{t \to t_0} \int_I d_j(x,t) \left| \nabla u \right|^p \, dx > 0 \quad \text{a.e. in } \Omega,
\]

then \( u \) is nonstationary.

**Proof.** In fact, if \( \frac{\partial u}{\partial t} \equiv 0 \) in \( Q \), we get the contradiction

\[
\int_{t_0} T \int_\Omega d_j(x,t) \left| \nabla u \right|^p \, dx - \sum_{j=2}^m \int_{t_0} T \int_\Omega d_j(x,t) \left| u \right|^p \, dx = \int_{t_0} T \int_\Omega \left| \Delta u \right|^p \, dx - \lambda \int_{t_0} T \int_\Omega \left| \nabla u \right|^p \, dx < 0 \quad \text{a.e. in } [0,T],
\]

\[
\lim_{t \to t_0} \int_{t_0} T \int_\Omega d_j(x,t) \left| \nabla u \right|^p \, dx - \sum_{j=2}^m \int_{t_0} T \int_\Omega d_j(x,t) \left| u \right|^p \, dx \geq 0. \quad \square
\]

Let us suppose \( d_j \) has the following structure (according to (3.36) and (3.37)):

\[
d_j(x,t) = d_{1j}(x,t) d_{12}(t), \quad d_{1j} \in L^\infty(\Omega) \text{ and } d_{1j} > 0 \text{ a.e. in } \Omega, d_{12} \in P_\ast(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\]

\[
d_{12} \sim 0 \quad \text{in } [0,T], \quad \int_0^T d_{12} \, dt < 0;
\]

there exist \( t_0 \in [0,T] \) and \( \varepsilon_0 > 0 \) as in \((c')\) such that \( d_{12} \in C^n([t_0 - \varepsilon_0, t_0 + \varepsilon_0]) \)

and \( d_{12}(t) < 0 \quad \forall t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0] \). (3.42)

In addition let us suppose:

\[
\forall j \in \{2, ..., m\} \quad d_j(x_1) \in C^n([t_0 - \varepsilon_0, t_0 + \varepsilon_0]) \quad \text{and } d_j(x,t_0) = 0 \text{ a.e. in } \Omega.
\] (3.43)
Proposition 3.13. Let (3.42) and (3.43) hold. Let \( \lambda \in \left[ \lambda^*, \lambda^* + \delta^* \right] \). If \( T < \left( d_{12} \left( t_0 \right) \right)^{-1} \int_0^T d_{12} dt \), then \( u \) is nonstationary.

Proof. It is sufficient (Prop. 2.2) to prove that

\[
\frac{\partial u}{\partial t} \equiv 0 \text{ in } Q, \text{ then there exists } \vartheta \in W' \text{ such that } < \partial D_1 \left( u \right), \vartheta > > 0 \text{ and }
\]

\[
< \partial H_1 \left( u \right), u > + q_1 \left( < \partial D_1 \left( u \right), \vartheta > \right)^{-1} < \partial H_1 \left( u \right), \vartheta > - \sum_{j=2}^{m} \left[ < \partial D_1 \left( u \right), u > + q_1 \left( < \partial D_1 \left( u \right), \vartheta > \right)^{-1} < \partial D_1 \left( u \right), \vartheta > \right] \neq 0.
\]

Let \( \omega \) be as in \( \left( c^* \right) \). We note that

\[
< \partial D_1 \left( u \right), \omega > = \left( \int_0^T d_{12} \omega \omega dt \right) \int_1^n \left( x \right) \nabla u^0 dx = -q_1 \left( \int_0^T d_{12} \omega \omega dt \right) \int_0^T d_{12} dt < 0.
\]

Then

\[
q_1 \left( < \partial D_1 \left( u \right), \omega > \right)^{-1} < \partial H_1 \left( u \right), \omega > = \left( \int_0^T d_{12} \omega \omega dt \right) \left( \int_0^T \omega \omega dt \right) \left( \int_0^T d_{12} dt \right) \left( \int_{\Omega} \Delta u^0 dx - \lambda \int_{\Omega} \nabla u^0 dx \right)
\]

and as \( j=2, \ldots, m \)

\[
q_1 \left( < \partial D_1 \left( u \right), \omega > \right)^{-1} < \partial D_1 \left( u \right), \omega > = \left( \int_0^T d_{12} \omega \omega dt \right) \left( \int_0^T \omega \omega dt \right) \left( \int_0^T d_{12} dt \right) \int_{\Omega} \left( x, t \right) u^0 \omega \left( t \right) dx dt.
\]

Let us note that

\[
\left( \int_0^T d_{12} \omega \omega dt \right)^{-1} \left( \int_0^T \omega \omega dt \right) \rightarrow \left( d_{12} \left( t_0 \right) \right)^{-1} \text{ as } \varepsilon \rightarrow 0^+;
\]

besides, set \( \eta = \min_{[\delta_0, \delta_0 + \delta]} d_{12} \), we get

\[
\left( \int_0^T d_{12} \omega \omega dt \right)^{-1} \int_{\Omega} \left( x, t \right) u^0 \omega \left( t \right) dx dt \leq \left( \int_0^T d_{12} \omega \omega dt \right)^{-1} \int_{\Omega} \left( x, t \right) u^0 \omega \left( t \right) dx dt \leq \eta \left[ \int_{\Omega} \nabla u^0 dx \right] \omega \left( t \right) dt \rightarrow \eta^{-1} \int_{\Omega} \omega \left( t \right) dx = 0 \text{ as } \varepsilon \rightarrow 0^+.
\]

Since

\[
< \partial H_1 \left( u \right), u > = \left( d_{12} \left( t_0 \right) \right)^{-1} \int_0^T d_{12} dt \int_{\Omega} \Delta u^0 dx - \lambda \int_{\Omega} \nabla u^0 dx = T - \left( d_{12} \left( t_0 \right) \right)^{-1} \int_0^T d_{12} dt \int_{\Omega} \Delta u^0 dx - \lambda \int_{\Omega} \nabla u^0 dx > 0,
\]

\[-\sum_{j=1}^{n} < \partial D_j (u), u > = \sum_{j=1}^{n} \int_d D_j (x,t) |u|^\gamma dxdt \geq 0,\]

with a suitable \( \varepsilon, \tilde{v} = \omega \varepsilon \) fulfills (3.44).

\[\Box\]

**Remark 3.2.** It is not difficult to set functions \( d_{2i} \) as in (3.42) such that \( T < \left( d_{2i} \{ t \} \right)^{-1} \int_0^T d_{2i} dt \).

**Application 3.4** (connected to Theor.2.2 (case (c1)) and Theor.2.3 (case (c9) with \( m_1 = m \)). Let us assume in the definition (1.1) of \( W_T, p_1 = p_2 = 2, n = 2 \) and \( V = W^2 (\Omega) \), then

\[\|\| = \left( \int_q \left( \frac{\partial v}{\partial t} \right)^2 dxdt + \sum_{j=1}^{m-2} \int_q \left( D_j v \right)^2 dxdt + \int_q v^2 dxdt \right)^{1/2} \forall v \in W_T,\]

and let us set as any \( v \in W_T \)

\[A(v) = 2^{-1} \left[ \int_q \left( \frac{\partial v}{\partial t} \right)^2 dxdt + \sum_{j=1}^{m-2} \int_q \left( D_j v \right)^2 dxdt \right],\]

\[B(v) = 2^{-1} \int_q \left( b_1 (x,t) v^+ - b_2 (x,t) v^- \right)^2 dxdt \quad \left( v^+ = \min \{ v, 0 \} \right),\]

\[D_j (v) = -q_j^{-1} \left( \int_q |\nabla v|^{-2} \nabla v dxdt \right)^{\gamma_j/\gamma_j} \text{ as } j = 1, \ldots, m-1,\]

\[D_m (v) = (2\alpha)^{-1} \left( \int_q \left( \int_{\gamma \Omega} (x,t) \right)^\alpha dxdt \right)^n,\]

where

\[1 < \gamma_j \leq 2, n \text{ is a positive odd integer, } 2 < q_1 < \ldots < q_{m-1} < q_m = 2n;\]

as \( i = 1, 2 \) and \( d_m \in \{ P_\gamma \Omega \Omega (Q) \} \setminus \{ 0 \}, b_1 (x,t) > 0 \text{ a.e. in } Q.\]  

(3.45)

**Problem \( P_T \)** becomes:

Find \( u \in W_T \setminus \{ 0 \} \) such that

\[\int_q \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} dxdt + \sum_{j=1}^{m-2} \int_q D_j^* u D_j^* v dxdt = \lambda \int_q \left( b_1 (x,t) u^+ - b_2 (x,t) u^- \right) dxdt - \]

\[\sum_{j=1}^{m-2} \int_q \left( \int_q |\nabla u|^{-2} \nabla v \nabla u dxdt \right) \gamma_j + \int_q \left( \int_q |\nabla u|^{-2} \nabla v dxdt \right)^{\gamma_j/\gamma_j} \int_q d_m (x,t) \right)^n \int q d_m (x,t) u v dxdt \forall v \in W_T.\]

(3.46)

Each solution \( u \) of (3.46) is for definition a weak solution of the problem:

\[-\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{m-2} D_j^* u = \lambda \left( b_1 (x,t) u^+ - b_2 (x,t) u^- \right) + \sum_{j=1}^{m-2} \int_q |\nabla u|^\gamma \nabla u dxdt + \int_q d_m (x,t) u v dxdt.\]
\[
\left( \int_Q d_m(x,t)u^2dxdt \right)^{-1}d_m(x,t)u \quad \text{in } Q,
\]

\[
\frac{\partial \Delta u}{\partial \nu} = \left| \sum_{j=1}^{n} \nu \int_Q \nabla u \cdot \nu \, dx \right|^{-1} |\nabla u|^{-2} \frac{\partial u}{\partial \nu} \quad \text{on } \Sigma,
\]

\[
\frac{\partial}{\partial \nu} \left( \frac{\partial u}{\partial x} \right) = 0 \quad \text{as } h = 1, \ldots, N \quad \text{on } \Sigma,
\]

\[
u(x,0) = \nu(x,T) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \quad \text{on } \Omega.
\] (3.47)

Let us introduce the conditions

\[
d_m^+ \sim 0 \quad \text{in } Q,
\] (3.48)

\[
\int_0^T d_m(x,t)dt < 0 \quad \text{a.e. in } \Omega.
\] (3.49)

Evidently

\[
\text{Proposition 3.14. (Theor.2.2 (case (c)), Theor.2.3 (case (c)) with } m_1 = m) . \text{Under conditions (3.45) we have: when (3.48) and (3.49) hold, with }
\]

\[
\text{Problem (3.47) has at least one nonstationary weak solution } u \quad (u = r_0v_0, \quad r_0 = \text{const.} > 0, v_0 \in S_0 \cap V(D_m));
\]

\[
\text{when (3.49) holds, with } \lambda \in [0, \delta^*_2) \text{ problem (3.47) has at least one weak solution } u \quad (u = r \nu, \quad r = \text{const.} > 0, \nu \in V_\lambda \cap S(D_m)).
\]

Consequently, when (3.48) and (3.49) hold, with \( \lambda \in [0, \min \{\delta^*_1, \delta^*_2\}] \) problem (3.47) has at least two different weak solutions.

Proposition 3.15. Let \( \lambda \in [0, \delta^*_2) \). Let be true one of the following conditions:

There exist a measurable set \( I \subseteq [0, T] \) with \( I \) \( > 0 \) and a limit point \( t_0 \) of \( I \) such that

\[
\lim_{i \to t_0} d_m(x,t) = 0 \quad \text{and} \quad \lim_{i \to t_0} b_i(x,t) \quad \text{a.e. in } \Omega \quad \text{as } i = 1, 2;
\] (3.50)

There exist a measurable set \( I \subseteq [0, T] \) with \( I \) \( > 0 \), a limit point \( t_0 \) of \( I \) and \( b_0 > 0 \) such that

\[
\lim_{i \to t_0} d_m(x,t) = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad b_i(x,t) \geq b_0 \quad \text{a.e. in } \Omega \times I \quad \text{as } i = 1, 2.
\] (3.51)

Then \( u \) is nonstationary.
Proof. If \( \frac{\partial u}{\partial t} \equiv 0 \) in \( Q \), then with \( \delta = \left( \int_Q d_m(x,t)(u^2) \, dxdt \right) > 0 \), we have

\[
\delta \int_\Omega d_m(x,t)u \, dx = -\lambda \int_\Omega \left( b_1(x,t)u^+ - b_2(x,t)u^- \right) \, dx \quad \text{a.e. in } [0,T]
\]

from which the contradictions

\[
0 = \lim_{t \to 0^+} \int_\Omega d_m(x,t)u \, dx = -\lambda \lim_{t \to 0^+} \int_\Omega \left( b_1(x,t)u^+ - b_2(x,t)u^- \right) \, dx < 0 \quad \text{when (3.50) holds,}
\]

\[
0 = \lim_{t \to 0^+} \int_\Omega d_m(x,t)u \, dx \leq -\lambda b_0 \int_\Omega (u^+ - u^-) \, dx < 0 \quad \text{when (3.51) holds.}
\]

Relations (3.48), (3.49) in particular fulfill when

\[
d_m(x,t) = d_{m_1}(x) d_{m_2}(t) \quad \text{with } d_{m_1} \in L^\infty(\Omega), d_{m_2} \in P_T(\mathbb{R}) \cap L^\infty(\mathbb{R}),
\]

\[
d_{m_1} > 0 \quad \text{a.e. in } \Omega, \quad d_{m_2} > 0 \quad \text{in } [0,T], \quad \int_0^T d_{m_2} \, dt < 0.
\]

(3.52)

**Proposition 3.16.** Let (3.52) holds. Let \( \lambda \in [0,\delta_2] \). Then \( u \) is nonstationary.

Proof. Reasoning by contradiction let \( \frac{\partial u}{\partial t} \equiv 0 \) in \( Q \). Since \( u \) \( \not\equiv 0 \) in \( \Omega \), set in (3.46) \( u = \bar{u} \) and \( v = 1 \), we have

\[
\left( \int_Q d_{m_1}(x) d_{m_2}(t) u^2 \, dxdt \right)^{n-1} \int_Q d_{m_1}(x)d_{m_2}(t)u \, dxdt = -\lambda \int_Q \left( b_1(x,t)u^+ - b_2(x,t)u^- \right) \, dxdt < 0.
\]

Then

\[
\left( \int_Q d_{m_1}(x)d_{m_2}(t)u^2 \, dxdt \right)^{n-1} > 0
\]

(3.53)

and moreover

\[
\int_\Omega d_{m_1}(x)u \, dx > 0
\]

(3.54)

since \( \int_0^T d_{m_2} \, dt < 0 \). Condition \( d_{m_2} > 0 \) in \( [0,T] \) implies there exists a compact set \( K \subset [0,T] \) with \( |K| > 0 \) and \( d_{m_2} > 0 \) in \( K \). Let \( \left( \varphi_\varepsilon \right)_{0 < \varepsilon < \varepsilon_0} \subset P_T(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) with \( 0 \leq \varphi_\varepsilon \leq 1, \text{supp } \varphi_\varepsilon \subset [0,T] \) and \( \varphi_\varepsilon \to \chi \) strongly in \( L^1(0,T) \)

as \( \varepsilon \to 0^+ \) \( \forall s \in [1, \infty) \), where \( \chi \) is the characteristic function of \( K \).

Since \( \lim_{\varepsilon \to 0} \int_K d_{m_2} \varphi_\varepsilon \, dt = \int \varphi_\varepsilon \, dt > 0 \), we choose \( \varepsilon \) such that \( \int_0^T d_{m_2} \varphi_\varepsilon \, dt > 0 \). Then taking into account (3.53), (3.54), from (3.46) with \( u = \bar{u} \) and \( v = \varphi_\varepsilon \) we get the contradiction

\[
0 < \left( \int_Q d_{m_1}(x) d_{m_2}(t) u^2 \, dxdt \right)^{n-1} \left( \int_0^T d_{m_2} \varphi_\varepsilon \, dt \right) \int_\Omega d_{m_1} u \, dx = -\lambda \int_Q \left( b_1(x,t)u^+ - b_2(x,t)u^- \right) \varphi_\varepsilon \, dxdt < 0.
\]
Application 3.5 (connected to Theor.2.2 (case (c) with \( m_{1}=m-1 \)). Let us assume in the definition (1.1) of \( W_{r} \) \( p_{1}=p_{2}=p \) and \( V=W_{u}^{m} (\Omega) \left( n=1,2,... \right) \), then

\[
\|v\| = \left( \int_{Q} \left| \frac{\partial v}{\partial t} \right|^{p} dxdt + \sum_{i=1}^{m} \int_{Q} \left| D^{\alpha} v \right|^{p} dxdt \right)^{\frac{1}{p}} \forall v \in W_{r},
\]

and let us set as any \( v \in W_{r} \)

\[
A(v) = p^{-1} \left[ \int_{Q} \left| \frac{\partial v}{\partial t} \right|^{p} dxdt + \sum_{i=1}^{m} \int_{Q} a(t) \left| D^{\alpha} v \right|^{p} dxdt \right],
\]

\[
B(v) = p^{-1} \int_{Q} a(t)b(x)(v^{+})^{p} dxdt,
\]

\[
D_{j}(v) = q_{j}^{-1} \int_{Q} d_{j}(x,t)|v^{+}|^{p} dxdt \quad \text{as} \ j=1,...,m-1,
\]

\[
D_{m}(v) = -q_{m}^{-1} \int_{Q} d_{m}(x,t)|v^{+}|^{p} dxdt,
\]

where

\[
1 < q_{1} < \ldots < q_{m-1} < q_{m} \leq p; \ a \in \mathcal{P}_{r}(\mathbb{R}) \cap C^{0}(\mathbb{R}) \quad \text{with} \ \ a(t) \geq a_{0} \ \forall t \in [0,T] \left( a_{0} = \text{const.} > 0 \text{and} 1 \right),
\]

\[
b \in L^{\infty}(\Omega) \setminus \{0\} \quad \text{with} \ b \geq 0 \ \text{a.e. in} \ \Omega, d_{j} \in \mathcal{P}_{r}(\Omega \times \mathbb{R}) \cap L^{\infty}(Q) \setminus \{0\} \quad \text{with} \ d_{j} \geq 0 \ \text{a.e. in} \ Q
\]

\[
as j=1,...,m.
\] (3.55)

Problem \((P^{T})\) becomes:

Find \( u \in W_{r} \setminus \{0\} \) such that

\[
\int_{Q} \left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} dxdt + \sum_{i=1}^{m} \int_{Q} a(t) \left| D^{\alpha} u \right|^{p-2} D^{\alpha} u D^{\alpha} \phi dxdt = \lambda \int_{Q} a(t)b(x)(u^{+})^{p-1} \phi dxdt + \sum_{j=1}^{m-1} d_{j}(x,t)|u^{+}|^{p-2} u \phi dxdt - \int_{Q} d_{m}(x,t)|u^{+}|^{p-2} \phi dxdt \quad \forall \phi \in W_{r}.
\] (3.56)

Each solution \( u \) of (3.56) is for definition a weak solution of the problem:

\[
-\frac{\partial}{\partial t} \left( \left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \right) + \sum_{i=1}^{m} \left( -1 \right)^{\alpha} a(t) D^{\alpha} \left( \left| D^{\alpha} u \right|^{p-2} D^{\alpha} u \right) = \lambda a(t)b(x)(u^{+})^{p-1} + \sum_{j=1}^{m-1} d_{j}(x,t)|u^{+}|^{p-2} u - d_{m}(x,t)|u^{+}|^{p-2} u \quad \text{in} \ Q,
\]

\[
D^{\alpha} u = 0 \quad \text{on} \ \Sigma \quad \text{as} \ 0 \leq \alpha \leq n-1,
\]

\[
u(x,0) = u(x,T) \text{ and} \ \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \text{ on} \ \Omega.
\] (3.57)

Evidently \((i_{u})\) holds if \( \lambda \in [-\infty,0] \). Let us add that set \( \overline{\Omega} = \left\{ \|u\|_{L^{\infty}([0,T])} \right\} \), there exists \( \delta^{*} > 0 \) satisfying the condition:
\[ \forall \lambda \in [0, \delta^+] \exists c(\lambda) > 0 : a_{\|v\|} \|P\| - \lambda \int q \, b(x) \left( u^+ \right)^p \, dx \, dt \geq c(\lambda) \|v\| \quad \forall v \in W_T. \] (3.58)

In fact, otherwise, for each \( k \in \mathbb{N} \) there exist \( v_k \in W_T \) and \( \lambda_k \in [0, k^{-1}] \) such that
\[
a_{\|v_k\|} \left( u^+ \right)^p \| P \| - \lambda_k \int q \, b(x) \left( u^+ \right)^p \, dx \, dt < k^{-1} \| v_k \|.
\]

Then with \( w_k = \| v_k \|^{-1} v_k \) we have
\[
a_{\|v_k\|} < \lambda_k \int q \, b(x) \left( w_k^+ \right)^p \, dx \, dt + k^{-1}
\]
from which, passing to limit as \( k \to +\infty \), we get \( a_0 \leq 0 \).

(3.58) implies \( (i_{21}) \) holds even if \( \lambda \in [0, \delta^+] \).

**Proposition 3.17.** (Theor.2.2 (case (c4) with \( m_1 = m-1 \))). Under conditions (3.55), with \( \lambda \in [-\infty, \delta^-] \) problem (3.57) has at least one weak solution \( \tilde{u}_0 \) \( (\tilde{u}_0 = \eta_0, \eta_0 = \text{const.} > 0, \tilde{v}_0 \in S_1 \cap V^+ (D_1, \ldots, D_{m-1})). \)

About the nonstationarity of \( \tilde{u}_0 \), let us introduce the conditions:

1. There exist a measurable set \( I \subseteq [0, T] \) with \( |I| > 0 \) and a limit point \( t_0 \) of \( I \) such that
   \[
   \lim_{t \to t_0} d_j (x, t) = 0 \quad \text{for almost any } x \in \Omega \quad \text{and as } j = 1, \ldots, m-1;
   \] (3.59)

2. There exist an open interval \( I \subseteq [0, T] \), \( g \in C^0 (I) \) with \( g(t) > 0 \) \( \forall t \in I \) and \( g^{-\eta, a} \) in \( I \quad \forall \eta > 0 \cdot g \in L^\infty (\Omega) \) with \( g_j > 0 \ a.e. \ in \Omega \) such that \( d_j (x, t) = g_j (x) g(t) \) as almost every \( x \in \Omega \) and as each
   \[
   t \in I \quad (j = 1, \ldots, m);
   \] (3.60)

3. There exist \( t_0 \in [0, T] \) and \( \varepsilon_0 > 0 \) as in \( (c^+) \) such that for almost any \( x \in \Omega \)
   \[
   d_j (x, t) \in C^0 \left( [t_0 - \varepsilon_0, t_0 + \varepsilon_0] \right) \quad \text{as } j = 1, \ldots, m
   \]
and
\[
\sum_{j=1}^{m-1} \int q \left[ d_j (x, t) - a \left( t \right) \left( a \left( t_0 \right) \right)^{-1} d_j (x, t_0) \right] \| u \| \| P \| \, dx \, dt -
\]
\[
\int q \left[ d_j (x, t) - a \left( t \right) \left( a \left( t_0 \right) \right)^{-1} d_j (x, t_0) \right] \| u \| \| P \| \, dx \, dt = 0.
\] (3.61)

**Remark 3.3.** It is easy to find assumptions on \( d \) and \( a \) such that the inequality in (3.61) holds.

**Proposition 3.18.** Let \( \lambda \in [-\infty, \delta^-] \). If one of the conditions (3.59) - (3.61) holds, then \( u_0 \) is nonstationary.

**Proof.** Reasoning by contradiction, let \( \frac{\partial u_0}{\partial t} \equiv 0 \) in \( Q \).

When (3.59) holds, we have the contradiction
\[
0 = \lim_{i \to i^+} \sum_{j=1}^{m-1} \int \left[ d_j (x, t) \right] u_\| \| \| P \| \, dx \geq \delta
\]
where
\[
\delta = a_0 \sum_{i} \int \left[ \rho^* u_\| \| \| P \| \, dx \right] > 0 \quad \text{if } \lambda \leq 0,
\]
\[ \delta = a_0 \sum_{m=1}^{\infty} \int_{\Omega} |D^r u_v| \, dx - \lambda \pi \int_{\Omega} b(x)(u_v) \, dx > 0 \quad \text{(from (3.58)) if } \lambda \in [0, \delta'] . \]

When (3.60) holds, we have \( g(t) = \eta_0 a(t) \forall t \in I \) with
\[
\eta_0 = \left[ \sum_{m=1}^{\infty} \int_{\Omega} |D^r u_v| \, dx - \lambda \pi \int_{\Omega} b(x)(u_v) \, dx \right] \left[ \sum_{j=1}^{\infty} \int_{\Omega} g_j(x)v_0 \, dx - \int_{\Omega} g_m(x)v_0 \, dx \right] > 0,
\]
and this contradicts hypothesis.

When (3.61) holds, it is sufficient (Prop. 2.1) to prove that

There exists \( \tilde{v} \in W_T \) such that \( \langle \partial H_\lambda(v_0), \tilde{v} \rangle > 0 \) and
\[
\sum_{j=1}^{m} \left| \langle \partial D_j(v_0), u_0 \rangle - p \langle \partial H_\lambda(v_0), \tilde{v} \rangle \right|^2 \langle \partial D_j(v_0), v_0 \rangle > 0 . \tag{3.62}
\]

With \( \omega_j \) as in \( \{e^j\} \) we have
\[
\langle \partial H_\lambda(v_0), \omega_j v_0 \rangle = \sum_{m=1}^{\infty} \int_{Q} a(t) |D^r v_0| \omega_j \omega \, dx dt - \lambda \pi \int_{Q} a(t) b(x)(u_v) \omega \, dx dt = \\
\left( \int_{0}^{T} a(t) \, dt \right) \sum_{j=1}^{m} \int_{\Omega} |D^r v_0| \, dx - \lambda \pi \int_{\Omega} b(x)(u_v) \, dx = p \left( \int_{0}^{T} a(t) \, dt \right) > 0.
\]

Then
\[
\sum_{j=1}^{m} \left| \langle \partial D_j(v_0), u_0 \rangle - p \langle \partial H_\lambda(v_0), \omega_j \rangle \right|^2 \langle \partial D_j(v_0), v_0 \rangle = \\
\sum_{j=1}^{m} \int_{Q} d_j(x,t) |u_0| \omega_j \, dx dt - \int_{0}^{T} a(t) \omega_j \omega_j \, dt \int_{Q} d_j(x,t) |u_0| \omega_j \omega_j \, dx dt - \\
\int_{0}^{T} a(t) \omega_j \omega_j \, dt \int_{Q} d_j(x,t) |u_0| \omega_j \omega_j \, dx dt.
\]

Since \( j=1, \ldots, m \)
\[
\lim_{j \to 0} \int_{0}^{T} a(t) \omega_j \omega_j \, dt = \\
\int_{0}^{T} a(t) \omega_j \omega_j \, dt \int_{Q} d_j(x,t) |u_0| \omega_j \omega_j \, dx dt ,
\]
with a suitable \( \epsilon, \tilde{v} = \omega_j v_0 \) fulfills (3.62).

\[\square\]

**Remark 3.4.** It is easy to prove that Propositions 3.17 and 3.18 also hold when
\[
A(v) = p^{-1} \int_{Q} \left| \frac{\partial v}{\partial t} \right|^2 + a(t) \left| D^r v \right|^p \, dx dt \quad \forall v \in W_T
\]
with \( 1 < \gamma < p \) and \( a \) as in (3.55).

**Application 3.6** (connected to Theor. 2.2 (case (c_0))). Let us assume in the definition (1.1) of \( W_T \) \( p_1 = p_2 = p, n=2 \) and \( V = W^{2,p} (\Omega) \), then
\[ \|v\| = \left( \int_Q \left| \frac{\partial u}{\partial t} \right|^p \, dx dt + \sum_{\alpha=1}^d \int_Q |D^\alpha v|^p \, dx dt + \int_Q |v|^p \, dx dt \right)^{1/p} \quad \forall v \in W_T, \]

and let us set as any \( v \in W_T \):

\[ A(v) = p^{-1} \int_Q \left[ \frac{\partial u}{\partial t} \right]^p \, dx dt + \sum_{\alpha=1}^d \int_Q a(t) |D^\alpha v|^p \, dx dt, \]

\[ B(v) = p^{-1} \int_Q a(t) |v|^p \, dx dt, \]

\[ D_j(v) = q_j^{-1} \int_Q d_j(x,t) \nabla v \, dx dt \quad \text{as } j = 1, \ldots, m - 2, \]

\[ D_{m-1}(v) = q_{m-1}^{-1} \int_Q d_{m-1}(x,t) \|v\|^{p-1} \, dx dt, \]

\[ D_m(v) = q_m^{-1} \int_Q d_m(x,t) \|v\|^{p-1} v \, dx dt, \]

where

\[ 1 < q_1 < \ldots < q_m < p; \quad a \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R}) \quad \text{with} \quad a(t) \geq a_0 \quad \forall t \in [0,T] \quad \{a_0 = \text{const.} > 0\}, \]

\[ d_j \in \left( P_T(\Omega \times \mathbb{R}) \cap L^\infty(Q) \right) \setminus \{0\} \quad \text{as } j = 1, \ldots, m, \quad d_j \geq 0 \quad \text{a.e. in } Q \quad \text{as } j = 1, \ldots, m - 2, \]

\[ d_{m-1} > 0 \quad \text{and} \quad d_m > 0 \quad \text{a.e. in } Q. \quad (3.63) \]

Problem \( P_T^* \) becomes:

Find \( u \in W_T \setminus \{0\} \) such that

\[ \int_Q \frac{\partial u}{\partial t} \left| \frac{\partial u}{\partial t} \right|^p \, dx dt + \sum_{\alpha=1}^d \int_Q a(t) \left| D^\alpha u \right|^p \, dx dt + \lambda \int_Q a(t) |u|^{p-2} u v \, dx dt + \sum_{j=1}^{m-2} \int_Q d_j(x,t) \left| \nabla u \right|^{p-2} \nabla u \nabla v \, dx dt + \]

\[ \int_Q d_{m-1}(x,t) \left| u \right|^{p-2} u v \, dx dt + \int_Q d_m(x,t) \left| u \right|^{p-1} v \, dx dt \quad \forall v \in W_T. \quad (3.64) \]

Each solution \( u \) of (3.64) is for definition a weak solution of the problem:

\[- \frac{\partial}{\partial t} \left( \left| \frac{\partial u}{\partial t} \right|^p \right) + \sum_{\alpha=1}^d a(t) \left| D^\alpha u \right|^p - \lambda a(t) |u|^{p-2} u = - \sum_{j=1}^{m-2} \text{div} \left( d_j(x,t) \left| \nabla u \right|^{p-2} \nabla u \right) + \]

\[ d_{m-1}(x,t) |u|^{p-2} u + d_m(x,t) |u|^{p-1} \quad \text{in } Q, \]

\[ a(t) \sum_{\lambda=1}^\nu \frac{\partial}{\partial x_\lambda} \left( \left| \frac{\partial^\nu u}{\partial x_\lambda \partial x_{\lambda}} \right|^{p-2} \frac{\partial^\nu u}{\partial x_\lambda \partial x_{\lambda}} \right) v_\lambda = - \sum_{j=1}^{m-2} d_j(x,t) \left| \nabla u \right|^{p-2} \frac{\partial u}{\partial \nu} \quad \text{on } \Sigma, \]

\[ \sum_{\lambda=1}^\nu \frac{\partial^\nu u}{\partial x_\lambda \partial x_{\lambda}} \left|^{p-2} \frac{\partial^\nu u}{\partial x_\lambda \partial x_{\lambda}} \right| v_\lambda = 0 \quad \text{on } \Sigma \quad \text{as } h = 1, \ldots, N. \]
\[ u(x,0) = u(x,T) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \quad \text{on} \quad \Omega. \]  

(3.65)

We note that

\[ V^+ \left( D_1, \ldots, D_{m-1} \right) = W_T \setminus \{0\} \quad \text{for almost any} \quad (i_{\eta}) \quad \text{if} \quad \lambda < 0. \]

Then

**Proposition 3.19.** (Theor.2.2 (case (c)\textsubscript{3})). Under conditions (3.63), with \( \lambda < 0 \) problem (3.65) has at least one weak solution \( u_0 \quad \text{(} u_0 = r_0 \nu_0, r_0 = \text{const.} > 0, \nu_0 \in S_\nu \).}

**Proposition 3.20.** Let \( \lambda < 0 \). Let one of the following conditions be fulfilled:

There exist a measurable set \( I \subseteq [0,T] \) with \( I \) and a limit point \( t_0 \) of \( I \) such that

\[ \lim_{\nu_j \to \infty} d_j(x,t) = 0 \quad \text{for almost any} \quad x \in \Omega \quad \text{and as} \quad j=1,\ldots,m; \]

There exist an open interval \( I \subseteq [0,T] \), \( g \in C^\infty(I) \) with \( g(t) > 0 \forall t \in I \) and \( g - \eta a \quad \text{in} \quad I \forall \eta > 0 \), \( g_j \in L^\infty(\Omega) \) with \( g_j > 0 \ a.e. \ \text{in} \ \Omega \) such that \( d_j(x,t) = g_j(x)g(t) \) as almost every \( x \in \Omega \) and as each \( t \in I \) \( j=1,\ldots,m \);

There exist \( t_0 \in ]0,T[ \) and \( \epsilon_0 > 0 \) as \( \epsilon \to 0 \) such that for almost any \( x \in \Omega \) \( d_j(x,t) \in C^\infty(\left[ t_0 - \epsilon_0, t_0 + \epsilon_0 \right]) \) as \( j=m,1,\ldots,m-1 \)

and

\[
\begin{align*}
\int_{t_0-\epsilon_0}^{t_0+\epsilon_0} \int_0^T d_j(x,t) - a \left( \frac{\partial u}{\partial t}(x,t) \right) \partial_{x^0} \left[ a \left( \frac{\partial u}{\partial t}(x,t) \right) \right] d_j(x,t) \ dx \ dt &+ \int_0^T \int_{t_0-\epsilon_0}^{t_0+\epsilon_0} d_{m-1}(x,t) - a \left( \frac{\partial u}{\partial t}(x,t) \right) \partial_{x^0} \left[ a \left( \frac{\partial u}{\partial t}(x,t) \right) \right] \ dx \ dt \ dx = 0,
\end{align*}
\]

\[ d_m(x,t) = g_m(x) \quad \text{as almost each} \quad x \in \Omega \quad \text{and for each} \quad t \in [0,T] \] \quad \text{where} \quad g_m \in L^\infty(\Omega) \quad \text{and} \quad g_m > 0 \ a.e. \ \text{in} \ \Omega.

Then \( u_{\nu_j} \) is nonstationary.

**Proof.** We reason as in Prop. (3.18). \qed