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Existence of Time Periodic Solutions of New Classes of Nonlinear Problems

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Abstract: We study the existence of one or more weak periodic solutions of nonlinear evolution PDEs in a cylinder of $\mathbb{R}^{N+1}$ with conditions on lateral surface by using the results connected to a general evolution variational equation depending on a parameter.

Keywords: Time Periodic, Evolution PDEs, Nonstationarity, Weak Periodic Solutions

1. Introduction

Let $0 < T < \infty$. Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded and connected set, with boundary $\partial\Omega \in C^{0,1}$.

For N=1, the condition "$\partial\Omega \in C^{0,1}$"means that Ω is a bounded open interval.

Let us set

$$Q = \Omega \times \left]0,T\right[\; and \, \Sigma = \partial\Omega \times \left]0,T\right[;$$

$\mathrm{P}_T\left(\Omega\times\mathbb{R}\right) =$ the class of the real functions $v\left(x,t\right)$ defined a.e. in $\Omega\times\mathbb{R}$, measurable and T-periodic with respect to t; with $v \in \mathrm{P}_T\left(\Omega\times\mathbb{R}\right)$ and $t \in \left[0,T\right]$ $\tilde{v}\left(t\right) = v\left(\cdot,t\right)$.

We denote by F the linear map $v \in \mathrm{P}_T\left(\Omega\times\mathbb{R}\right) \rightarrow \tilde{v}$.

Let $1 < p_1 < \infty, 1 < p_2 < \infty$ and V be a closed subspace of $W^{n,p_1}\left(\Omega\right)$ $\left(n = 1,2,...\right)$ such that $C_0^\infty\left(\Omega\right) \subseteq V$. We do not exclude $V = W^{n,p_1}\left(\Omega\right)$. Let $\left\|\cdot\right\|_V$ be a norm equivalent to the one of $W^{n,p_1}\left(\Omega\right)$ on V.

Set $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}}$, we consider the normed spaces:

$$W_T^0 = \Big\{v \in \mathrm{P}_T\left(\Omega\times\mathbb{R}\right) \cap L^{p_1}\left(Q\right) : as\, 0 < \left|\alpha\right| \le n \; there\; exists\; the\, weak\, derivative$$

$$D^\alpha v\, with\, D^\alpha v \in L^{p_1}\left(Q\right), v\left(\cdot,t\right) \in V \;\; for\, a.e.\, t \in \mathbb{R}\Big\}, \left\|v\right\|_0 = \left(\int_0^T \left\|v\left(\cdot\;,t\right)\right\|_V^{p_1} dt\right)^{1\backslash p_1} \;\; \forall v \in W_T^0;$$

$$W_T = \left\{v \in W_T^0 : there\;\; exists\;\; the\, weak\, derivative\, \frac{\partial v}{\partial t}\, with\, \frac{\partial v}{\partial t} \in L^{p_2}\left(Q\right)\right\},$$

$$\left\|v\right\| = \left\|v\right\|_0 + \left(\int_Q \left|\frac{\partial v}{\partial t}\right|^{p_2} dxdt\right)^{1\backslash p_2} \quad \forall v \in W_T\,; \tag{1.1}$$

$$\tilde{W}^0 = L^{p_1}\left(0,T;V\right) \quad , \left\|w\right\|_{\tilde{W}^0} = \left(\int_0^T \left\|w\left(t\right)\right\|_V^{p_1} dt\right)^{1\backslash p_1} \quad \forall w \in \tilde{W}^0;$$

$$\tilde{W} = \left\{w \in \tilde{W}^0 : w' \in L^{p_2}\left(0,T;L^{p_2}\left(\Omega\right)\right)\right\}, \left\|w\right\|_{\tilde{W}} = \left\|w\right\|_{\tilde{W}^0} + \left(\int_0^T \left\|w'\left(t\right)\right\|_{L^{p_2}\left(\Omega\right)}^{p_2} dt\right)^{1\backslash p_2} \quad \forall w \in \tilde{W}.$$

Remark 1.1. When $p_1 = p_2 = p$ we assume

$$\left\|v\right\| = \left(\int_0^T \left\|v\left(\cdot,t\right)\right\|_V^p dt + \int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt\right)^{1\backslash p} \quad \forall v \in W_T \;\; and \left\|w\right\|_{\tilde{W}} = \left(\int_0^T \left\|w\left(t\right)\right\|_V^p dt + \int_0^T \left\|w'\left(t\right)\right\|_{L^p\left(\Omega\right)}^p dt\right)^{1\backslash p} \quad \forall w \in \tilde{W}.$$

We recall ([16], Chap.23) $\tilde{W}^0 \; and \, \tilde{W}$ are reflexive and separable Banach spaces. It is not difficult to prove that $W_T^0 \; and \; W_T$ are Banach spaces; the restriction of F to $W_T^0 \left[\, resp. W_T\right]$ is a norm-preserving linear transformation into $\tilde{W}^0 \; \left[resp. \tilde{W}\right]$.

Consequently $W_T^0 \; and \; W_T$ are reflexive and separable spaces.

Remark 1.2. These conclusions hold even if $V = L^{p_1}\left(\Omega\right)$ .In this case $W_T^0 = \mathrm{P}_T\left(\Omega \times \mathbb{R}\right) \cap L^{p_1}\left(Q\right)$ and $\left\|v\right\|_0 = \left(\int_Q \left|v\right|^{p_1} dxdt\right)^{1\backslash p_1}$.

Let us denote by $< \cdot,\cdot >$ the duality between W_T^* (dual space of W_T) and W_T, and by "∂" Fréchet differential operator. Let A~0, D_j~0 (j=1,…,m; $m \geq 1$) and B be real functionals defined in W_T satisfying the conditions

$$\left(i_{11}\right)\begin{cases} A\, is\;\; weakly\, lower\;\; semicontinuous\;\; in\, W_T\; and\, C^1\left(W_T\right), \\ B\, is\, weakly\, continuous\;\; in\, W_T\; and\, C^1\left(W_T\right), \\ \exists\, p > 1 : A\left(rv\right) = r^p A\left(v\right) \;\; and \;\;\; B\left(rv\right) = r^p B\left(v\right) \forall r \geq 0 \;\; and\, \forall v \in W_T; \end{cases}$$

$$\left(i_{12}\right)\begin{cases} D_j \;\; is \;\; weakly\, continuous \;\; in\, W_T \; and\, C^1\left(W_T\right), \exists\, q_j > 1 : \\ D_j\left(rv\right) = r^{q_j} D_j\left(v\right) \;\; \forall r \geq 0 \;\; and\, \forall v \in W_T, q_1 < \ldots < q_m \;\; if\, m > 1. \end{cases}$$

Let us consider the following problem.

Problem $\left(P^T\right)$.Find $u \in W_T \setminus \left\{0\right\}$ such that

$$< \partial A\left(u\right), v >= \lambda < \partial B\left(u\right), v > + \sum_{j=1}^m < \partial D_j\left(u\right), v > \;\forall v \in W_T,$$

where λ is a real parameter,

Problem $\left(P^T\right)$ is a particular case of Problem(P) studied in ([10], [12]) by using the Lagrange multipliers and the "algebraic" approach which is based on the fibering method [14]. In ([11], [12]) many applications of the results connected to Problem (P) related to nonlinear elliptic systems are present.

In section 2 we considered convenient about Problem $\left(P^T\right)$ to state existence theorems (Theorems 2.1-2.3) included in the results of ([4], [7] [8], [10], [12]) whose validity, by the way, depends on λ. Furthermore we added Propositions 2.1 and 2.2 useful in some concrete cases in order to establish the nonstationarity of the found solutions.

In section 3 we study some evolution PDEs in the cylinder Q, with also nonlocal nonlinearities and with different conditions on

Σ. About these problems, whose variational formulation is included in Problem $\left(P^T\right)$, we find the existence of one or more weak periodic solutions, giving also some sufficient conditions to their nonstationarity.

It is known that the search for periodic solutions of nonlinear problems has attracted the attention of many researchers. In particular Pohozaev in ([13], [15]) introduced "the separation variables method" for nonlinear equations in which it is possible to find weak periodic solutions in the form $u\left(x,t\right)=u_1\left(x\right)u_2\left(t\right)$. As far as we know, the methods developed in literature are not applicable in this article.

2. Existence Theorems Preliminary Results

As $\lambda\in\mathbb{R}, v\in W_T, r\geq 0, j\in\left\{1,\dots,m\right\}$ *and* $\left\{j_1,\dots,j_s\right\}\subseteq\left\{1,\dots,m\right\}\left(j_1<\dots<j_s \text{ if } s>1\right)$ we set:

$$H_\lambda\left(v\right)=A\left(v\right)-\lambda B\left(v\right), E\left(v\right)=H_\lambda\left(v\right)-\sum_{j=1}^{m}D_j\left(v\right), \tilde{E}\left(r,v\right)=E\left(rv\right)=r^pH_\lambda\left(v\right)-\sum_{j=1}^{m}r^{q_j}D_j\left(v\right),$$

$$\Psi\left(r,v\right)=\frac{\partial\tilde{E}}{\partial r}\left(r,v\right)\ , S_\lambda=\left\{v\in W_T: H_\lambda\left(v\right)=1\right\}, V_\lambda^-=\left\{v\in W_T: H_\lambda\left(v\right)<0\right\},$$

$$S\left(D_j\right)=\left\{v\in W_T: D_j\left(v\right)=-1\right\}, V^+\left(D_{j_1},\dots,D_{j_s}\right)=\left\{v\in W_T: D_{j_1}\left(v\right)+\dots+D_{j_s}\left(v\right)>0\right\}.$$

We find the solvability of Problem $\left(P^T\right)$ in the cases based on the following assumptions

$$\left(i_{21}\right)\ \exists c\left(\lambda\right)>0:\left\|v\right\|^p\leq c\left(\lambda\right)H_\lambda\left(v\right)\ \forall v\in W_T;$$

$$\left(i_{22}\right)\ \exists c\left(\lambda\right)>0:\left\|v\right\|^p\leq c\left(\lambda\right)H_\lambda\left(v\right)\ \forall v\in V^+\left(D_1\right);$$

$$\left(i_{23}\right)\ \exists c\left(\lambda\right)>0:\left\|v\right\|^p\leq c\left(\lambda\right)H_\lambda\left(v\right)\ \forall v\in V^+\left(D_m\right);$$

$$\left(i_{24}\right)\ \exists m_1\in\left\{1,\dots,m\right\}: V_\lambda^-\cap S\left(D_{m_1}\right) \text{ is nonempty and bounded in } W_T.$$

At first we consider the cases in which one of the assumptions $\left(i_{21}\right)-\left(i_{23}\right)$ is present:

$\left(c_1\right)\ m=1, q_1\neq p, V^+\left(D_1\right)\neq\varnothing,\left(i_{22}\right) holds;$

$\left(c_2\right)\ m>1, q_1<p, V^+\left(D_1\right)\neq\varnothing, D_j\leq 0\ \forall j\geq 2,\ \left(i_{22}\right) holds;$

$\left(c_3\right)\ m>1, q_m>p, V^+\left(D_m\right)\neq\varnothing, D_j\leq 0\ \forall j\leq m-1,\ \left(i_{23}\right) holds;$

$\left(c_4\right)\ m>1,\exists m_1\in\left\{2,\dots,m\right\}: q_{m_1}<p, D_j\geq 0\ \forall j\leq m_1\ and\ D_j\leq 0\ \forall j>m_1\ if\ m_1<m,\left(i_{21}\right) holds;$

$\left(c_5\right)\ m>1,\exists m_1\in\left\{1,\dots,m-1\right\}: q_{m_1}>p, D_j\geq 0\ \forall j\geq m_1\ and\ D_j\leq 0\ \forall j<m_1\ if\ m_1>1,\left(i_{21}\right) holds;$

$\left(c_6\right)\ m>1, q_m<p, D_j\geq 0\ \forall j<m,\ D_m$ changes sign, $\mathrm{V}^+\left(D_1,\dots,D_{m-1}\right)=W_T\setminus\left\{0\right\},\left(i_{21}\right) holds;$

$\left(c_7\right)\ m>1, q_1>p,\ D_1$ changes sign, $D_j\geq 0\ \forall j>1, \mathrm{V}^+\left(D_2,\dots,D_m\right)=W_T\setminus\left\{0\right\},\left(i_{21}\right) holds.$

Let us introduce the open set $\mathcal{Q}$ of the space W_T:

$$\mathcal{Q}=V^+\left(D_1\right)\ in\left(c_1\right)\ and\ \left(c_2\right),\ \mathcal{Q}=V^+\left(D_m\right)\ in\left(c_3\right),\mathcal{Q}=V^+\left(D_1,\dots,D_{m_1}\right)\ in\left(c_4\right),$$

$$\mathcal{Q}=V^+\left(D_{m_1},\dots,D_m\right)\ in\left(c_5\right),\mathcal{Q}=W_T\setminus\left\{0\right\}in\left(c_6\right)\ and\ \left(c_7\right).$$

Theorem 2.1 ([10], Section 2). Under assumptions $\left(i_{11}\right) and \left(i_{12}\right)$, in case $\left(c_1\right)$ we have:

$$\exists v_0 \in S_\lambda \cap \mathcal{A} : D_1(v_0) = \sup\left\{D_1(v) : v \in S_\lambda \cap \mathcal{A}\right\};$$

$$with\ r_0 = \left(q_1 p^{-1} D_1(v_0)\right)^{\frac{1}{p-q_1}}\ ,\ u_0 = r_0 v_0\ is\ solution\ of\ \text{Problem}\left(P^T\right).$$

When W_T is a vector lattice, if $H_\lambda(v) = H_\lambda(|v|)$ and $D_1(v) \le D_1(|v|)$ $\left[resp. D_1(v) = D_1(|v|)\right]$ $\forall v \in W_T$, then $r_0|v_0|$ $\left[resp.\, r_0|v_0|\, and - r_0|v_0|\right]$ is solution $\left[resp.\ are\ solutions\right]$ of Problem $\left(P^T\right)$. Consequently we can assume $v_0 \ge 0$ $i.e. u_0 \ge 0$.

When m>1, for any $v \in \mathcal{A}$ the equation $\Psi(r,v) = 0$ has only one positive root $r(v)$ and we have $\frac{\partial \Psi}{\partial r}\left(r(v), v\right) \ne 0.$

Evidently

functionals $r(v)\, and\, \tilde{\tilde{E}}(v) = \tilde{E}\left(r(v), v\right) = \left(r(v)\right)^p H_\lambda(v) - \sum_{j=1}^{m}\left(r(v)\right)^{q_j} D_j(v)$

belong to $C^1(\mathcal{A})$. (2.1)

Theorem 2.2 ([10],Section 2;[12],Section 2).Under assumptions $(i_{11})\, and\, (i_{12})$, in cases $(c_2)-(c_7)$ we have:

$$\exists v_0 \in S_\lambda \cap \mathcal{A} : \tilde{\tilde{E}}(v_0) = \inf\left\{\tilde{\tilde{E}}(v) : v \in S_\lambda \cap \mathcal{A}\right\};$$

$$with\ r_0 = r(v_0)\ ,\ u_0 = r_0 v_0\ is\ solution\ of\ \text{Problem}\left(P^T\right).$$

When W_T is a vector lattice, if $H_\lambda(v) = H_\lambda(|v|)$ and $D_j(v) = D_j(|v|)$ $\forall v \in W_T\, and\, \forall j \in \{1,...,m\}$, then $r_0|v_0|$ $and - r_0|v_0|$ are solutions of Problem $\left(P^T\right)$. Consequently we can assume $v_0 \ge 0$ $i.e. u_0 \ge 0$.

Proposition 2.1. In cases $(c_1)-(c_7)$ let u_0, v_0, r_0 be as in Theorems 2.1 and 2.2. If $\tilde{v} \in W_T$ is such that $< \partial H_\lambda(v_0), \tilde{v} > \ne 0$, then

$$\sum_{j=1}^{m}\left[< \partial D_j(u_0), u_0 > -p\left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1} < \partial D_j(u_0), r_0\tilde{v} >\right] = 0\ as\, m \ge 1. \tag{2.2}$$

Proof. Let us set $f(s,\tau) = H_\lambda(sv_0 + \tau\tilde{v})$ $\forall s > 0\, and\, \forall \tau \in \mathbb{R}$. We note that $f \in C^1\left(]0,+\infty[\times\mathbb{R}\right)$, $\frac{\partial f}{\partial s}(s,\tau) =$ $< \partial H_\lambda(sv_0 + \tau\tilde{v}), v_0 >$ and $\frac{\partial f}{\partial \tau}(s,\tau) = < \partial H_\lambda(sv_0 + \tau\tilde{v}), \tilde{v} >$ $\forall (s,\tau) \in]0,+\infty[\times\mathbb{R}$. Since $f(1,0) = 1$ and $\frac{\partial f}{\partial \tau}(1,0) =$ $< \partial H_\lambda(v_0), \tilde{v} > \ne 0$, there exist $\delta \in]0,1[$ and only one function $\tau(s) \in C^1\left(]1-\delta, 1+\delta[\right)$ such that $\tau(1) = 0$ and $f\left(s,\tau(s)\right) = 1$ $\forall s \in]1-\delta, 1+\delta[$; moreover $\tau'(1) = - < \partial H_\lambda(v_0), v_0 > \left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1} = -pH_\lambda(v_0)$ $\left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1} = -p\left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1}$.

Let $\delta_0 \in]0,\delta]$ such that $v(s) = sv_0 + \tau(s)\tilde{v} \in \mathcal{A}$ $\forall s \in]1-\delta_0, 1+\delta_0[$; then

$$v(s) \in S_\lambda \cap \mathcal{A}\ \ \forall s \in]1-\delta_0, 1+\delta_0[. \tag{2.3}$$

In (c_1) $(2.3) \Rightarrow D_1\left(v(s)\right) \le D_1(v_0) = D_1\left(v(1)\right)$ $\forall s \in]1-\delta_0, 1+\delta_0[$. Then $\left[\frac{d}{ds}D_1\left(v(s)\right)\right]_{s=1} = 0$, from which (2.2) since

$$\left[\frac{d}{ds}D_1\left(v(s)\right)\right]_{s=1} = < \partial D_1(v_0), v_0 > -p\left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1} < \partial D_1(v_0), \tilde{v} >=$$

$$r_0^{-q_1}\left[< \partial D_1\left(u_0\right), u_0 > -p\left(< \partial H_\lambda\left(v_0\right), \tilde{v} >\right)^{-1} < \partial D_1\left(u_0\right), r_0\tilde{v} >\right].$$

In $\left(c_2\right)-\left(c_7\right)$ $\left(2.1\right) and \left(2.3\right) \Rightarrow$ $\tilde{\tilde{E}}\left(v\left(s\right)\right) \in C^1\left(\left]1-\delta_0, 1+\delta_0\right[\right)$, $\left(2.3\right) \Rightarrow$ $\tilde{\tilde{E}}\left(v\left(1\right)\right) = \tilde{\tilde{E}}\left(v_0\right) \leq$ $\tilde{\tilde{E}}\left(v\left(s\right)\right)$ $\forall s \in \left]1-\delta_0, 1+\delta_0\right[$.

Consequently $\left[\frac{d}{ds}\tilde{\tilde{E}}\left(v\left(s\right)\right)\right]_{s=1} = 0$, i.e. (2.2). In fact, since

$$H_\lambda\left(v\left(s\right)\right) = 1 \; and \; p\left(r\left(v\left(s\right)\right)\right)^{p-1} - \sum_{j=1}^{m} q_j\left(r\left(v\left(s\right)\right)\right)^{q_j-1} D_j\left(v\left(s\right)\right) = 0 \; \forall s \in \left]1-\delta_0, 1+\delta_0\right[,$$

we have

$$\left[\frac{d}{ds}\tilde{\tilde{E}}\left(v\left(s\right)\right)\right]_{s=1} = \left[-\sum_{j=1}^{m} r\left(v\left(s\right)\right)^{q_j} < \partial D_j\left(v\left(s\right)\right), v_0 + \tau'\left(s\right)\tilde{v} >\right]_{s=1} =$$

$$-\sum_{j=1}^{m}\left[r_0^{q_j} < \partial D_j\left(v_0\right), v_0 > -p r_0^{q_j}\left(< \partial H_\lambda\left(v_0\right), \tilde{v} >\right)^{-1} < \partial D_j\left(v_0\right), \tilde{v} >\right] =$$

$$-\sum_{j=1}^{m}\left[< \partial D_j\left(u_0\right), u_0 > -p\left(< \partial H_\lambda\left(v_0\right), \tilde{v} >\right)^{-1} < \partial D_j\left(u_0\right), r_0\tilde{v} >\right]. \qquad \square$$

Let us pass to the cases in which $\left(i_{24}\right)$ is present:

$$\left(c_8\right) \;\; m = 1, q_1 \neq p, \left(i_{24}\right) holds \left(with \; m_1 = 1\right);$$

$$\left(c_9\right) \;\; m > 1, \left(i_{24}\right) holds, either \; p < q_1 \;\; or \; q_m < p, D_j \leq 0 \;\; as \; j \neq m_1.$$

In $\left(c_9\right)$ for any $v \in V_\lambda^- \cap S\left(D_{m_1}\right)$ the equation $\Psi\left(r, v\right) = 0$ has only one positive root $r\left(v\right)$ with $\frac{\partial \Psi}{\partial r}\left(r\left(v\right), v\right) \neq 0$. Let us set $\tilde{\tilde{E}}\left(v\right) = \tilde{E}\left(r\left(v\right), v\right) \; \forall v \in V_\lambda^- \cap S\left(D_{m_1}\right)$.

Theorem 2.3 ([10], Section 4).Let $\left(i_{11}\right) and \left(i_{12}\right)$ hold. In case $\left(c_8\right)$

$$\exists \underline{v} \in V_\lambda^- \cap S\left(D_1\right) : H_\lambda\left(\underline{v}\right) = \inf\left\{H_\lambda\left(v\right) : v \in V_\lambda^- \cap S\left(D_1\right)\right\};$$

$$with \;\; \underline{r} = \left(-p q_1^{-1} H_\lambda\left(\underline{v}\right)\right)^{\frac{1}{q_1-p}} \;\; , \underline{u} = \underline{r}\,\underline{v} \; is \;\; solution \;\; of \; \text{Problem}\left(P^T\right).$$

In case $\left(c_9\right)$

$$\exists \underline{v} \in V_\lambda^- \cap S\left(D_{m_1}\right) : \tilde{\tilde{E}}\left(\underline{v}\right) = \inf\left\{\tilde{\tilde{E}}\left(v\right) : v \in V_\lambda^- \cap S\left(D_{m_1}\right)\right\};$$

$$with \;\; \underline{r} = r\left(\underline{v}\right), \; \underline{u} = \underline{r}\,\underline{v} \; is \;\; solution \;\; of \; \text{Problem}\left(P^T\right).$$

When W_T is a vector lattice, if $H_\lambda\left(v\right) = H_\lambda\left(\left|v\right|\right)$ and $D_j\left(v\right) = D_j\left(\left|v\right|\right) \; \forall v \in W_T \; and \; \forall j \in \left\{1, \ldots, m\right\}$, then in $\left(c_8\right)$ and $\left(c_9\right)$ $\underline{r}\left|\underline{v}\right| \; and - \underline{r}\left|\underline{v}\right|$ are solutions of Problem $\left(P^T\right)$. Consequently we can assume $\underline{v} \geq 0 \quad i.e. \underline{u} \geq 0$.

Proposition 2.2. Let $\underline{u}, \underline{v}, \underline{r}$ be as in Theor. 2.3. In case $\left(c_8\right)$ if $\tilde{v} \in W_T$ is such that $< \partial D_1\left(\underline{v}\right), \tilde{v} > \neq 0$, then

$$< \partial H_\lambda\left(\underline{u}\right),\underline{u} > +q_1\left(< \partial D_1\left(\underline{v}\right),\tilde{v} >\right)^{-1} < \partial H_\lambda\left(\underline{u}\right),\underline{r}\tilde{v} >= 0. \tag{2.4}$$

In case $\left(c_9\right)$ if $\tilde{v}\in W_T$ is such that $< \partial D_{m_1}\left(\underline{v}\right),\tilde{v} >\neq 0$, then

$$\left[< \partial H_\lambda\left(\underline{u}\right),\underline{u} > +q_{m_1}\left(< \partial D_{m_1}\left(\underline{v}\right),\tilde{v} >\right)^{-1} < \partial H_\lambda\left(\underline{u}\right),\underline{r}\tilde{v} >\right]-$$

$$\sum_{j\neq m_1}\left[< \partial D_j\left(\underline{u}\right),\underline{u} > +q_{m_1}\left(< \partial D_{m_1}\left(\underline{v}\right),\tilde{v} >\right)^{-1} < \partial D_j\left(\underline{u}\right),\underline{r}\tilde{v} >\right]= 0\,. \tag{2.5}$$

Proof. As $j_0\in\left\{1,m_1\right\}$ let us consider the function of $C^1\left(\left]0,+\infty\right[\times\mathbb{R}\right)$ $f\left(s,\tau\right)=D_{j_0}\left(s\underline{v}+\tau\tilde{v}\right)$. Since $f\left(1,0\right)=D_{j_0}\left(\underline{v}\right)=-1$ and $\frac{\partial f}{\partial\tau}\left(1,0\right)=\ < \partial D_{j_0}\left(\underline{v}\right),\tilde{v} >\neq 0$, there exist $\delta\in\left]0,1\right[$ and only one function $\tau\left(s\right)\in C^1\left(\left]1-\delta,1+\delta\right[\right)$ such that $f\left(s,\tau\left(s\right)\right)=-1\ \forall s\in\left]1-\delta,1+\delta\right[$, $\tau\left(1\right)=0$ and we have

$$\tau'\left(1\right)=-< \partial D_{j_0}\left(\underline{v}\right),\underline{v} >\left(< \partial D_{j_0}\left(\underline{v}\right),\tilde{v} >\right)^{-1}=-q_{j_0}D_{j_0}\left(\underline{v}\right)\left(< \partial D_{j_0}\left(\underline{v}\right),\tilde{v} >\right)^{-1}=q_{j_0}\left(< \partial D_{j_0}\left(\underline{v}\right),\tilde{v} >\right)^{-1}.$$

In $\left(c_8\right)$, where $j_0=1$, with $\delta_0\in\left]0,\delta\right]$ such that $v\left(s\right)=s\underline{v}+\tau\left(s\right)\tilde{v}\in V_\lambda^-\ \ \forall s\in\left]1-\delta_0,1+\delta_0\right[$ we have

$$H_\lambda\left(v\left(s\right)\right)\geq H_\lambda\left(\underline{v}\right)=H_\lambda\left(v\left(1\right)\right)\ \forall s\in\left]1-\delta_0,1+\delta_0\right[\,;$$

then

$$0=\left[\frac{d}{ds}H_\lambda\left(v\left(s\right)\right)\right]_{s=1}=< \partial H_\lambda\left(\underline{v}\right),\underline{v} > +q_1\left(< \partial D_1\left(\underline{v}\right),\tilde{v} >\right)^{-1} < \partial H_\lambda\left(\underline{v}\right),\tilde{v} >=$$

$$\underline{r}^{-p}\left[< \partial H_\lambda\left(\underline{u}\right),\underline{u} > +q_1\left(< \partial D_1\left(\underline{v}\right),\tilde{v} >\right)^{-1} < \partial H_\lambda\left(\underline{u}\right),\underline{r}\tilde{v} >\right]$$

from which (2.4) follows.

In $\left(c_9\right)$, where $j_0=m_1$, we note that relations $\Psi\left(\underline{r},\underline{v}\right)=0\,and\,\frac{\partial\Psi}{\partial r}\left(\underline{r},\underline{v}\right)\neq 0$ imply that there exist an open ball B^* centered in $\underline{v}$ included in V_λ^- and only one functional $r^*\left(v\right)\in C^1\left(B^*\right)$ such that $\Psi\left(r^*\left(v\right),v\right)=0$ for any $v\in B^*$. Since $r^*\left(v\right)=r\left(v\right)\forall v\in B^*\cap S\left(D_{m_1}\right)$, set $\delta_0\in\left]0,\delta\right]$ such that $v\left(s\right)=s\underline{v}+\tau\left(s\right)\tilde{v}\in B^*\ \ \forall s\in\left]1-\delta_0,1+\delta_0\right[$, we have

$$r^*\left(v\left(s\right)\right)=r\left(v\left(s\right)\right)\ \forall s\in\left]1-\delta_0,1+\delta_0\right[\,.$$

Then functional

$$\tilde{\tilde{E}}\left(v\left(s\right)\right)=\left(r\left(v\left(s\right)\right)\right)^p H_\lambda\left(v\left(s\right)\right)-\sum_{j=1}^{m}\left(r\left(v\left(s\right)\right)\right)^{q_j}D_j\left(v\left(s\right)\right)$$

belongs to $C^1\left(\left]1-\delta_0,1+\delta_0\right[\right)$.

Taking into account that

$$D_{m_1}\left(v\left(s\right)\right)=-1\ and\ p\left(r\left(v\left(s\right)\right)\right)^{p-1}H_\lambda\left(v\left(s\right)\right)-\sum_{j=1}^{m}q_j\left(r\left(v\left(s\right)\right)\right)^{q_j-1}D_j\left(v\left(s\right)\right)=0\ \ \forall s\in\left]1-\delta_0,1+\delta_0\right[,$$

we have

$$\left[\frac{d}{ds}\tilde{\tilde{E}}\left(v\left(s\right)\right)\right]_{s=1} = \underline{r}^{p}\left[<\partial H_{\lambda}\left(\underline{v}\right),\underline{v}> +q_{m_1}\left(<\partial D_{m_1}\left(\underline{v}\right),\tilde{v}>\right)^{-1}<\partial H_{\lambda}\left(\underline{v}\right),\tilde{v}>\right]-$$

$$\sum_{j\neq m_1}\underline{r}^{q_j}\left[<\partial D_j\left(\underline{v}\right),\underline{v}> +q_{m_1}\left(<\partial D_{m_1}\left(\underline{v}\right),\tilde{v}>\right)^{-1}<\partial D_j\left(\underline{v}\right),\tilde{v}>\right]=$$

$$\left[<\partial H_{\lambda}\left(\underline{u}\right),\underline{u}> +q_{m_1}\left(<\partial D_{m_1}\left(\underline{v}\right),\tilde{v}>\right)^{-1}<\partial H_{\lambda}\left(\underline{u}\right),\underline{r}\tilde{v}>\right]-$$

$$\sum_{j\neq m_1}\left[<\partial D_j\left(\underline{u}\right),\underline{u}> +q_{m_1}\left(<\partial D_{m_1}\left(\underline{v}\right),\tilde{v}>\right)^{-1}<\partial D_j\left(\underline{u}\right),\underline{r}\tilde{v}>\right]$$

from which (2.5) since $\tilde{\tilde{E}}\left(v\left(s\right)\right)\geq\tilde{\tilde{E}}\left(\underline{v}\right)=\tilde{\tilde{E}}\left(v\left(1\right)\right)\ \forall s\in\left]1-\delta_0,1+\delta_0\right[$. □

3. Some Applications

In this section we suppose $N\geq 2$ and set

$\left|.\right|_1$ the Lebesgue measure on $\mathbb{R}$;

$\nu=\left(\nu_1,...,\nu_N\right)=$ the outward orthogonal unitary vector to $\partial\Omega$;

$\mathrm{P}_T\left(\mathbb{R}\right)=$ the class of the real functions defined a.e.in $\mathbb{R}$, measurable and T-periodic;

$\forall\varphi\in\mathrm{P}_T\left(\mathbb{R}\right)\cap C^0\left(\mathbb{R}\right)\ \operatorname*{supp}_{[0,T]}\varphi=$ the support of the restriction of $\varphi\ to\left[0,T\right]$.

We warn that the weak continuity in W_T of the functionals $B\ and\ D_j$ present in the applications can be easily proved by using embedding Sobolev theorems [1],a compactness lemma ([6], Theor.5.1 page 58) and the isomorphism F (section 1).

We add the following clarification:

$$\left(c^*\right)\begin{cases}With\ t_0\in\left]0,T\right[\ and\ \varepsilon_0>0\left[resp.with\ \varepsilon_0>0\right]we\ \text{suppose } 0<t_0-\varepsilon_0\ and\ t_0+\varepsilon_0<T\left[resp.\varepsilon_0<T-\varepsilon_0\right].\\ Additionally\ as\ 0<\varepsilon<\varepsilon_0/2\ we\ denote\ by\ \omega_\varepsilon\ \text{a nonnegative function belonging to } \mathrm{P}_T\left(\mathbb{R}\right)\cap\mathrm{C}^\infty\left(\mathbb{R}\right)\\ \text{such that} \operatorname*{supp}_{[0,T]}\ \omega_\varepsilon\subseteq\left]t_0-2\varepsilon,t_0+2\varepsilon\right[\ \text{ and } \omega_\varepsilon=1\ in\left[t_0-\varepsilon,t_0+\varepsilon\right]\\ \left[resp.\operatorname*{supp}_{[0,T]}\ \omega_\varepsilon\subseteq\left[0,2\varepsilon\right[\cup\left]T-2\varepsilon,T\right]and\ \omega_\varepsilon=1\ in\left[0,\varepsilon\right]\cup\left[T-\varepsilon,T\right]\right].\end{cases}$$

Application 3.1 (connected to Theor.2.1). Let us assume in the definition (1.1) of $W_T\ \ n=1\ and\ V=W_0^{1,p_1}\left(\Omega\right)$, then

$$\left\|v\right\|=\left(\int_Q\left|\frac{\partial v}{\partial t}\right|^{p_2}dxdt\right)^{1/p_2}+\left(\int_Q\left|\nabla v\right|^{p_1}dxdt\right)^{1/p_1}\ \forall v\in W_T,$$

and let us set as any $v\in W_T$

$$A\left(v\right)=p^{-1}\left[\left(\int_Q\left|\frac{\partial v}{\partial t}\right|^{p_2}dxdt\right)^{\frac{p}{p_2}}+\left(\int_Q a\left(x,t\right)\left|\nabla v\right|^{p_1}dxdt\right)^{\frac{p}{p_1}}\right],$$

$$B\left(v\right)=p^{-1}\left(\int_Q b\left(x,t\right)\left|v\right|^{p_1}dxdt\right)^{\frac{p}{p_1}},\qquad D_1\left(v\right)=q_1^{-1}\left|\int_Q d_1\left(x,t\right)vdxdt\right|^{q_1},$$

where

$$1 < p_1 \le p,\ 1 < p_2 \le p, 1 < q_1, q_1 \ne p; a, b \in \left(\mathbb{P}_T\left(\Omega \times \mathbb{R}\right) \cap L^\infty\left(Q\right)\right) \setminus \left\{0\right\},\ a\left(x,t\right) \ge a_0\ and\ \ 0 \le b\left(x,t\right) \le b_0$$

$$a.e.\ in\ Q\left(a_0, b_0 = const. > 0\right),\ d_1 \in \left(\mathbb{P}_T\left(\Omega \times \mathbb{R}\right) \cap L^{p_1'}\left(Q\right)\right) \setminus \left\{0\right\}\ \left(p_1' = p_1 / \left(p_1 - 1\right)\right). \tag{3.1}$$

Problem $\left(P^T\right)$ becomes:

Find $u \in W_T \setminus \left\{0\right\}$ such that

$$\left(\int_Q \left|\frac{\partial u}{\partial t}\right|^{p_2} dxdt\right)^{\frac{p}{p_2}-1} \int_Q \left|\frac{\partial u}{\partial t}\right|^{p_2-2} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} dxdt + \left(\int_Q a\left(x,t\right)\left|\nabla u\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1} \int_Q a\left(x,t\right)\left|\nabla u\right|^{p_1-2} \nabla u \nabla v\, dxdt =$$

$$\lambda\left(\int_Q b\left(x,t\right)\left|u\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1} \int_Q b\left(x,t\right)\left|u\right|^{p_1-2} uv\, dxdt + \left|\int_Q d_1\left(x,t\right) u dxdt\right|^{q_1-2} \left(\int_Q d_1\left(x,t\right) u dxdt\right) \int_Q d_1\left(x,t\right) v\, dxdt \quad \forall v \in W_T. \tag{3.2}$$

Each solution u of (3.2) is for definition a weak solution of the problem:

$$-\left(\int_Q \left|\frac{\partial u}{\partial t}\right|^{p_2} dxdt\right)^{\frac{p}{p_2}-1} \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p_2-2} \frac{\partial u}{\partial t}\right) - \left(\int_Q a\left(x,t\right)\left|\nabla u\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1} div\left(a\left(x,t\right)\left|\nabla u\right|^{p_1-2} \nabla u\right) =$$

$$\lambda\left(\int_Q b\left(x,t\right)\left|u\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1} b\left(x,t\right)\left|u\right|^{p_1-2} u + \left|\int_Q d_1\left(x,t\right) u dxdt\right|^{q_1-2} \left(\int_Q d_1\left(x,t\right) u dxdt\right) d_1\left(x,t\right)\ \ in\ Q,$$

$$u = 0\ \ on\ \ \Sigma,\ u\left(x,0\right) = u\left(x,T\right)\ \ and\ \frac{\partial u}{\partial t}\left(x,0\right) = \frac{\partial u}{\partial t}\left(x,T\right)\ \ on\ \ \Omega. \tag{3.3}$$

Evidently

$$V^+\left(D_1\right) \ne \varnothing\ . \tag{3.4}$$

Let $\lambda^*\ and\ z^*$ be the first eigenvalue and the first eigenfunction of the problem:

$$z \in W_0^{1,p_1}\left(\Omega\right):\ \ -a_0 div\left(\left|\nabla z\right|^{p_1-2} \nabla z\right) = \theta b_0 \left|z\right|^{p_1-2} z \quad in\ \Omega.$$

We remember that [3] $z^* > 0\ in\ \Omega$ and

$$\lambda^* = \left(a_0 \int_\Omega \left|\nabla z^*\right|^{p_1} dx\right)\left(b_0 \int_\Omega \left(z^*\right)^{p_1} dx\right)^{-1} \le \left(a_0 \int_\Omega \left|\nabla z\right|^{p_1} dx\right)\left(b_0 \int_\Omega \left|z\right|^{p_1} dx\right)^{-1} \forall z \in W_0^{1,p_1}\left(\Omega\right) \setminus \left\{0\right\}. \tag{3.5}$$

(3.5) implies that

$$b_0 \int_\Omega \left|v\left(x,t\right)\right|^{p_1} dx \le \left(\lambda^*\right)^{-1} a_0 \int_\Omega \left|\nabla v\left(x,t\right)\right|^{p_1} dx\ \ a.e.\ in\ \left[0,T\right]\ \forall\, v \in W_T$$

;

then as $0 < \lambda < \left(\lambda^*\right)^{\frac{p}{p_1}}$

$$\left(a_0\int_Q \left|\nabla v\right|^{p_1} dxdt\right)^{\frac{p}{p_1}} - \lambda\left(b_0\int_Q \left|v\right|^{p_1} dxdt\right)^{\frac{p}{p_1}} \geq \left(1-\lambda\left(\lambda^*\right)^{-\frac{p}{p_1}}\right)\left(a_0\int_Q \left|\nabla v\right|^{p_1} dxdt\right)^{\frac{p}{p_1}} \quad \forall\, v\in W_T \tag{3.6}$$

from which

$$H_\lambda\left(v\right) = A\left(v\right) - \lambda B\left(v\right) \geq p^{-1}\min\left\{1, a_0^{\frac{p}{p_1}}\right\}\left(1-\lambda\left(\lambda^*\right)^{-\frac{p}{p_1}}\right)2^{-p}\left\|v\right\|^p \quad \forall\, v\in W_T.$$

Then, since

$$as\ \lambda\leq 0 \quad \left\|v\right\|^p \leq p2^p H_\lambda\left(v\right) \quad \forall\, v\in W_T,$$

we have

$$\left(i_{21}\right),\ in\ particular\ \left(i_{22}\right), holds\ \ if\ \ \lambda\in\left]-\infty,\left(\lambda^*\right)^{\frac{p}{p_1}}\right[. \tag{3.7}$$

Relations (3.4),(3.7) and Theor.2.1 let us to state the following proposition.

Proposition 3.1. Under conditions (3.1), with λ as in (3.7) problem (3.3) has at least two weak solutions u_0 and $-u_0$ ($u_0 = r_0 v_0$, $r_0 = const. > 0$, $v_0 \in S_\lambda \cap V^+\left(D_1\right)$).

Remark 3.1. If $d_1 \geq 0$, we have $D_1\left(v\right) \leq D_1\left(\left|v\right|\right) \forall v\in W_T$. Then, since $H_\lambda\left(v\right) = H_\lambda\left(\left|v\right|\right) \forall v\in W_T$, it results in $u_0 \geq 0$. Additionally, when $a\left(x,t\right)\equiv a_0\ and\, b\left(x,t\right)\equiv b_0\ \ in\, Q$, since $z^*\in V^+\left(D_1\right) and\ \ H_\lambda\left(z^*\right)\leq 0\ \ as\,\lambda\geq\left(\lambda^*\right)^{\frac{p}{p_1}}$, $\left(i_{22}\right)$ holds if and only if $\lambda\in\left]-\infty,\left(\lambda^*\right)^{\frac{p}{p_1}}\right[$.

Proposition 3.2. Let $p_2 \leq Np_1/\left(N-p_1\right)$ if $N > p_1$. If there exist a measurable set $I\subseteq\left[0,T\right]$ with $\left|\,I\,\right|_1 > 0$, a limit point $t_0\ of\ I\ \ and\, g\in L^{p_0}\left(\Omega\right)\ \left(p_0 = \min\left\{p_1', p_2'\right\}, p_i' = p_i/\left(p_i-1\right)\right)$ such that $\lim\limits_{\substack{t\to t_0\\ t\in I}} d_1\left(x,t\right) = 0$ a.e. in Ω and $\left|d_1\left(x,t\right)\right| \leq g\left(x\right)$ a.e. in $\Omega\times I$, then u_0 is nonstationary.

Proof. The additional assumption on p_2 implies that

$$W_0^{1,p_1}\left(\Omega\right)\subseteq L^{p_2}\left(\Omega\right)\ \ \text{with continuous embedding.} \tag{3.8}$$

Reasoning by contradiction, let $\dfrac{\partial u_0}{\partial t}\equiv 0\, in\, Q$. Set $\omega\left(t\right) = \int_\Omega d_1\left(x,t\right)u_0 dx$ a.e. in$\left[0,T\right]$, a Lebesgue theorem assures that

$$\lim_{\substack{t\to t_0\\ t\in I}} \omega\left(t\right) = 0\ . \tag{3.9}$$

Let $\varphi\in \mathrm{P}_T\left(\mathbb{R}\right)\cap C^\infty\left(\mathbb{R}\right) with\, \varphi\geq 0\, and \operatorname*{supp}_{[0,T]}\varphi\subset\left]0,T\right[$. Since by (3.8) $\varphi u_0\in W_T$, from (3.2) with $u = u_0$ and $v = \varphi u_0$ we get

$$\left(\int_Q a\left(x,t\right)\left|\nabla u_0\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1}\int_Q a\left(x,t\right)\left|\nabla u_0\right|^{p_1}\varphi dxdt - \lambda\left(\int_Q b\left(x,t\right)\left|u_0\right|^{p_1} dxdt\right)^{\frac{p}{p_1}-1}\int_Q b\left(x,t\right)\left|u_0\right|^{p_1}\varphi dxdt =$$

$$\left|\int_Q d_1(x,t)\,u_0 dxdt\right|^{q_1-2}\left(\int_Q d_1(x,t)\,u_0 dxdt\right)\left(\int_Q d_1(x,t)\,u_0\varphi dxdt\right). \tag{3.10}$$

Let us add that

$$\text{the left side of (3.10) is } \geq (T)^{-1}\delta\int_0^T \varphi dt \tag{3.11}$$

where

$$\delta=\left(\int_Q a_0\left|\nabla u_0\right|^{p_1} dxdt\right)^{\frac{p}{p_1}}>0 \quad if\, \lambda\leq 0,$$

$$\delta=\left(\int_Q a_0\left|\nabla u_0\right|^{p_1} dxdt\right)^{\frac{p}{p_1}}-\lambda\left(\int_Q b_0\left|u_0\right|^{p_1} dxdt\right)^{\frac{p}{p_1}}>0\,(\text{from } (3.6)) \text{ if } 0<\lambda<(\lambda^*)^{\frac{p}{p_1}}.$$

Since $u_0\in V^+(D_1)\Rightarrow\int_0^T \omega dt\neq 0$, from (3.10),(3.11) we get

$$\omega(t)\geq const.>0 \; if\int_0^T \omega dt>0,\; \omega(t)\leq const.<0 \; if\int_0^T \omega dt<0 \quad a.e.\, in\,[0,T]. \tag{3.12}$$

Then (3.12) contradicts (3.9). □

Application 3.2 (connected to Theor.2.1 and Theor.2.3 (case (c_8))). Let us assume in the definition (1.1) of W_T $p_1=p_2=p, n=1$ $and\, V=W^{1,p}(\Omega)$, then

$$\|v\|=\left(\int_Q\left|\frac{\partial v}{\partial t}\right|^p dxdt+\int_Q |v|^p dxdt+\int_Q |\nabla v|^p dxdt\right)^{1/p} \quad \forall v\in W_T,$$

and let us set as any $v\in W_T$

$$A(v)=p^{-1}\left[\int_Q\left|\frac{\partial v}{\partial t}\right|^p dxdt+\int_Q a(x,t)\left|\nabla v\right|^p dxdt\right],$$

$$B(v)=p^{-1}\int_Q b(x,t)|v|^p dxdt, \qquad D_1(v)=q_1^{-1}\int_Q d_1(x,t)|v|^{q_1} dxdt,$$

where

$$1<q_1<p;\; a,b,d_1\in\left(\mathrm{P}_T(\Omega x\mathbb{R})\cap L^\infty(Q)\right)\setminus\{0\},\; a(x,t)\geq a_0 \; and \;\; 0<b(x,t)\leq b_0$$
$$a.e.\, in\, Q\,(a_0,b_0=const.>0). \tag{3.13}$$

Problem (P^T) becomes:

Find $u\in W_T\setminus\{0\}$ such that

$$\int_Q \left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} dxdt + \int_Q a(x,t)|\nabla u|^{p-2} \nabla u \nabla v\, dxdt = \lambda \int_Q b(x,t)|u|^{p-2} u\, v\, dxdt + \int_Q d_1(x,t)|u|^{q_1-2} u\, v\, dxdt \quad \forall v \in W_T. \tag{3.14}$$

Each solution u of (3.14) is for definition a weak solution of the problem:

$$-\frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\right) - div\left(a(x,t)|\nabla u|^{p-2} \nabla u\right) = \lambda b(x,t)|u|^{p-2} u + d_1(x,t)|u|^{q_1-2} u \;\; in\;\; Q,$$

$$a(x,t)|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0\, on\, \Sigma,\; u(x,0) = u(x,T) \;and\; \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T)\, on\, \Omega. \tag{3.15}$$

Let us introduce the conditions

$$d_1^+ = \max\{d_1, 0\} \not\equiv 0 \text{ in Q}, \tag{3.16}$$

$$\int_Q d_1(x,t)\, dxdt < 0. \tag{3.17}$$

Evidently

$$(3.16) \Rightarrow V^+(D_1) \neq \varnothing \;\; ,(3.17) \Rightarrow V_\lambda^- \cap S(D_1) \neq \varnothing \;\forall \lambda > 0.$$

Proposition 3.3. Under conditions (3.13) (with p>1 and not necessarily >q_1),(i_{21}) holds if λ<0.

Proof. Let λ<0. Reasoning by contradiction, as any $k \in \mathbb{N}$ there exists $v_k \in W_T$ such that

$$H_\lambda(v_k) < k^{-1}\|v_k\|^p.$$

Then with $w_k = \|v_k\|^{-1} v_k$ we have

$$\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt + \int_Q a(x,t)|\nabla w_k|^p dxdt - \lambda \int_Q b(x,t)|w_k|^p dxdt \le p\, k^{-1};$$

moreover there exists $w \in W_T$ such that (within a subsequence) $w_k \to w \; weakly\; in\; W_T$.

Consequently

$$\int_Q \left|\frac{\partial w}{\partial t}\right|^p dxdt = \lim_{k\to+\infty} \int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt = 0, \int_Q a(x,t)|\nabla w|^p dxdt = \lim_{k\to+\infty} \int_Q a(x,t)|\nabla w_k|^p dxdt = 0,$$

$$\int_Q b(x,t)|w|^p dxdt = \lim_{k\to+\infty} \int_Q b(x,t)|w_k|^p dxdt = 0,$$

from which $w_k \to 0 \; strongly \;\; in\, W_T$ and the contradiction $1 = \lim_{k\to+\infty}\|w_k\| = 0.$ □

Proposition 3.4. Under conditions (3.13), (3.16) and (3.17), there exists $\delta_1^* > 0$ satisfying the condition

$$\forall \lambda \in [0, \delta_1^*]\; \exists c(\lambda) > 0 : \int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + a_0 \int_Q |\nabla v|^p dxdt - \lambda b_0 \int_Q |v|^p dxdt \ge c(\lambda)\|v\|^p \;\; \forall v \in V^+(D_1).$$

Consequently (i_{22}) holds if $\lambda \in [0, \delta_1^*]$.

Proof. Reasoning by contradiction, as any $k \in \mathbb{N}\; there\, exist\, v_k \in V^+(D_1)\, and\, \lambda_k \in [0, k^{-1}]$ such that

$$\int_Q \left|\frac{\partial v_k}{\partial t}\right|^p dxdt + a_0 \int_Q \left|\nabla v_k\right|^p dxdt < \lambda_k b_0 \int_Q \left|v_k\right|^p dxdt + k^{-1}\left\|v_k\right\|^p .$$

Then with $w_k = \left\|v_k\right\|^{-1} v_k$ *there exists* $w \in W_T$ such that (within a subsequence)

$$w_k \to w \text{ weakly in } W_T \ ,$$

$$\int_Q \left|\frac{\partial w}{\partial t}\right|^p dxdt = \lim_{k\to+\infty}\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt = 0, \int_Q \left|\nabla w\right|^p dxdt = \lim_{k\to+\infty}\int_Q \left|\nabla w_k\right|^p dxdt = 0,$$

$$\int_Q d_1(x,t)\left|w\right|^{q_1} dxdt = \lim_{k\to+\infty}\int_Q d_1(x,t)\left|w_k\right|^{q_1} dxdt \geq 0,$$

from which since (3.17) we get $w \equiv 0 \text{ in } Q$. Then $w_k \to 0$ *strongly in* W_T and the contradiction $1 = \lim_{k\to+\infty}\left\|w_k\right\| = 0.$ □

Proposition 3.5. Under conditions (3.13) and (3.17), there exists $\delta_2^* > 0$ such that (i_{24}) holds if $\lambda \in \left]0, \delta_2^*\right]$.

Proof. Reasoning by contradiction, as any $k \in \mathbb{N}$ *there exist* $\lambda_k \in \left]0, k^{-1}\right]$ *and* $\left(v_{k,h}\right)_{h\in\mathbb{N}} \subseteq W_T$ such that

$$D_1\left(v_{k,h}\right) = -1, \tag{3.18}$$

$$H_{\lambda_k}\left(v_{k,h}\right) < 0, \tag{3.19}$$

$$\lim_{h\to+\infty}\left\|v_{k,h}\right\|^p = +\infty. \tag{3.20}$$

Relation (3.20) implies there exists $\left(h_k\right)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ strictly increasing such that

$$\lim_{k\to+\infty}\left\|v_{k,h_k}\right\|^p = +\infty .$$

Set $w_k = \left\|v_{k,h_k}\right\|^{-1} v_{k,h_k}$, from (3.18), (3.19) we get

$$\int_Q d_1(x,t)\left|w_k\right|^{q_1} dxdt = -q_1\left\|v_{k,h_k}\right\|^{-q_1}, \tag{3.21}$$

$$\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt + \int_Q a(x,t)\left|\nabla w_k\right|^p dxdt < \lambda_k \int_Q b(x,t)\left|w_k\right|^p dxdt. \tag{3.22}$$

Let $w \in W_T$ such that (within a subsequence) $w_k \to w \text{ weakly in } W_T$.

From (3.21), (3.22) we get

$$\int_Q d_1(x,t)\left|w\right|^{q_1} dxdt = \lim_{k\to+\infty}\int_Q d_1(x,t)\left|w_k\right|^{q_1} dxdt = 0, \int_Q \left|\frac{\partial w}{\partial t}\right|^p dxdt = \lim_{k\to+\infty}\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt = 0,$$

$$\int_Q a(x,t)\left|\nabla w\right|^p dxdt = \lim_{k\to+\infty}\int_Q a(x,t)\left|\nabla w_k\right|^p dxdt = 0,$$

from which since (3.17) $w_k \to 0$ *strongly in* W_T and the contradiction $1 = \lim_{k\to+\infty}\left\|w_k\right\| = 0.$ □

Propositions 3.3-3.5, Theorems 2.1 and 2.3 (case c_8) allow the following proposition.

Proposition 3.6. Under assumptions (3.13) we have:

when (3.16) and (3.17) hold, with $\lambda \in \left]-\infty, \delta_1^*\right]$ problem (3.15) has at least two weak solutions $u_0 \geq 0$ and $-u_0$ ($u_0 = r_0 v_0$, $r_0 = const. > 0$, $v_0 \in S_\lambda \cap V^+\left(D_1\right)$);

when (3.17) holds, with $\lambda \in \left]0, \delta_2^*\right]$ problem (3.15) has at least two weak solutions $\underline{u} \geq 0$ and $-\underline{u}$ ($\underline{u} = \underline{r}\,\underline{v}$, $\underline{r} = const. > 0, \underline{v} \in V_\lambda^- \cap S\left(D_1\right)$).

Consequently, when (3.16) and (3.17) hold, with $\lambda \in \left]0, \min\left\{\delta_1^*, \delta_2^*\right\}\right]$ problem (3.15) has at least four different weak solutions.

*Proposition 3.7.*Let (3.13), (3.16) and (3.17) hold. Let $\lambda \in \left]-\infty, \delta_1^*\right]$. If there exists a measurable set $I \subseteq \left[0, T\right]$ with $\left| I \right|_1 > 0$ such that $d_1\left(x, t\right) \leq 0$ a.e. in $\Omega \times I$, then u_0 is nonstationary.

Proof. Reasoning by contradiction let $\dfrac{\partial u_0}{\partial t} \equiv 0 \, in \, Q$. With $\omega\left(t\right) = \int\limits_\Omega d_1\left(x, t\right)\left(u_0\right)^{q_1} dx$ a.e. in$\left[0, T\right]$ we have

$$\omega\left(t\right) \leq 0 \;\; a.e. \, in \, I\,. \tag{3.23}$$

Let $\varphi \in \mathbb{P}_T\left(\mathbb{R}\right) \cap C^\infty\left(\mathbb{R}\right) with\, \varphi \geq 0\, and\; \operatorname*{supp}_{[0,T]} \varphi \subset \left]0, T\right[$. Setting in (3.14) $u = u_0 \, and\, v = \varphi u_0 \in W_T$, we have

$$\int\limits_Q a\left(x, t\right)\left|\nabla u_0\right|^p \varphi \, dxdt - \lambda \int\limits_Q b\left(x, t\right)\left(u_0\right)^p \varphi \, dxdt = \int\limits_0^T \omega\varphi dt$$

from which

$$\left(T\right)^{-1} \delta \int\limits_0^T \varphi dt \leq \int\limits_0^T \omega\varphi dt,$$

where

$$\delta = a_0 \int\limits_Q \left|\nabla u_0\right|^p dxdt > 0 \; \left(\text{since } \mathrm{u}_0 \in V^+\left(D_1\right)\right) if \; \lambda < 0,$$

$$\delta = a_0 \int\limits_Q \left|\nabla u_0\right|^p dxdt - \lambda b_0 \int\limits_Q \left(u_0\right)^p dxdt > 0 \,(\text{since Prop.3.4}) \text{ if } \lambda \in \left[0, \delta_1^*\right].$$

Then $\omega\left(t\right) \geq \left(T\right)^{-1} \delta$ a.e.in$\left[0, T\right]$, and this contradicts (3.23). □

Proposition 3.8. Let (3.13) and (3.17) hold. Let $\lambda \in \left]0, \delta_2^*\right]$. Let one of the following conditions holds:

There exists a measurable set $I \subseteq \left[0, T\right]$ with $\left|I\right|_1 > 0$ such that $d_1\left(x, t\right) \geq 0$ a.e. in $\Omega \times I$; (3.24)

There exist a measurable set $I \subseteq \left[0, T\right]$ with $\left|I\right|_1 > 0$, $c_0 > 0$ and a limit point t_0 of I such that $b\left(x, t\right) \geq c_0$ a.e. in $\Omega \times I$, $\lim\limits_{\substack{t \to t_0 \\ t \in I}} d_1\left(x, t\right) = 0 \;\; a.e. \, in \, \Omega$; (3.25)

There exist $t_0 \in \left]0, T\right[\; and\, \varepsilon_0 > 0$ [resp.there exists $\varepsilon_0 > 0$] as in $\left(c^*\right)$, $\eta > 0, h > 1$ and $\theta \in \mathbb{P}_T\left(\mathbb{R}\right) \cap W^{1,p}\left(\left]0, T\right[\right)$, with $\theta\left(t\right) > 0 \quad \forall t \in \left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right] \setminus \left\{t_0\right\} \quad \left[resp. \forall t \in \left]0, \varepsilon_0\right] \cup \left[T - \varepsilon_0, T\right[\right]$ and $\theta\left(t_0\right) = 0 \left[resp. \theta\left(0\right) = 0\right]$, such that

$b(x,t) \geq \theta(t)$ and $-\eta(\theta(t))^h \leq d_1(x,t) < 0$ $a.e.\ in\, \Omega \times [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ $[resp.\, a.e.\, in\, \Omega \times ([0,\varepsilon_0] \cup [T-\varepsilon_0, T])]$. (3.26)

Then $\underline{u}$ is nonstationary.

Proof. Under condition (3.24) [resp. (3.25)], reasoning by contradiction let $\dfrac{\partial \underline{u}}{\partial t} \equiv 0\, in\, Q$.

Let $\varphi \in \mathbb{P}_T(\mathbb{R}) \cap C^\infty(\mathbb{R})\, with\ \operatorname{supp}_{[0,T]} \varphi \subset]0,T[$. Relation (3.14) with $u = \underline{u}\ and\, v = \varphi \in W_T$ becomes

$$\int_0^T \left(\int_\Omega d_1(x,t)(\underline{u})^{q_1-1}\, dx \right) \varphi dt = -\lambda \int_0^T \left(\int_\Omega b(x,t)(\underline{u})^{p-1}\, dx \right) \varphi dt$$

from which

$$\int_\Omega d_1(x,t)(\underline{u})^{q_1-1}\, dx = -\lambda \int_\Omega b(x,t)(\underline{u})^{p-1}\, dx\ \ a.e.\, in [0,T]$$

and then the contradiction

$$0 \leq \int_\Omega d_1(x,t)(\underline{u})^{q_1-1}\, dx < 0\ \ a.e.\, in\, I$$

$$\left[resp. \int_\Omega d_1(x,t)(\underline{u})^{q_1-1}\, dx \leq -\lambda c_0 \int_\Omega (\underline{u})^{p-1}\, dx\ \ a.e.\, in\, I\ \ and \lim_{\substack{t \to t_0 \\ t \in I}} \int_\Omega d_1(x,t)(\underline{v})^{q_1-1}\, dx = 0 \right].$$

Under condition (3.26), according to Prop.2.2 it is sufficiently to prove that

If $\dfrac{\partial \underline{u}}{\partial t} \equiv 0\, in\, Q$, then there exists $\tilde{v} \in W_T$ such that $< \partial D_1(\underline{v}), \tilde{v} > \neq 0$ and

$$< \partial H_\lambda(\underline{u}), \underline{u} > + q_1 \left(< \partial D_1(\underline{v}), \tilde{v} > \right)^{-1} < \partial H_\lambda(\underline{u}), \underline{r}\tilde{v} > \neq 0. \tag{3.27}$$

Let ω_ε be as in (c^*). Since $\theta\omega_\varepsilon \in W_T$, we have

$$< \partial D_1(\underline{v}), \theta\omega_\varepsilon > = \int_Q d_1(x,t)(\underline{v})^{q_1-1} \theta\omega_\varepsilon\, dxdt < 0\ ,$$

$$< \partial D_1(\underline{v}), \theta\omega_\varepsilon > \geq -\eta \int_Q (\underline{v})^{q_1-1} \theta^{h+1}\omega_\varepsilon\, dxdt = -\eta \left(\int_\Omega (\underline{v})^{q_1-1}\, dx \right) \int_0^T \theta^{h+1}\omega_\varepsilon dt.$$

Then

$$< \partial H_\lambda(\underline{u}), \underline{u} > + q_1 \left(< \partial D_1(\underline{v}), \theta\omega_\varepsilon > \right)^{-1} < \partial H_\lambda(\underline{u}), \underline{r}\theta\omega_\varepsilon > =$$

$$pH_\lambda(\underline{u}) - \lambda \underline{r} q_1 \left(< \partial D_1(\underline{v}), \theta\omega_\varepsilon > \right)^{-1} \left(\int_Q b(x,t)(\underline{u})^{p-1} \theta\omega_\varepsilon\, dxdt \right) \geq$$

$$pH_\lambda(\underline{u}) + \lambda \underline{r} q_1 \left[\eta \left(\int_0^T \theta^{h+1}\omega_\varepsilon dt \right) \int_\Omega (\underline{v})^{q_1-1}\, dx \right]^{-1} \left(\int_0^T \theta^2 \omega_\varepsilon dt \right) \int_\Omega (\underline{u})^{p-1}\, dx\,. \tag{3.28}$$

Let us add

$$\left(\int_0^T \theta^{h+1}\omega_\varepsilon dt \right)^{-1} \int_0^T \theta^2\omega_\varepsilon dt = \left(\theta^{h-1}(t_\varepsilon) \right)^{-1} \text{where } t_\varepsilon \in [t_0 - 2\varepsilon, t_0 + 2\varepsilon]$$

$$\left[resp. \left(\int_0^T \theta^{h+1}\omega_\varepsilon dt \right)^{-1} \int_0^T \theta^2\omega_\varepsilon dt \geq \left(\max\left\{ \theta^{h-1}(t'_\varepsilon), \theta^{h-1}(t''_\varepsilon) \right\} \right)^{-1} \text{where } t'_\varepsilon \in [0,2\varepsilon]\ and\ t''_\varepsilon \in [T-2\varepsilon, T] \right].$$

Since as $\varepsilon \to 0^+$

$$\theta^{h-1}\left(t_\varepsilon\right) \to \theta^{h-1}\left(t_0\right) = 0\left[resp.\theta^{h-1}\left(t'_\varepsilon\right) \to \theta^{h-1}\left(0\right) = 0 \ and\, \theta^{h-1}\left(t''_\varepsilon\right) \to \theta^{h-1}\left(T\right) = 0\right],$$

from (3.28) we get that it is possible to choose ε such that (3.27) holds with $\tilde{v} = \theta\omega_\varepsilon$. □

Application 3.3 (connected to Theor.2.2 (case (c_2)) and Theor.2.3 (case (c_9) with m_1=1)). We premise some clarifications. Let

$$\frac{2N}{N+2} \le p \le \frac{2N}{N-2} \ if \ N > 2 \ , 1 < p < \infty \ if \ N = 2. \tag{3.29}$$

We note

$$\left(3.29\right) \Rightarrow W_0^1\left(\Omega\right) \subseteq L^p\left(\Omega\right) \cap L^{p'}\left(\Omega\right) \ \text{ with continuous embedding } \left(p' = p / \left(p-1\right)\right). \tag{3.30}$$

Let $X = W^{2,p}\left(\Omega\right) \cap W_0^1\left(\Omega\right)$.We note the norm on X

$$\left\|z\right\|_X = \left(\int_\Omega \left|\Delta z\right|^p dx\right)^{\frac{1}{p}}$$

is equivalent to the natural one on X. It is also equivalent to the norm of $W^{2,p}\left(\Omega\right) on\, X$. In fact, there exist $c_1, c_2, c_3 > 0$ such that for any $z \in X$

$\left\|z\right\|_{W^{2,p}(\Omega)} \le c_1\left(\left\|\Delta z\right\|_{L^p(\Omega)} + \left\|z\right\|_{L^p(\Omega)}\right)$ ([2], Theor.8.2 page 444),

$\left\|z\right\|_{L^p(\Omega)} \le c_2 \left\|\nabla z\right\|_{L^2(\Omega)} \ \le c_3 \left\|\Delta z\right\|_{L^p(\Omega)}$ (from (3.30)).

Therefore X is a closed subspace of $W^{2,p}\left(\Omega\right)$. It is easy to verify that

$$\exists\, z^* \in X: \int_\Omega \left|\nabla z^*\right|^p dx = 1 \ \ and \ \ \int_\Omega \left|\Delta z^*\right|^p dx = \inf\left\{\int_\Omega \left|\Delta z\right|^p dx : z \in X \ and \int_\Omega \left|\nabla z\right|^p dx = 1\right\};$$

then, set $\lambda^* = \int_\Omega \left|\Delta z^*\right|^p dx > 0$,we have

$$\int_\Omega \left|\Delta z\right|^p dx \quad \ge \lambda^* \int_\Omega \left|\nabla z\right|^p dx \ \forall z \in X, \tag{3.31}$$

$$\int_\Omega \left|\Delta z\right|^p dx \quad - \lambda \int_\Omega \left|\nabla z\right|^p dx \ge \left(1 - \lambda\left(\lambda^*\right)^{-1}\right)\int_\Omega \left|\Delta z\right|^p dx \quad \forall \lambda \in \left]0, \lambda^*\right[\, and \ \forall z \in X. \tag{3.32}$$

Let us assume in the definition (1.1) of W_T $p_1 = p_2 = p, n = 2$ and V=X, then

$$\left\|v\right\| = \left(\int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + \int_Q \left|\Delta v\right|^p dxdt\right)^{1/p} \quad \forall v \in W_T \ ,$$

and let us set for any $v \in W_T$

$$A\left(v\right) = p^{-1}\left\|v\right\|^p, \ \ B\left(v\right) = p^{-1}\int_Q \left|\nabla v\right|^p dxdt,$$

$$D_1(v) = q_1^{-1} \int_Q d_1(x,t) \left|\nabla v\right|^{q_1} dxdt \ , \ \ D_j(v) = -q_j^{-1} \int_Q d_j(x,t) \left|v\right|^{q_j} dxdt \quad as \ j = 2,\ldots,m,$$

where

$$1 < q_1 < \ldots < q_m < p \ \ ; \ d_j \in \left(\mathbb{P}_T\left(\Omega x \mathbb{R}\right) \cap L^\infty(Q)\right) \setminus \{0\} \ as \ j = 1,\ldots m \ ,$$

$$d_j \geq 0 \ a.e. \ in \ Q \ as \ j = 2,\ldots m. \tag{3.33}$$

Problem $\left(P^T\right)$ becomes:

Find $u \in W_T \setminus \{0\}$ such that

$$\int_Q \left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} dxdt + \int_Q \left|\Delta u\right|^{p-2} \Delta u \Delta v dxdt = \lambda \int_Q \left|\nabla u\right|^{p-2} \nabla u \nabla v dxdt + \int_Q d_1(x,t) \left|\nabla u\right|^{q_1-2} \nabla u \nabla v dxdt -$$

$$\sum_{j=2}^{m} \int_Q d_j(x,t) \left|u\right|^{q_j-2} uv dxdt \ \ \forall v \in W_T. \tag{3.34}$$

Each solution u of (3.34) is for definition a weak solution of the problem:

$$-\frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\right) + \Delta\left(\left|\Delta u\right|^{p-2} \Delta u\right) = -\lambda div\left(\left|\nabla u\right|^{p-2} \nabla u\right) - div\left(d_1(x,t)\left|\nabla u\right|^{q_1-2} \nabla u\right) - \sum_{j=2}^{m} d_j(x,t)\left|u\right|^{q_j-2} u \quad in \ Q,$$

$$u = 0 \ and \ \left|\Delta u\right|^{p-2} \Delta u = 0 \qquad on \, \Sigma,$$

$$u(x,0) = u(x,T) \ and \ \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \ \ on \quad \Omega. \tag{3.35}$$

Let us introduce the conditions

There exist a compact set $\mathbb{K} \subseteq \left]0,T\right[$ with $\left|\mathbb{K}\right|_1 > 0$ and an open

$$\text{set } \ \Omega^+ \subseteq \Omega \text{ such that } \ d_1(x,t) > 0 \ a.e. \, in \, \Omega^+ x \mathbb{K} \, , \tag{3.36}$$

$$\int_0^T d_1(x,t) dt < 0 \ a.e. \ in \ \ \Omega. \tag{3.37}$$

We note

$$(3.36) \Rightarrow V^+(D_1) \neq \varnothing \ \ ,(3.37) \Rightarrow \frac{\partial v}{\partial t} \sim 0 \text{ in Q } \ \forall v \in V^+(D_1);$$

besides

$$(3.37) \Rightarrow V_\lambda^- \cap S(D_1) \neq \varnothing \ \forall \, \lambda > \lambda^*,$$

since as $v = \left|D_1(z^*)\right|^{-\frac{1}{q_1}} z^*$ we have $D_1(v) = -1 \ and \ H_\lambda(v) < 0.$

Taking into account from (3.32)

$$p^{-1}\left[\int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + \int_Q \left|\Delta v\right|^p dxdt - \lambda \int_Q \left|\nabla v\right|^p dxdt\right] \geq p^{-1}\left[\int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + \left(1 - \lambda\left(\lambda^*\right)^{-1}\right)\int_Q \left|\Delta v\right|^p dxdt\right]$$

$$\forall \lambda \in \left]0, \lambda^*\right[\, and \ \forall v \in W_T,$$

we have

$$\left(i_{21}\right), in \ \ particular\left(i_{22}\right), holds \ \ if \, \lambda \in \left]-\infty, \lambda^*\right[.$$

Proposition 3.9. Under conditions (3.36) and (3.37), there exists $\delta_1^* > 0$ such that

$$\left(i_{22}\right) \ holds \ \ if \, \lambda \in \left[\lambda^*, \lambda^* + \delta_1^*\right].$$

Proof. Reasoning by contradiction, for any $k \in \mathbb{N}$ there exist $v_k \in V^+\left(D_1\right) and \, \lambda_k \in \left[\lambda^*, \lambda^* + k^{-1}\right]$ such that

$$\left\|v_k\right\|^p - \lambda_k \int_Q \left|\nabla v_k\right|^p dxdt < pk^{-1}\left\|v_k\right\|^p.$$

Then with $w_k = \left\|v_k\right\|^{-1} v_k \, and \, w \in W_T$ such that (within a subsequence) $w_k \to w \, weakly \, in \, W_T$, from the relations

$$\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt + \int_Q \left|\Delta w_k\right|^p dxdt - \lambda_k \int_Q \left|\nabla w_k\right|^p dxdt < pk^{-1}, \int_Q d_1\left(x,t\right)\left|\nabla w_k\right|^{q_1} dxdt > 0,$$

passing to limit as $k \to +\infty$ we get

$$\int_Q \left|\frac{\partial w}{\partial t}\right|^p dxdt + \int_Q \left|\Delta w\right|^p dxdt - \lambda^* \int_Q \left|\nabla w\right|^p dxdt \le 0 , \tag{3.38}$$

$$\int_Q d_1\left(x,t\right)\left|\nabla w\right|^{q_1} dxdt \ge 0. \tag{3.39}$$

Since from (3.31) $\int_Q \left|\Delta w\right|^p dxdt \ge \lambda^* \int_Q \left|\nabla w\right|^p dxdt,$ according to (3.38) we get $\frac{\partial w}{\partial t} \equiv 0$ in Q. Consequently

$$\left(3.37\right) \ and \ \left(3.39\right) \Rightarrow \left|\nabla w\right| \equiv 0 \ \ in \, Q, that \, is \, w \equiv 0 \ \ in \, Q,$$

from which the contradiction $1 = \lim_{k\to+\infty} \left\|w_k\right\|^p \le \lim_{k\to+\infty} \left(\lambda_k \int_Q \left|\nabla w_k\right|^p dxdt + pk^{-1}\right) = 0.$ □

Proposition 3.10. Under conditions (3.37), there exists $\delta_2^* > 0$ such that

$$\left(i_{24}\right) \ holds \ \ with \ \ m_1 = 1 \ \ if \, \lambda \in \left]\lambda^*, \lambda^* + \delta_2^*\right].$$

Proof. Reasoning by contradiction, as in Prop.3.5 there exist $\left(\lambda_k\right)_{k\in\mathbb{N}}$,with $\lambda_k \in \left]\lambda^*, \lambda^* + k^{-1}\right]$,and $\left(v_{k,h_k}\right)_{k\in\mathbb{N}} \subseteq W_T$ such that

$$H_{\lambda_k}\left(v_{k,h_k}\right) < 0, D_1\left(v_{k,h_k}\right) = -1, \ \lim_{k\to+\infty}\left\|v_{k,h_k}\right\| = +\infty.$$

Then, set $w_k = \left\|v_{k,h_k}\right\|^{-1} v_{k,h_k}$, we have

$$\int_Q \left|\frac{\partial w_k}{\partial t}\right|^p dxdt + \int_Q \left|\Delta w_k\right|^p dxdt < \lambda_k \int_Q \left|\nabla w_k\right|^p dxdt, \tag{3.40}$$

$$\int_Q d_1(x,t)\left|\nabla w_k\right|^{q_1} dxdt = -q_1\left\|v_{k,h_k}\right\|^{-q_1} \to 0 \ as \ k \to +\infty. \tag{3.41}$$

Let $w \in W_T$ such that (within a subsequence) $w_k \to w \, weakly \, in \, W_T$. Relations (3.40), (3.41) imply

$$\int_Q \left|\frac{\partial w}{\partial t}\right|^p dxdt + \int_Q \left|\Delta w\right|^p dxdt - \lambda^* \int_Q \left|\nabla w\right|^p dxdt \le 0 \ , \int_Q d_1(x,t)\left|\nabla w\right|^{q_1} dxdt = 0,$$

from which, taking into account (3.37), we deduce that $w \equiv 0$ in Q and the contradiction $1 = \lim_{k\to+\infty}\left\|w_k\right\|^p \le \lim_{k\to+\infty}\lambda_k \int_Q \left|\nabla w_k\right|^p dxdt = 0.$ □

How established far allows the following result.

Proposition 3.11 (Theor.2.2(case (c_2)); Theor.(2.3) (case $(c_9)\, with \, m_1 = 1$)). Under assumptions (3.29) and (3.33) we have: when (3.36) and (3.37) hold, with $\lambda \in \left]-\infty, \lambda^* + \delta_1^*\right]$ problem (3.35) has at least two nonstationary weak solutions u_0 and $-u_0$ ($u_0 = r_0 v_0$, $r_0 = const. > 0$, $v_0 \in S_\lambda \cap V^+(D_1)$);

when (3.37) holds, with $\lambda \in \left]\lambda^*, \lambda^* + \delta_2^*\right]$ problem (3.35) has at least two weak solutions $\underline{u}$ and $-\underline{u}$ ($\underline{u} = \underline{r}\,\underline{v}$, $\underline{r} = const. > 0, \underline{v} \in V_\lambda^- \cap S(D_1)$).

Consequently, when (3.36) and (3.37) hold, with $\lambda \in \left]\lambda^*, \lambda^* + \min\left\{\delta_1^*, \delta_2^*\right\}\right]$ problem (3.35) has at least four different weak solutions.

Proposition 3.12. Let $\lambda \in \left]\lambda^*, \lambda^* + \delta_2^*\right]$. If there exist a measurable set $I \subseteq [0,T]$ with $\left|I\right|_1 > 0$ and a limit point t_0 of I such that $\lim_{\substack{t\to t_0\\ t\in I}} d_1(x,t) \ge 0$ and as j=2,…,m $\lim_{\substack{t\to t_0\\ t\in I}} d_j(x,t) = 0$ a.e. in Ω, then $\underline{u}$ is nonstationary.

Proof. In fact, if $\dfrac{\partial \underline{u}}{\partial t} \equiv 0 \, in \, Q$, we get the contradiction

$$\int_\Omega d_1(x,t)\left|\nabla \underline{u}\right|^{q_1} dx - \sum_{j=2}^m \int_\Omega d_j(x,t)\left|\underline{u}\right|^{q_j} dx = \int_\Omega \left|\Delta \underline{u}\right|^p dx - \lambda \int_\Omega \left|\nabla \underline{u}\right|^p dx (< 0) \, a.e. \, in \, [0,T],$$

$$\lim_{\substack{t\to t_0\\ t\in I}} \left(\int_\Omega d_1(x,t)\left|\nabla \underline{u}\right|^{q_1} dx - \sum_{j=2}^m \int_\Omega d_j(x,t)\left|\underline{u}\right|^{q_j} dx\right) \ge 0. \qquad □$$

Let us suppose d_1 has the following structure (according to (3.36) and (3.37)):

$$d_1(x,t) = d_{11}(x) d_{12}(t), d_{11} \in L^\infty(\Omega) \, and \, d_{11} > 0 \ a.e. \, in \, \Omega, d_{12} \in \mathrm{P}_T(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

$$d_{12}^+ \tilde{\ } 0 \ \ in \ [0,T], \ \int_0^T d_{12} dt < 0;$$

there exist $t_0 \in \left]0,T\right[\, and \, \varepsilon_0 > 0$ as in (c^*) such that $d_{12} \in C^0\left(\left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right]\right)$

and $d_{12}(t) < 0 \ \ \forall t \in \left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right]$. (3.42)

In addition let us suppose:

$$\forall \, j \in \{2,\ldots,m\} \, d_j(x,\cdot) \in C^0\left(\left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right]\right) \ and \ d_j(x,t_0) = 0 \ a.e. \, in \ \Omega \ . \tag{3.43}$$

Proposition 3.13. Let (3.42) and (3.43) hold. Let $\lambda \in \left]\lambda^*, \lambda^* + \delta_2^*\right]$.If $T < \left(d_{12}\left(t_0\right)\right)^{-1} \int_0^T d_{12} dt$, then $\underline{u}$ is nonstationary.

Proof. It is sufficient (Prop.2.2) to prove that

If $\frac{\partial \underline{u}}{\partial t} \equiv 0 \, in \, Q$, then there exists $\tilde{v} \in W_T$ such that $< \partial D_1\left(\underline{v}\right), \tilde{v} > \neq 0$ and

$$\left[< \partial H_\lambda\left(\underline{u}\right), \underline{u} > + q_1\left(< \partial D_1\left(\underline{v}\right), \tilde{v} >\right)^{-1} < \partial H_\lambda\left(\underline{u}\right), \underline{r}\tilde{v} >\right] -$$

$$\sum_{j=2}^{m}\left[< \partial D_j\left(\underline{u}\right), \underline{u} > + q_1\left(< \partial D_1\left(\underline{v}\right), \tilde{v} >\right)^{-1} < \partial D_j\left(\underline{u}\right), \underline{r}\tilde{v} >\right] \neq 0. \tag{3.44}$$

Let ω_ε be as in $\left(c^*\right)$. We note that

$$< \partial D_1\left(\underline{v}\right), \omega_\varepsilon \underline{v} > = \left(\int_0^T d_{12}\omega_\varepsilon dt\right)\int_\Omega d_{11}\left(x\right)\left|\nabla \underline{v}\right|^{q_1} dx = -q_1\left(\int_0^T d_{12}\omega_\varepsilon dt\right)\left(\int_0^T d_{12} dt\right)^{-1} < 0.$$

Then

$$q_1\left(< \partial D_1\left(\underline{v}\right), \omega_\varepsilon \underline{v} >\right)^{-1} < \partial H_\lambda\left(\underline{u}\right), \underline{r}\omega_\varepsilon \underline{v} > = -\left(\int_0^T d_{12}\omega_\varepsilon dt\right)^{-1}\left(\int_0^T \omega_\varepsilon dt\right)\left(\int_0^T d_{12} dt\right)\left(\int_\Omega \left|\Delta \underline{u}\right|^p dx - \lambda\int_\Omega \left|\nabla \underline{u}\right|^p dx\right)$$

and as j=2,…,m

$$q_1\left(< \partial D_1\left(\underline{v}\right), \omega_\varepsilon \underline{v} >\right)^{-1} < \partial D_j\left(\underline{u}\right), \underline{r}\omega_\varepsilon \underline{v} > = \left(\int_0^T d_{12}\omega_\varepsilon dt\right)^{-1}\left(\int_0^T d_{12} dt\right)\int_Q d_j\left(x,t\right)\left|\underline{u}\right|^{q_j} \omega_\varepsilon\left(t\right) dx dt.$$

Let us note that

$$\left(\int_0^T d_{12}\omega_\varepsilon dt\right)^{-1}\left(\int_0^T \omega_\varepsilon dt\right) \to \left(d_{12}\left(t_0\right)\right)^{-1} \quad as \ \varepsilon \to 0^+;$$

besides, set $\eta = \min_{\left[t_0-\varepsilon_0, t_0+\varepsilon_0\right]}\left|d_{12}\right|$, we get

$$\left|\left(\int_0^T d_{12}\omega_\varepsilon dt\right)^{-1}\int_Q d_j\left(x,t\right)\left|\underline{u}\right|^{q_j} \omega_\varepsilon\left(t\right) dx dt\right| = \left(\int_0^T \left|d_{12}\right|\omega_\varepsilon dt\right)^{-1}\int_Q d_j\left(x,t\right)\left|\underline{u}\right|^{q_j} \omega_\varepsilon\left(t\right) dx dt \leq$$

$$\eta^{-1}\left(\int_0^T \omega_\varepsilon dt\right)^{-1}\int_0^T\left(\int_\Omega d_j\left(x,t\right)\left|\underline{u}\right|^{q_j} dx\right)\omega_\varepsilon\left(t\right) dt \to \eta^{-1}\int_\Omega d_j\left(x,t_0\right) dx = 0 \ as \, \varepsilon \to 0^+.$$

Since

$$< \partial H_\lambda\left(\underline{u}\right), \underline{u} > - \left(d_{12}\left(t_0\right)\right)^{-1}\int_0^T d_{12} dt\left(\int_\Omega \left|\Delta \underline{u}\right|^p dx - \lambda\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) =$$

$$\left(T - \left(d_{12}\left(t_0\right)\right)^{-1}\int_0^T d_{12} dt\right)\left(\int_\Omega \left|\Delta \underline{u}\right|^p dx - \lambda\int_\Omega \left|\nabla \underline{u}\right|^p dx\right) > 0,$$

$$-\sum_{j=2}^{m} < \partial D_j\left(\underline{u}\right), \underline{u} > \; = \sum_{j=2}^{m} \int_Q d_j\left(x,t\right)\left|\underline{u}\right|^{q_j} dxdt \geq 0,$$

with a suitable $\varepsilon, \tilde{v} = \omega_\varepsilon \underline{v}$ fulfills (3.44). □

Remark 3.2. It is not difficult to set functions d_{12} as in (3.42) such that $T < \left(d_{12}\left(t_0\right)\right)^{-1} \int_0^T d_{12} dt$.

Application 3.4 (connected to Theor.2.2 (case (c_3)) and Theor.2.3 (case (c_9) with m_1=m)). Let us assume in the definition (1.1) of W_T $p_1 = p_2 = 2$, n=2 and $V = W^2\left(\Omega\right)$, then

$$\left\|v\right\| = \left(\int_Q \left(\frac{\partial v}{\partial t}\right)^2 dxdt + \sum_{|\alpha|=2} \int_Q \left(D^\alpha v\right)^2 dxdt + \int_Q v^2 dxdt\right)^{1/2} \quad \forall v \in W_T,$$

and let us set as any $v \in W_T$

$$A\left(v\right) = 2^{-1}\left[\int_Q \left(\frac{\partial v}{\partial t}\right)^2 dxdt + \sum_{|\alpha|=2} \int_Q \left(D^\alpha v\right)^2 dxdt\right],$$

$$B\left(v\right) = 2^{-1}\int_Q \left(b_1\left(x,t\right)\left(v^+\right)^2 - b_2\left(x,t\right)\left(v^-\right)^2\right) dxdt \quad \left(v^- = \min\left\{v, 0\right\}\right),$$

$$D_j\left(v\right) = -q_j^{-1}\left(\int_Q \left|\nabla v\right|^{\gamma_j} dxdt\right)^{q_j/\gamma_j} \quad as \;\; j = 1, \ldots, m-1,$$

$$D_m\left(v\right) = \left(2n\right)^{-1}\left(\int_Q d_m\left(x,t\right) v^2 dxdt\right)^n,$$

where

$1 < \gamma_j \leq 2$, n is a positive odd integer, $2 < q_1 < \ldots < q_{m-1} < q_m = 2n$;

$$as \;\; i = 1,2 \;\; b_i \; and \; d_m \in \left(\mathrm{P}_T\left(\Omega x \mathbb{R}\right) \cap L^\infty\left(Q\right)\right) \setminus \left\{0\right\}, \; b_i\left(x,t\right) > 0 \;\; a.e. \; in \; Q. \tag{3.45}$$

Problem $\left(P^T\right)$ becomes:

Find $u \in W_T \setminus \left\{0\right\}$ such that

$$\int_Q \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} dxdt + \sum_{|\alpha|=2} \int_Q D^\alpha u D^\alpha v \, dxdt = \lambda \int_Q \left(b_1\left(x,t\right) u^+ v - b_2\left(x,t\right) u^- v\right) dxdt -$$

$$\sum_{j=1}^{m-1}\left(\int_Q \left|\nabla u\right|^{\gamma_j} dxdt\right)^{\frac{q_j}{\gamma_j}-1}\left(\int_Q \left|\nabla u\right|^{\gamma_j - 2} \nabla u \nabla v dxdt\right) + \left(\int_Q d_m\left(x,t\right) u^2 \, dxdt\right)^{n-1} \int_Q d_m\left(x,t\right) uv dxdt \quad \forall v \in W_T. \tag{3.46}$$

Each solution u of (3.46) is for definition a weak solution of the problem:

$$-\frac{\partial^2 u}{\partial t^2} + \sum_{|\alpha|=2} D^\alpha D^\alpha u = \lambda\left(b_1\left(x,t\right)u^+ - b_2\left(x,t\right)u^-\right) + \sum_{j=1}^{m-1}\left(\int_Q \left|\nabla u\right|^{\gamma_j} dxdt\right)^{\frac{q_j}{\gamma_j}-1} div\left(\left|\nabla u\right|^{\gamma_j-2} \nabla u\right) +$$

$$\left(\int_Q d_m\left(x,t\right)u^2dxdt\right)^{n-1} d_m\left(x,t\right)u \quad in\ Q,$$

$$\frac{\partial\Delta u}{\partial\nu}=\left[\sum_{j=1}^{m-1}\left(\int_Q\left|\nabla u\right|^{\gamma_j}dxdt\right)^{\frac{q_j}{\gamma_j}-1}\left|\nabla u\right|^{\gamma_j-2}\right]\frac{\partial u}{\partial\nu}\quad on\,\Sigma,$$

$$\frac{\partial}{\partial\nu}\left(\frac{\partial u}{\partial x_h}\right)=0\quad as\ \ h=1,...,N\quad on\,\Sigma,$$

$$u\left(x,0\right)=u\left(x,T\right)\ and\ \ \frac{\partial u}{\partial t}\left(x,0\right)=\frac{\partial u}{\partial t}\left(x,T\right)\ \ on\quad\Omega. \tag{3.47}$$

Let us introduce the conditions

$$d_m^+ \sim 0\ \text{ in Q}, \tag{3.48}$$

$$\int_0^T d_m\left(x,t\right)dt<0\ \ a.e.\ in\ \ \Omega. \tag{3.49}$$

Evidently

$\left(3.48\right)\Rightarrow V^+\left(D_m\right)\neq\varnothing$ $,\left(3.49\right)\Rightarrow\frac{\partial v}{\partial t}\sim 0$ in Q $\forall v\in V^+\left(D_m\right)$, $\left(3.49\right)\Rightarrow V_\lambda^-\cap S\left(D_m\right)\neq\varnothing\ \forall\lambda>0.$

Reasoning as in Prop.3.9 and Prop.3.10, we prove that
under conditions (3.48) and (3.49), there exists $\delta_1^*>0$ such that $\left(i_{23}\right)$ holds if $\left|\lambda\right|\leq\delta_1^*$;
under condition (3.49), there exists δ_2^* such that $\left(i_{24}\right)$ holds with $m_1=m\ \ if\,\lambda\in\left]0,\delta_2^*\right]$.

Then

Proposition 3.14. (Theor.2.2 (case (c_3)), Theor.2.3 (case (c_9) with m_1=m)).Under conditions (3.45) we have:
when (3.48) and (3.49) hold, with $\left|\lambda\right|\leq\delta_1^*$ problem (3.47) has at least one nonstationary weak solution u_0 ($u_0=r_0v_0$, $r_0=const.>0$, $v_0\in S_\lambda\cap V^+\left(D_m\right)$);
when (3.49) holds, with $\lambda\in\left]0,\delta_2^*\right]$ problem (3.47) has at least one weak solution $\underline{u}$ ($\underline{u}=$ $\underline{r}\,\underline{v}$, $\underline{r}=const.>0,\underline{v}\in V_\lambda^-\cap S\left(D_m\right)$).

Consequently, when (3.48) and (3.49) hold, with $\lambda\in\left]0,\min\left\{\delta_1^*,\delta_2^*\right\}\right]$ problem (3.47) has at least two different weak solutions.

Proposition 3.15. Let $\lambda\in\left]0,\delta_2^*\right]$. Let be true one of the following conditions:

There exist a measurable set $I\subseteq\left[0,T\right]with\left|\,I\,\right|_1>0$ and a limit point t_0 of I such that

$$\lim_{\substack{t\to t_0\\t\in I}}d_m\left(x,t\right)=0\ \ and\ \lim_{\substack{t\to t_0\\t\in I}}b_i\left(x,t\right)>0\ \ a.e.\,in\ \Omega\ \ as\ \ i=1,2; \tag{3.50}$$

There exist a measurable set $I\subseteq\left[0,T\right]with\left|\,I\,\right|_1>0$,a limit point t_0 of I and $b_0>0$ such that

$$\lim_{\substack{t\to t_0\\t\in I}}d_m\left(x,t\right)=0\ \ a.e.\,in\ \Omega\quad and\,b_i\left(x,t\right)\geq b_0\ \ \ a.e.\,in\ \ \Omega\times I\ \ as\ \ i=1,2. \tag{3.51}$$

Then $\underline{u}$ is nonstationary.

Proof. If $\frac{\partial \underline{u}}{\partial t} \equiv 0\, in\, Q$, then with $\delta = \left(\int_Q d_m\left(x,t\right)\left(\underline{u}\right)^2 dxdt \right)^{n-1} > 0$ we have

$$\delta \int_\Omega d_m\left(x,t\right)\underline{u}dx = -\lambda \int_\Omega \left(b_1\left(x,t\right)\underline{u}^+ - b_2\left(x,t\right)\underline{u}^-\right)dx \;\; a.e.\, in \left[0,T\right]$$

from which the contradictions

$$0 = \lim_{\substack{t\to t_0\\ t\in I^0}} \delta \int_\Omega d_m\left(x,t\right)\underline{u}dx = -\lambda \lim_{\substack{t\to t_0\\ t\in I^0}} \int_\Omega \left(b_1\left(x,t\right)\underline{u}^+ - b_2\left(x,t\right)\underline{u}^-\right)dx < 0 \;\; when \left(3.50\right) holds,$$

$$0 = \lim_{\substack{t\to t_0\\ t\in I^0}} \delta \int_\Omega d_m\left(x,t\right)\underline{u}dx \le -\lambda b_0 \int_\Omega \left(\underline{u}^+ - \underline{u}^-\right)dx < 0 \;\; when \left(3.51\right) holds.$$

□

Relations (3.48), (3.49) in particular fulfill when

$$d_m\left(x,t\right) = d_{m1}\left(x\right)d_{m2}\left(t\right) \quad with\, d_{m1} \in L^\infty\left(\Omega\right), d_{m2} \in \mathrm{P}_T\left(\mathbb{R}\right) \cap L^\infty\left(\mathbb{R}\right),$$

$$d_{m1} > 0 \;\; a.e. \;\, in \;\, \Omega, \; d_{m2}^+ \tilde{\ } 0\, in \left[0,T\right], \int_0^T d_{m2}dt < 0. \tag{3.52}$$

Proposition 3.16. Let (3.52) holds. Let $\lambda \in \left]0,\delta_2^*\right]$. Then $\underline{u}$ is nonstationary.

Proof. Reasoning by contradiction let $\frac{\partial \underline{u}}{\partial t} \equiv 0\, in\, Q.$ Since $\underline{u} \tilde{\ } 0$ in **Ω**, set in (3.46) $u = \underline{u} \;\, and \;\, v = 1,$ we have

$$\left(\int_Q d_{m1}\left(x\right)d_{m2}\left(t\right)\underline{u}^2 dxdt \right)^{n-1} \int_Q d_{m1}\left(x\right)d_{m2}\left(t\right)\underline{u}dxdt = -\lambda \int_Q \left(b_1\left(x,t\right)\underline{u}^+ - b_2\left(x,t\right)\underline{u}^-\right)dxdt < 0.$$

Then

$$\left(\int_Q d_{m1}\left(x\right)d_{m2}\left(t\right)\underline{u}^2 dxdt \right)^{n-1} > 0 \tag{3.53}$$

and moreover

$$\int_\Omega d_{m1}\left(x\right)\underline{u}dx > 0 \tag{3.54}$$

since $\int_0^T d_{m2}dt < 0.$ Condition $d_{m2}^+ \tilde{\ } 0 \;\; in \left[0,T\right]$ implies there exists a compact set $\mathrm{K} \subseteq \left]0,T\right[$ with $\left|\mathrm{K}\right|_1 > 0$ and $d_{m2} > 0 \;\, in \;\, \mathrm{K}$ .Let $\left(\varphi_\varepsilon\right)_{0<\varepsilon<\varepsilon_0} \subseteq \mathrm{P}_T\left(\mathbb{R}\right) \cap C^\infty\left(\mathbb{R}\right) \quad with \;\; 0 \le \varphi_\varepsilon \le 1, \operatorname*{supp}_{[0,T]} \varphi_\varepsilon \subseteq \left]0,T\right[and \quad \varphi_\varepsilon \to \chi\, strongly\, in\, L^s\left(\left]0,T\right[\right)$ $as \;\, \varepsilon \to 0^+ \;\forall s \in \left[1,+\infty\right[,$ where $\chi\, is \;\, the\, chacteristic\, function\, of\, \mathrm{K}$.

Since $\lim_{\varepsilon\to 0^+} \int_0^T d_{m2}\varphi_\varepsilon dt = \int_\mathrm{K} d_{m2}dt > 0,$ we choose ε such that $\int_0^T d_{m2}\varphi_\varepsilon dt > 0.$ Then taking into account (3.53), (3.54), from (3.46) with $u = \underline{u} \;\, and\, v = \varphi_\varepsilon$ we get the contradiction

$$0 < \left(\int_Q d_{m1}\left(x\right)d_{m2}\left(t\right)\underline{u}^2 dxdt \right)^{n-1} \left(\int_0^T d_{m2}\varphi_\varepsilon dt \right) \int_\Omega d_{m1}\underline{u}dx = -\lambda \int_Q \left(b_1\left(x,t\right)\underline{u}^+ - b_2\left(x,t\right)\underline{u}^-\right)\varphi_\varepsilon dxdt < 0.$$

□

Application 3.5 (connected to Theor.2.2 (case (c_4) with m_1=m-1)). Let us assume in the definition (1.1) of W_T $p_1 = p_2 = p$ *and* $V = W_0^{n,p}(\Omega)$ $(n = 1,2,...)$, then

$$\|v\| = \left(\int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + \sum_{|\alpha|=n} \int_Q |D^\alpha v|^p \, dxdt\right)^{1/p} \quad \forall v \in W_T,$$

and let us set as any $v \in W_T$

$$A(v) = p^{-1}\left[\int_Q \left|\frac{\partial v}{\partial t}\right|^p dxdt + \sum_{|\alpha|=n} \int_Q a(t)|D^\alpha v|^p \, dxdt\right],$$

$$B(v) = p^{-1}\int_Q a(t)b(x)(v^+)^p \, dxdt \ ,$$

$$D_j(v) = q_j^{-1}\int_Q d_j(x,t)|v|^{q_j} \, dxdt \quad as \ j = 1,...,m-1,$$

$$D_m(v) = -q_m^{-1}\int_Q d_m(x,t)|v|^{q_m} \, dxdt,$$

where

$$1 < q_1 < ... < q_{m-1} < q_m \le p; \, a \in \mathrm{P}_T(\mathbb{R}) \cap C^0(\mathbb{R}) \ with \ \ a(t) \ge a_0 \ \forall t \in [0,T] (a_0 = const. > 0 \, and \le 1),$$
$$b \in L^\infty(\Omega) \setminus \{0\} \, with \ b \ge 0 \ a.e. \ in \ \Omega, d_j \in (\mathrm{P}_T(\Omega \times \mathbb{R}) \cap L^\infty(Q)) \setminus \{0\} \, with \, d_j \ge 0 \ a.e. \ in \ Q$$
$$as \, j = 1,...,m. \tag{3.55}$$

Problem (P^T) becomes:

Find $u \in W_T \setminus \{0\}$ such that

$$\int_Q \left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} dxdt + \sum_{|\alpha|=n} \int_Q a(t)|D^\alpha u|^{p-2} D^\alpha u D^\alpha v \, dxdt = \lambda \int_Q a(t)b(x)(u^+)^{p-1} v \, dxdt +$$
$$\sum_{j=1}^{m-1} \int_Q d_j(x,t)|u|^{q_j-2} uv \, dxdt - \int_Q d_m(x,t)|u|^{q_m-2} uvdxdt \quad \forall v \in W_T. \tag{3.56}$$

Each solution u of (3.56) is for definition a weak solution of the problem:

$$-\frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p-2} \frac{\partial u}{\partial t}\right) + \sum_{|\alpha|=n} (-1)^n a(t) D^\alpha \left(|D^\alpha u|^{p-2} D^\alpha u\right) = \lambda a(t)b(x)(u^+)^{p-1} +$$
$$\sum_{j=1}^{m-1} d_j(x,t)|u|^{q_j-2} u - d_m(x,t)|u|^{q_m-2} u \ \ in \ Q,$$
$$D^\alpha u = 0 \ on \ \Sigma \ as \ 0 \le |\alpha| \le n-1,$$
$$u(x,0) = u(x,T) \ and \ \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T) \ on \ \ \Omega. \tag{3.57}$$

Evidently (i_{21}) *holds if* $\lambda \in]-\infty, 0]$. Let us add that set $\overline{a} = \|a\|_{C^0([0,T])}$, there exists $\delta^* > 0$ satisfying the condition:

$$\forall\lambda \in \left]0,\delta^*\right] \exists c(\lambda) > 0 : a_0 \left\|v\right\|^p - \lambda\overline{a}\int_Q b(x)\left(v^+\right)^p dxdt \geq c(\lambda)\left\|v\right\|^p \ \forall v \in W_T. \tag{3.58}$$

In fact, otherwise, for each $k \in \mathbb{N}$ there exist $v_k \in W_T \ and\, \lambda_k \in \left]0, k^{-1}\right] such\ that$

$$a_0 \left\|v_k\right\|^p - \lambda_k \overline{a}\int_Q b(x)\left(v_k^+\right)^p dxdt < k^{-1}\left\|v_k\right\|^p .$$

Then with $w_k = \left\|v_k\right\|^{-1} v_k$ we have

$$a_0 < \lambda_k \overline{a}\int_Q b(x)\left(w_k^+\right)^p dxdt + k^{-1}$$

from which, passing to limit as $k \to +\infty$, we get $a_0 \leq 0$.

(3.58) implies $\left(i_{21}\right)$ holds even if $\lambda \in \left]0,\delta^*\right]$.

Proposition 3.17. (Theor.2.2 (case (c_4) with m_1=m-1)).Under conditions (3.55), with $\lambda \in \left]-\infty,\delta^*\right]$ problem (3.57) has at least one weak solution u_0 ($u_0 = r_0 v_0$, $r_0 = const. > 0$, $v_0 \in S_\lambda \cap V^+\left(D_1,\ldots,D_{m-1}\right)$).

About the nonstationarity of u_0 ,let us introduce the conditions:

There exist a measurable set $I \subseteq \left[0,T\right] with \left| I \right|_1 > 0$ and a limit point t_0 of I such that

$$\lim_{\substack{t\to t_0\\ t\in I}} d_j(x,t) = 0 \ \text{ for almost any } x \in \Omega \text{ and as j=1,…,m-1;} \tag{3.59}$$

There exist an open interval $I \subseteq \left[0,T\right]$, $g \in C^0(I) with\, g(t) > 0\, \forall t \in I$ and $g \sim \eta\, a$ in I $\forall \eta > 0$, $g_j \in L^\infty(\Omega)$ with $g_j > 0\ a.e.\ in\,\Omega$ such that $d_j(x,t) = g_j(x)g(t)$ as almost every $x \in \Omega$ and as each

$$t \in I\ \left(j = 1,\ldots,m\right); \tag{3.60}$$

There exist $t_0 \in \left]0,T\right[and\, \varepsilon_0 > 0\ as\, in\left(c^*\right)$ such that for almost any $x \in \Omega$ $d_j(x,\cdot) \in C^0\left(\left[t_0 - \varepsilon_0, t_0 + \varepsilon_0\right]\right) as\ j = 1,\ldots,m$ and

$$\sum_{j=1}^{m-1}\int_Q \left[d_j(x,t) - a(t)\left(a(t_0)\right)^{-1} d_j(x,t_0)\right]\left|u_0\right|^{q_j} dxdt -$$

$$\int_Q \left[d_m(x,t) - a(t)\left(a(t_0)\right)^{-1} d_m(x,t_0)\right]\left|u_0\right|^{q_m} dxdt \neq 0. \tag{3.61}$$

Remark 3.3. It is easy to find assumptions on d_j and *a* such that the inequality in (3.61) holds.

Proposition 3.18. Let $\lambda \in \left]-\infty,\delta^*\right]$. If one of the conditions (3.59) - (3.61) holds, then u_0 is nonstationary.

Proof. Reasoning by contradiction, let $\dfrac{\partial u_0}{\partial t} \equiv 0$ in Q.

When (3.59) holds, we have the contradiction

$$0 = \lim_{\substack{t\to t_0\\ t\in I}} \sum_{j=1}^{m-1}\int_\Omega d_j(x,t)\left|u_0\right|^{q_j} dx \geq \delta$$

where

$$\delta = a_0 \sum_{|\alpha|=n}\int_\Omega \left|D^\alpha u_0\right|^p dx > 0\ if\, \lambda \leq 0\ ,$$

$$\delta = a_0 \sum_{|\alpha|=n} \int_\Omega \left|D^\alpha u_0\right|^p dx - \lambda \overline{a} \int_\Omega b(x)\left(u_0^+\right)^p dx > 0 \quad \text{(from (3.58)) if } \lambda \in \left]0, \delta^*\right].$$

When (3.60) holds, we have $g(t) = \eta_0 a(t)\ \forall t \in I$ with

$$\eta_0 = \left[\sum_{|\alpha|=n} \int_\Omega \left|D^\alpha u_0\right|^p dx - \lambda \int_\Omega b(x)\left(u_0^+\right)^p dx\right]\left[\sum_{j=1}^{m-1} \int_\Omega g_j(x)\left|u_0\right|^{q_j} dx - \int_\Omega g_m(x)\left|u_0\right|^{q_m} dx\right]^{-1} > 0,$$

and this contradicts hypothesis.

When (3.61) holds, it is sufficient(Prop.2.1) to prove that

There exists $\tilde{v} \in W_T$ such that $< \partial H_\lambda(v_0), \tilde{v} > \neq 0$ and

$$\sum_{j=1}^{m}\left[< \partial D_j(u_0), u_0 > - p\left(< \partial H_\lambda(v_0), \tilde{v} >\right)^{-1} < \partial D_j(u_0), r_0 \tilde{v} >\right] \neq 0 . \tag{3.62}$$

With $\omega_\varepsilon\ as\ in\left(c^*\right)$ we have

$$< \partial H_\lambda(v_0), \omega_\varepsilon v_0 > = \sum_{|\alpha|=n} \int_Q a(t)\left|D^\alpha v_0\right|^p \omega_\varepsilon dxdt - \lambda \int_Q a(t) b(x)\left(v_0^+\right)^p \omega_\varepsilon dxdt =$$

$$\left(\int_0^T a\omega_\varepsilon dt\right)\left[\sum_{|\alpha|=n} \int_\Omega \left|D^\alpha v_0\right|^p dx - \lambda \int_\Omega b(x)\left(v_0^+\right)^p dx\right] = p\left(\int_0^T a\omega_\varepsilon dt\right)\left(\int_0^T adt\right)^{-1} > 0.$$

Then

$$\sum_{j=1}^{m}\left[< \partial D_j(u_0), u_0 > - p\left(< \partial H_\lambda(v_0), \omega_\varepsilon v_0 >\right)^{-1} < \partial D_j(u_0), r_0 \omega_\varepsilon v_0 >\right] =$$

$$\sum_{j=1}^{m-1}\left[\int_Q d_j(x,t)\left|u_0\right|^{q_j} dxdt - \left(\int_0^T adt\right)\left(\int_0^T a\omega_\varepsilon dt\right)^{-1} \int_Q d_j(x,t)\left|u_0\right|^{q_j} \omega_\varepsilon(t) dxdt\right] -$$

$$\left[\int_Q d_m(x,t)\left|u_0\right|^{q_m} dxdt - \left(\int_0^T adt\right)\left(\int_0^T a\omega_\varepsilon dt\right)^{-1} \int_Q d_m(x,t)\left|u_0\right|^{q_m} \omega_\varepsilon(t) dxdt\right].$$

Since as j=1,…,m

$$\lim_{\varepsilon \to 0^+}\left(\int_0^T adt\right)\left(\int_0^T a\omega_\varepsilon dt\right)^{-1} \int_Q d_j(x,t)\left|u_0\right|^{q_j} \omega_\varepsilon(t) dxdt =$$

$$\left(\int_0^T adt\right)\left(a(t_0)\right)^{-1} \int_\Omega d_j(x,t_0)\left|u_0\right|^{q_j} dx = \int_Q a(t)\left(a(t_0)\right)^{-1} d_j(x,t_0)\left|u_0\right|^{q_j} dxdt ,$$

with a suitable ε, $\tilde{v} = \omega_\varepsilon v_0$ fulfills (3.62). □

Remark 3.4. It is easy to prove that Propositions 3.17 and 3.18 also hold when

$$A(v) = p^{-1} \int_Q \left[\left|\frac{\partial v}{\partial t}\right|^\gamma + \left(a(t)\right)^{\frac{\gamma}{p}} \sum_{|\alpha|=n} \left|D^\alpha v\right|^\gamma\right]^{\frac{p}{\gamma}} dxdt \quad \forall v \in W_T$$

with $1 < \gamma < p$ and a as in (3.55).

Application 3.6 (connected to Theor.2.2 (case (c_6))). Let us assume in the definition (1.1) of W_T $p_1 = p_2 = p$, n=2 and $V = W^{2,p}(\Omega)$, then

$$\|v\| = \left(\int_Q \left| \frac{\partial v}{\partial t} \right|^p dxdt + \sum_{|\alpha|=2} \int_Q \left| D^\alpha v \right|^p dxdt + \int_Q |v|^p dxdt \right)^{1/p} \quad \forall v \in W_T ,$$

and let us set as any $v \in W_T$

$$A(v) = p^{-1} \left[\int_Q \left| \frac{\partial v}{\partial t} \right|^p dxdt + \sum_{|\alpha|=2} \int_Q a(t) \left| D^\alpha v \right|^p dxdt \right],$$

$$B(v) = p^{-1} \int_Q a(t) |v|^p dxdt ,$$

$$D_j(v) = q_j^{-1} \int_Q d_j(x,t) |\nabla v|^{q_j} dxdt \quad as \ j = 1,\dots,m-2,$$

$$D_{m-1}(v) = q_{m-1}^{-1} \int_Q d_{m-1}(x,t) |v|^{q_{m-1}} dxdt,$$

$$D_m(v) = q_m^{-1} \int_Q d_m(x,t) |v|^{q_m - 1} vdxdt,$$

where

$$1 < q_1 < \dots < q_m < p; \ a \in \mathbb{P}_T(\mathbb{R}) \cap C^0(\mathbb{R}) \, with \ a(t) \ge a_0 \ \forall t \in [0,T] \left(a_0 = const. > 0 \right),$$

$$d_j \in \left(\mathbb{P}_T(\Omega \times \mathbb{R}) \cap L^\infty(Q) \right) \setminus \{0\} \ as \ j = 1,\dots,m, d_j \ge 0 \ a.e. \ in \ Q \ as \ j = 1,\dots,m-2,$$

$$d_{m-1} > 0 \ and \ d_m > 0 \quad a.e. \ in Q. \tag{3.63}$$

Problem (P^T) becomes:

Find $u \in W_T \setminus \{0\}$ such that

$$\int_Q \left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} dxdt + \sum_{|\alpha|=2} \int_Q a(t) \left| D^\alpha u \right|^{p-2} D^\alpha u D^\alpha v \, dxdt = \lambda \int_Q a(t) |u|^{p-2} u \, v \, dxdt + \sum_{j=1}^{m-2} \int_Q d_j(x,t) |\nabla u|^{q_j - 2} \nabla u \nabla v \, dxdt +$$

$$\int_Q d_{m-1}(x,t) |u|^{q_{m-1}-2} uvdxdt + \int_Q d_m(x,t) |u|^{q_m - 1} vdxdt \quad \forall v \in W_T. \tag{3.64}$$

Each solution u of (3.64) is for definition a weak solution of the problem:

$$-\frac{\partial}{\partial t} \left(\left| \frac{\partial u}{\partial t} \right|^{p-2} \frac{\partial u}{\partial t} \right) + \sum_{|\alpha|=2} a(t) D^\alpha \left(\left| D^\alpha u \right|^{p-2} D^\alpha u \right) - \lambda a(t) |u|^{p-2} u = -\sum_{j=1}^{m-2} div \left(d_j(x,t) |\nabla u|^{q_j - 2} \nabla u \right) +$$

$$d_{m-1}(x,t) |u|^{q_{m-1}-2} u + d_m(x,t) |u|^{q_m - 1} \ in \ Q,$$

$$a(t) \sum_{h,k=1}^N \frac{\partial}{\partial x_k} \left(\left| \frac{\partial^2 u}{\partial x_h \partial x_k} \right|^{p-2} \frac{\partial^2 u}{\partial x_h \partial x_k} \right) \nu_h = -\sum_{j=1}^{m-2} d_j(x,t) |\nabla u|^{q_j - 2} \frac{\partial u}{\partial \nu} \ on \ \Sigma,$$

$$\sum_{k=1}^N \left| \frac{\partial^2 u}{\partial x_h \partial x_k} \right|^{p-2} \frac{\partial^2 u}{\partial x_h \partial x_k} \nu_k = 0 \ on \ \Sigma \ as \ h = 1,\dots,N,$$

$$u\left(x,0\right)=u\left(x,T\right)\ and\ \frac{\partial u}{\partial t}\left(x,0\right)=\frac{\partial u}{\partial t}\left(x,T\right)\ \ on\ \ \ \Omega. \tag{3.65}$$

We note that

$$V^{+}\left(D_1,...,D_{m-1}\right)=W_T\setminus\left\{0\right\}\ ,\left(i_{21}\right)holds\ if\,\lambda<0.$$

Then

Proposition 3.19. (Theor.2.2 (case (c_6))).Under conditions (3.63), with $\lambda<0$ problem (3.65) has at least one weak solution u_0 ($u_0=r_0v_0$, $r_0=const.>0$, $v_0\in S_\lambda$).

Proposition 3.20. Let $\lambda<0$. Let one of the following conditions be fulfilled:

There exist a measurable set $I\subseteq\left[0,T\right]with\left|I\right|_1>0$ and a limit point t_0 of I such that

$\lim\limits_{\substack{t\to t_0\\ t\in I}} d_j\left(x,t\right)=0$ for almost any $x\in\Omega$ and as j=1,…,m;

There exist an open interval $I\subseteq\left]0,T\right[$, $g\in C^0\left(I\right)with\,g\left(t\right)>0\,\forall t\in I$ and $g\sim\eta\, a$ in I $\forall\eta>0$, $g_j\in L^\infty\left(\Omega\right)$ with $g_j>0\ a.e.\ in\,\Omega$ such that $d_j\left(x,t\right)=g_j\left(x\right)g\left(t\right)$ as almost every $x\in\Omega$ and as each $t\in I\ \left(j=1,...,m\right)$;

There exist $t_0\in\left]0,T\right[\,and\,\varepsilon_0>0\ as\,in\left(c^*\right)$such that for almost any $x\in\Omega$ $d_j\left(x,\cdot\right)\in C^0\left(\left[t_0-\varepsilon_0,t_0+\varepsilon_0\right]\right)$ $as\,j=.1,...,m-1$ and $\sum_{j=1}^{m-2}\int_Q\left[d_j\left(x,t\right)-a\left(t\right)\left(a\left(t_0\right)\right)^{-1}d_j\left(x,t_0\right)\right]\left|\nabla u_0\right|^{q_j}dxdt+\int_Q\left[d_{m-1}\left(x,t\right)-a\left(t\right)\left(a\left(t_0\right)\right)^{-1}d_{m-1}\left(x,t_0\right)\right]\left|u_0\right|^{q_{m-1}}dxdt\neq0$,

$d_m\left(x,t\right)=g_m\left(x\right)a\left(t\right)$ as almost each $x\in\Omega$ and for each $t\in\left[0,T\right]$where $g_m\in L^\infty\left(\Omega\right)\ and\ g_m>0\ a.e.\,in\,\Omega.$

Then u_0 is nonstationary.

Proof. We reason as in Prop. (3.18). □

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