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Fuzzy Derivations BCC-Ideals on BCC-Algebras

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Abstract: In the theory of rings, the properties of derivations are important. In [15], Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some properties of its .In this manuscript, the concept of fuzzy left (right) derivations BCC-ideals in BCC-algebras is introduced and then investigate their basic properties. In connection with the notion of homomorphism, the authors study how the image and the pre-image of fuzzy left (right) derivations BCC-ideals under homomorphism of BCC-algebras become fuzzy left (right) derivations BCC-ideals. Furthermore, the Cartesian product of fuzzy left (right) derivations BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

Keywords: BCC-Ideals, Fuzzy Left (Right)-Derivations, the Cartesian Product of Fuzzy Derivations

1. Introduction

In 1966 Iami and Iseki [13, 14] introduced the notion of BCK-algebras .Iseki [11, 12] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. A BCC-algebra is an important class of logical algebras introduced by Y. Komori [16] and was extensively investigated by many researcher's see [1, 3, 4, 5, 6, 7, 8]. The concept of fuzzy sets was introduced by Zadeh [21]. O. G. Xi [20] applied the concept of fuzzy set s to BCK-algebras. In the theory of rings, the properties of derivations are important. In [15], Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some of its properties, defined a d-derivation ideal and gave conditions for an ideal to be d-derivation. Two years later, Hamza and Al-Shehri [9, 10] studied derivation in BCK-algebras, a left derivation in BCI-algebras and investigated a regular left derivation of BCI-algebras. C. Prabpayak, U. Leerawat [18] applied the notion of a regular derivation to BCC algebras and investigated some related properties.

In this paper, the authors consider the the concept of fuzzy left (right) derivations BCC-ideals in BCC-algebras and investigate some properties of it. Moreover, the concepts of the image and the pre-image of fuzzy left (right) derivations BCC-ideals under homomorphism of BCC-algebras is given and studies some its properties. The Cartesian product of fuzzy left (right) derivations BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1 [16] A BCC-algebra (X , $*$, 0) is a non-empty set X with a constant 0 and a binary operation $*$ such that for all $x , y , z \in$ X satisfying the following axioms:

$$\text{(BCC-1)}\ ((x*y)*(z*y))*(x*z)=0.$$

$$\text{(BCC-2)}\ x*\ 0=x\,.$$

$$\text{(BCC-3)}\ x*x=0.$$

$$\text{(BBC-4)}\ 0*x=0.$$

$$\text{(BCC-5)}\quad x*y=y*x=0\ \ implies\ \ x=y.$$

Definition 2.2 [5] Let (X , $*$, 0) be a BCC-algebra, we can define a binary relation $\leq$ on X as, $x \leq y$ if and only if x

$* y = 0$, this makes $(X, \leq)$ as a partially ordered set.

Proposition 2.3 [8] Let (X , $*$, 0) be a BCC-algebra. Then the following hold $\forall\ x, y, z \in X$.

1. $(x * y) * x = 0.$
2. $x \leq y \quad implies \quad x * z \leq y * z.$
3. $x \leq y \quad implies \quad z * y \leq z * x.$
4. $(x * y) * (z * y) \leq x * z.$

For elements x and y of a BCC-algebra $X = (X, *, 0)$ denote $x \wedge y = y * (y * x)$.

Lemma 2.4 [8] Let $(X, *, 0)$ be a BCC-algebra. Then the following hold $\forall\ x, y \in X$.

1. $0 \wedge x = 0.$
2. $x \wedge y \leq y.$

Example 2.5 [8] Let $X = \{0,1,2,3\}$ be a set in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Then $(X, *, 0)$ is a BCC-algebra.

Definition 2.6 [8] Let (X , $*$, 0) be a BCC-algebra and S be a non-empty subset of X , then S is called subalgebra of X if $x * y \in S \ \ \forall x, y \in S$.

Definition 2.7 [8] Let $(X, *, 0)$ be a BCC-algebra and A be a non-empty subset of X , then A is called ideal of X if it satisfied the following conditions:

1. $0 \in A,$
2. $x * y \in A \ , y \in A \ \ implies \ x \in A \qquad \forall x, y \in X.$

Definition 2.8 [8] Let $(X, *, 0)$ be a BCC-algebra and A be a non-empty subset of X , then A is called BCC- ideal of X if it satisfied the following conditions:

1. $0 \in A,$
2. $(x * y) * z \in A, y \in A \ implies \ x * z \in A \quad \forall x, y, z \in X.$

Definition 2.9 [6] Let $(X, *, 0)$ be a BCC-algebra, a fuzzy set μ in X is called a fuzzy subalgebra if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \qquad \forall\, x, y \in X.$

Definition 2.10 [6] Let (X , $*$, 0) be a BCC-algebra, a fuzzy set μ in X is called a fuzzy BCC-ideal of X if it satisfied the following conditions:

(F_1) $\mu(0) \geq \mu(x),$

(F_2) $\mu(x * z) \geq \min \{\mu((x * y) * z), \mu(y)\} \qquad \forall x, y, z \in X.$

Definition 2.11 [8] Let $(X, *, 0)$ is a BCC-algebra, $x, y \in X$,we denote $x \wedge y = y * (y * x)$.

Definition 2.12 [18] Let $(X, *, 0)$ be a BCC-algebra. A map d: $X \rightarrow X$ is called a left- right derivation (briefly (l, r)-derivation) of X if

$d(x * y) = (d(x) * y) \wedge (x * d(y)) \ \ \forall\ x, y \in X$.

Similarly, a map d: $X \rightarrow X$ is called a right- left derivation (briefly (r, l) -derivation) of X if

d ($x * y) = (x * d(y)) \wedge (d(x) * y) \quad \forall\ x, y \in X$.

A map d: $X \rightarrow X$ is called a derivation of X if d is both a (l, r)-derivation and a (r, l) -derivation of X .

Example 2.13 [18] Let $X = \{0,1,2,3\}$ be a BCC-algebra, in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Define a map d: $X \rightarrow X$ by

$d(x) = \begin{cases} 0 & if\ x = 0,1,3 \\ 2 & if\ x = 2. \end{cases}$ .Then it is clear that d is a derivation of X .

Example 2.14 Let $X = \{0,1,2,3\}$ be a BCC-algebra, in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Define a map d: $X \rightarrow X$ by

$d(x) = \begin{cases} x & if\ x = 1,3 \\ 0 & if\ x = 0,2. \end{cases}$

Then d is a (r, l) -derivation of X but is not a (l, r)-derivation of X .

Definition 2.15 [18] Let be a BCC-algebra and d: $X \rightarrow X$ be a map of a QS-algebra X , then d is called regular if d (0)=0.

Lemma 2.16 [18] A derivation d of BCC algebra X is regular.

Proposition 2.17 [18] Let $(X, *, 0)$ be a BCC-algebra with partial order $\leq$, and let d be a derivation of X. Then the following hold for all $x, y \in X$:

1. $d(x) \leq x,$
2. $d(x * y) \leq d(x) * y,$
3. $d(x * y) \leq x * d(y),$
4. $d(d(x)) \leq x,$
5. $d(x * d(x)) = 0,$
6. $d^{-1}(0) = \{x \in X : d(x) = 0\}$ is a sub-algebra of X.

Definition 2.18 Let X be a BCC-algebra and be a derivation of X.

Denote $Fix_d(X) = \{x \in X : d(x) = x\}$.

Proposition 2.19 [18] X be a BCC-algebra and d be a derivation of X. Then $Fix_d(X)$ is a subalgebra of X.

3. Fuzzy Derivations BCC-Ideals on BCC-Algebras

In this section, we will discuss and investigate a new notion is called fuzzy left (right) derivations BCC-ideals on BCC-algebras and study several basic properties which are related to fuzzy left (right) derivations BCC-ideals.

Definition 3.1 Let $(X,*,0)$ be a BCC-algebra. and $d : X \to X$ be a self map. A non-empty subset A of a BCC-algebra X is called left derivations BCC-ideal of

If it satisfies the following conditions:

1. $0 \in A$,
2. $d(x*y)*z \in A, d(y) \in A$ *implies* $d(x*z) \in A$ $\forall x, y, z \in X$.

Definition 3.2 Let (X , $*$,0) be a BCC-algebra. and $d : X \to X$ be a self map. A non-empty subset A of a BCC-algebra X is called right derivations BCC-ideal of X

If it satisfies the following conditions:

1. $0 \in A$,
2. $(x*y)*d(z) \in A$,$d(y) \in A$ *implies* $d(x*z) \in A$ $\forall x, y, z \in X$.

Definition 3.3 Let $(X,*,0)$ be a BCC-algebra. and $d : X \to X$ be a self map. A non-empty subset A of a BCC-algebra X is called derivations BCC-ideal of X

If it satisfies the following conditions:

1. $0 \in A$,
2. $d((x*y)*z) \in A$,$d(y) \in A$ *implies* $d(x*z) \in A$ $\forall x, y, z \in X$.

Definition 3.4 Let $(X,*,0)$ be a BCC-algebra. and $d : X \to X$ be a self map. A fuzzy set $\mu : X \to [0,1]$ in X is called a fuzzy left derivations BCC-ideal (briefly (F,l) - derivation) of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x) \quad \forall x \in X$,

(FL_2) $\mu(d(x*z)) \geq \min\{\mu(d(x*y)*z), \mu(d(y))\}$ $\forall x, y, z \in X$.

Definition 3.5 Let $(X,*,0)$ be a BCC-algebra. and $d : X \to X$ be a self map. A fuzzy set $\mu : X \to [0,1]$ in X is called a fuzzy right derivations BCC-ideal (briefly (F,r) - derivation) of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x) \quad \forall x \in X$,

(FR_2) $\mu(d(x*z)) \geq \min\{\mu((x*y)*d(z)), \mu(d(y))\}$ $\forall x, y, z \in X$.

Definition 3.6 Let $(X,*,0)$ be a BCC-algebra. and $d : X \to X$ be a self map.

A fuzzy set $\mu : X \to [0,1]$ in X is called a fuzzy derivations BCC-ideal of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(d(x)) \quad \forall x \in X$,

(F_2) $\mu(d(x*z)) \geq \min\{\mu(d((x*y)*z)), \mu(d(y))\}$ $\forall x, y, z \in X$.

Remark 3.7

1. If d is fixed, the definitions (3.1., 3.2., 3.3.) gives the definition BCC-ideal.
2. If d is fixed, the definitions (3.4., 3.5., 3.6.) gives the definition fuzzy BCC-ideal.

Example 3.8 Let $X = \{0,1,2,3\}$ be a BCC-algebra, in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & if\ x = 0,1,3 \\ 2 & if\ x = 2 \end{cases}$$

Define a fuzzy set $\mu : d(x) \to [0,1]$, by $\mu(d(0)) = t_0$, $\mu(d(1)) = t_1$, $\mu(d(2)) = \mu(d(3)) = t_2$, where t_0, t_1, $t_2 \in [0,1]$ with $t_0 > t_1 > t_2$. Routine calculations give that μ is not fuzzy left (right)-derivations BCC-ideal of BCC-algebra.

Example 3.9 Let $X = \{0,1,2,3,4,5\}$ be a BCC-algebra, in which the operation $*$ is defined as follows:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & if\ x = 0,1,2,3,4 \\ 5 & if\ x = 5 \end{cases}$$

Define a fuzzy set $\mu : d(x) \to [0,1]$, by $\mu(d(0)) = t_0$, $\mu(d(1)) = t_1$, $\mu(d(2)) = \mu(d(3)) = t_2$, $\mu(d(4)) = \mu(d(5)) = t_2$, where t_0, t_1, $t_2 \in [0,1]$ with

$t_0 > t_1 > t_2$. Routine calculations give that μ is fuzzy left (right)-derivations BCC-ideal of BCC-algebra.

Theorem 3.10 Let μ be a fuzzy left derivations BCC-ideal of BCC-algebra X.

1. If $x \le d(y)$, *then* $\mu(d(x)) \ge \mu(d(y))$
2. If $x * y \le d(x)$, *then* $\mu(d(x*y)) \ge \mu(d(x))$
3. If $(x*y)*(z*y) \le d(x*z)$, *then* $\mu(d((x*y)*(z*y))) \ge \mu(d(x*z))$
4. If $\mu(d(x*y)) = \mu(d(0))$, *then* $\mu(d(x)) \ge \mu(d(y))$.

Pr*oof*. 1. Let $x \le d(y)$ and since $\overbrace{d(y) \le y}^{from\ \mathrm{Pro}\ 2.16.\ 1}$,

hence $x \le y$ *i.e.* $x * y = 0$, then

$$\mu(d(x)) = \mu(d(x*0)) \ge \overbrace{\min\{\mu(d(x*y)*0), \mu(d(y))\}}^{from\ Def\ 3.4.\ (FL_2)}$$
$$= \min\{\mu(d(x*y)), \mu(d(y))\}$$
$$= \min\{\mu(d(0)), \mu(d(y))\}$$
$$= \min\{\mu(0), \mu(d(y))\} = \mu(d(y)).$$

2. Let $x * y \le d(x)$, then by Theorem 3.10.1, we get $\mu(d(x*y)) \ge \mu(d(x))$.

3. Let $(x*y)*(z*y) \le d(x*z)$, then by theorem 3.10.1, we get $\mu(d((x*y)*(z*y))) \ge \mu(d(x*z))$.

4. Let $\mu(d(x*y)) = \mu(d(0))$, then

$$\mu(d(x)) = \mu(d(x*0))$$
$$\ge \overbrace{\min\{\mu(d(x*y)*0), \mu(d(y))\}}^{from\ Def\ 3.4.\ (FL_2)}$$
$$= \min\{\mu(d(x*y)), \mu(d(y))\}$$
$$= \min\{\mu(d(0)), \mu(d(y))\} = \min\{\mu(0), \mu(d(y))\} = \mu(d(y)).$$

Proposition 3.11 The intersection of any set of fuzzy left derivations BCC-ideals of BCC-algebra X is also fuzzy left derivations BCC-ideal.

Pr*oof*. Let $\{\mu_i\}$ be a family of fuzzy left derivations BCC-ideals of BCC-algebra X,

then $\forall$ $x, y, z \in X$,

$(\cap\mu_i)(0) = \inf(\mu_i(0)) \ge \inf(\mu_i(d(x))) = (\cap\mu_i)(d(x))$ and

$$(\cap\mu_i)(d(x*z)) = \inf(\mu_i(d(x*z))$$
$$\ge \inf(\min\{\mu_i(d(x*y)*z), \mu_i(d(y))\}$$
$$= \min\{\inf(\mu_i(d(x*y)*z)), \inf(\mu_i(d(y)))\}$$
$$= \min\{(\cap\mu_i)(d(x*y)*z), (\cap\mu_i)(d(y))\}.$$

Lemma 3.12 The intersection of any set of fuzzy right derivations BCC-ideals of BCC-algebra X is also fuzzy right derivations BCC-ideal.

Pr*oof*. Clear

Theorem 3.13 Let μ be a fuzzy set in X, then μ is a fuzzy left derivations BCC-ideal of X if and only if it satisfies : $\forall$ $\alpha \in [0, 1]$), U(μ , α) ≠ φ implies U(μ ,α) is BCC-ideal of X …(A), where U (μ , α) = {x ∈ X / μ (d(x)) ≥ α}.

Pr*oof*. Assume that μ is a fuzzy left derivations BCC-ideal of X, let $\alpha \in [0,1]$ be such that $\mathrm{U}(\mu,\alpha) \ne \phi$ and $x, y \in X$ such that $x \in \mathrm{U}(\mu,\alpha)$, then $\mu(d(x)) \ge \alpha$ and so by (FL_2),

$$\mu(d(0)) = \mu(d(0*y)) \ge \min\{\mu(d(0*x)*y), \mu(d(x))\}$$
$$= \min\{\mu(d(0)*y), \mu(d(x))\}$$
$$= \min\{\mu(0*y), \mu(d(x))\} = \min\{\mu(0), \mu(d(x))\} = \alpha,$$

hence $0 \in \mathrm{U}(\mu,\alpha)$.

Let $d(x*y)*z \in \mathrm{U}(\mu,\alpha)$ and $d(y) \in (\mu,\alpha)$,

it follows from (FL_2) that

$$\mu(d(x*z)) \ge \min\{\mu(d(x*y)*z), \mu(d(y))\} = \alpha,$$

so that $x * z \in \mathrm{U}(\mu,\alpha)$. Hence $\mathrm{U}(\mu,\alpha)$ is BCC-ideal of X.

Conversely, suppose that μ satisfies (A), let $x, y, z \in X$ be such that

$\mu(d(x*z)) < \min\{\mu(d(x*y)*z), \mu(d(y))\}$, taking

$\beta_0 = 1/2\{\mu(d(x*z)) + \min\{\mu(d(x*y)*z), \mu(d(y))\}\}$,

we have $\beta_0 \in [0,1]$ and

$$\mu(d(x*z)) < \beta_0 < \min\{\mu(d(x*y)*z), \mu(d(y))\},$$

it follows that $d(x*y)*z \in \mathrm{U}(\mu,\beta_0)$ and $\mathrm{U}(\mu,\beta_0)$, this is a contradiction and therefore μ is a fuzzy left derivations BCC-ideal of X.

Theorem 3.14 Let μ be a fuzzy set in X, then μ is a fuzzy right derivations BCC-ideal of X if and only if it satisfies : $\forall$ $\alpha \in [0, 1]$), U(μ , α) ≠ φ implies U(μ ,α) is BCC-ideal of X …(A), where U (μ , α) = {x ∈ X / μ (d(x)) ≥ α}.

Pr*oof*. Clear

Definition 3.15 Let μ be a fuzzy derivations BCC-ideal of BCC-algebra X, the BCC-ideals μ_t, $t \in [0,1]$ are called level BCC-ideal of μ.

4. Image (Pre-image) of Fuzzy Derivations BCC-Ideals Under Homomorphism

In this section, we introduce the concepts of the image and the pre-image of fuzzy left and right derivations BCC-ideals in BCC-algebras under homomorphism of BCC-algebras

Definition 4.1 Let f be a mapping from the set X to a set

Y. If μ is a fuzzy subset of X, then the fuzzy subset β of Y is defined by

$$f(\mu)(y)=\beta(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu(x), & if\ f^{-1}(y)=\{x\in X, f(x)\equiv y\}\neq\varphi \\ 0 & otherwise\end{cases}$$ Is

said to be the image of μ under f

Similarly if β is a fuzzy subset of Y, then the fuzzy subset $\mu=\beta\circ f$ in X (i.e. the fuzzy subset is defined by $\mu(x)=\beta(f(x))\quad\forall\ x\in X$) is called the preimage of β under f.

Theorem 4.2 An onto homomorphic preimage of a fuzzy right derivations BCC-ideal is also a fuzzy right derivations BCC-ideal under homomorphism of BCC-algebras

Pr*oof*. Let $f: X\to X'$ be an onto homomorphism of BCC-algebras, β a fuzzy right derivations BCC-ideal of X' and μ the preimage of β under f, then $\beta(f(d(x)))=\mu(d(x)),\ \forall\ x\in X$. Let $x\in X$, we have $\mu(d(0))=\beta(f(d(0)))\geq\beta(f(d(x)))=\mu(d(x))$.

Now let $x,y,z\in X$, then

$$\mu(d(x*z))=\beta(f(d(x*z)))$$
$$\geq\min\{\beta((f(x)*'f(y))*'f(d(z))),\beta(f(d(y)))\}$$
$$=\min\{\beta((x*y)*f(d(z))),\beta(f(d(y)))\}$$
$$=\min\{\mu((x*y)*d(z)),\mu(d(y))\}.$$

The proof is completed.

Theorem 4.3 An onto homomorphic preimage of a fuzzy left derivations BCC-ideal is also a fuzzy left derivations BCC-ideal.

Pr*oof*. Clear

Definition 4.4 [2] A fuzzy subset μ of X has sup property if for any subset T of X,

there exist $t_0\in T$ such that, $\mu(t_0)=\sup\limits_{t\in T}\mu(t)$.

Theorem 4.5 Let $f: X\to Y$ be a homomorphism between BCC-algebras X and Y. For every fuzzy left derivations BCC-ideal μ in X, $f(\mu)$ is a fuzzy left derivations BCC-ideal of Y.

Pr*oof*. By definition

$$\beta(d(y'))=f(\mu)(d(y'))=\sup_{d(x)\in f^{-1}(d(y'))}\mu(d(x))$$

$\forall\ y'\in Y$ and $\sup\varphi=0$ .We have to prove that $\beta(d(x'*z'))\geq\min\{\beta(d(x'*y')*z'),\beta(d(y'))\}$, $\forall\ x',y',z'\in Y$.

Let $f: X\to Y$ be an onto a homomorphism of BCC-algebras, μ a fuzzy left derivations BCC-ideal of X with sup property and β the image of μ under f, since μ is a fuzzy left derivations BCC-ideal of X, we have μ(d(0)) ≥ μ(d(x))

$\forall$ x∈ X. Note that $0\in f^{-1}(0')$, where 0, $0'$ are the zero of X and Y respectively. Thus,

$$\beta(d(0'))=\sup_{d(t)\in f^{-1}(d(0'))}\mu(d(t))=\mu(d(0))=\mu(0)\geq\mu(d(x)),$$

$\forall\ x\in X$, which implies that

$\beta(d(0'))\geq\sup\limits_{d(t)\in f^{-1}(d(x'))}\mu(d(t))=\beta(d(x')),\ \forall\ x'\in Y$. $\forall$ $x', y', z'\in Y$, let

$d(x_0)\in f^{-1}(d(x'))$,
$d(y_0)\in f^{-1}(d(y'))$,
$d(z_0)\in f^{-1}(d(z'))$

be such that

$$\mu(d(x_0*z_0))=\sup_{d(t)\in f^{-1}(d(x'*z'))}\mu(d(t)),\ \mu(y_0)=\sup_{d(t)\in f^{-1}(d(y'))}\mu(d(t))$$ and

$$\mu(d(x_0*y_0)*z_0)=\beta\{f(d(x_0*y_0)*z_0)\}$$
$$=\beta(d(x'*y')*z')$$
$$=\sup_{(d(x_0*y_0)*z_0)\in f^{-1}(d(x'*y')*z')}\mu(d(x_0*y_0)*z_0)$$
$$=\sup_{d(t)\in f^{-1}(d(x'*y')*z')}\mu(d(t)).$$

Then

$$\beta(d(x'*z'))=\sup_{d(t)\in f^{-1}(d(x'*z'))}\mu(d(t))=\mu(d(x_0*z_0))$$
$$\geq\min\{\mu(d(x_0*y_0)*z_0),\mu(d(y_0))\}=$$
$$\min\left\{\sup_{d(t)\in f^{-1}(d(x'*y')*z')}\mu(d(t))\ ,\ \sup_{d(t)\in f^{-1}(d(y'))}\mu(d(t))\right\}=$$
$$\min\{\beta(d(x'*y')*z'),\ \ \beta(d(y'))\}.$$

Hence β is a fuzzy left derivations BCC-ideal of Y.

Theorem 4.6 Let $f: X\to Y$ be a homomorphism between BCC-algebras X and Y. For every fuzzy right derivations BCC-ideal μ in X, $f(\mu)$ is a fuzzy right derivations BCC-ideal of Y.

Pr*oof*. Clear

5. Cartesian Product of Fuzzy Left Derivations BCC-ideals

Definition 5.1 [2] A fuzzy μ is called a fuzzy relation on any set S, if μ is a fuzzy subset $\mu: S\times S\to[0,1]$.

Definition 5.2 [2] If μ is a fuzzy relation on a set S and is a fuzzy subset of S, then μ is a fuzzy relation on β if $\mu(x,y)\leq\min\{\beta(x),\beta(y)\},\quad\forall\ x,y\in S$.

Definition 5.3 [2] Let μ and β be a fuzzy subset of a set S, the Cartesian product of μ and β is defined by $(\mu\times\beta)(x,y)=\min\{\mu(x),\beta(y)\},\quad\forall\ x,y\in S$.

Lemma 5.4 [2] Let μ and β be a fuzzy subset of a set S, then

(i) $\mu\times\beta$ is a fuzzy relation on S.

$(ii)(\mu\times\beta)_t=\mu_t\times\beta_t \quad \forall\ t\in[0,1]$.

Definition 5.5 If μ is a fuzzy derivations relation on a set S and β is a fuzzy derivations subset of S, then μ is a fuzzy derivations relation on β if

$$\mu(d(x,y))\le\min\{\beta(d(x)),\beta(d(y))\},\quad \forall\ x,y\in S.$$

Definition 5.6 Let μ and β be a fuzzy derivations subset of a set S, the Cartesian product of μ and β is defined by $(\mu\times\beta)(d(x,y))=\min\{\mu(d(x)),\beta(d(y))\},\quad \forall\ x,y\in S$.

Definition 5.7 If β is a fuzzy derivations subset of a set S, the strongest fuzzy relation on S, that is a fuzzy derivations relation on β is μ_β given by

$$\mu_\beta(d(x,y))=\min\{\beta(d(x)),\beta(d(y))\},\quad \forall\ x,y\in S.$$

Lemma 5.8 [2] For a given fuzzy derivations subset of a set S, let μ_β be the strongest fuzzy derivations relation on S, then for $t\in[0,1]$, we have $(\mu_\beta)_t=\beta_t\times\beta_t$.

Proposition 5.9 For a given fuzzy derivations subset β of BCC-algebra X, let μ_β be the strongest fuzzy derivations relation on X. If μ_β is a fuzzy derivations BCC-ideal of $X\times X$, then $\beta(d(x))\le\beta(d(0))=\beta(0) \quad \forall\ x\in X$.

Proof. Since μ_β is a fuzzy derivations BCC-ideal of $X\times X$, it follows from (F_1) that

$$\mu_\beta(x,x)=\min\{\beta(d(x)),\beta(d(x))\}\le\beta(d(0,0))$$
$$=\min\{\beta(d(0)),\beta(d(0))\}$$

where $(0,0)\in X\times X$, then $\beta(d(x))\le\beta(d(0))=\beta(0)$.

Remark 5.10 Let X and Y be BCC-algebras, we define $*$ on $X\times Y$ by $(x,y)*(u,v)=(x*u,y*v) \quad \forall\ (x,y),(u,v)\in X\times Y$, then clearly $(X\times Y,*,(0,0))$ is a BCC-algebra.

Theorem 5.11 Let μ and β be a fuzzy derivations BCC-ideals of BCC-algebra X, then $\mu\times\beta$ is a fuzzy derivations BCC-ideal of $X\times X$.

Proof. 1.

$$(\mu\times\beta)(d(0,0))=\min\{\mu(d(0)),\beta(d(0))\}=\min\{\mu(0),\beta(0)\}$$
$$\ge\min\{\mu(d(x)),\beta(d(x))\}=(\mu\times\beta)(d(x,y))$$
$$\forall\ (x,y)\in X\times X.$$

2. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$, then

$$(\mu\times\beta)(d(x_1*z_1,x_2*z_2))=\min\{\mu(d(x_1,z_1)),\beta(d(x_2,z_2))\}$$
$$\ge\min\begin{Bmatrix}\min\{\mu(d((x_1*y_1)*z_1)),\mu(d(y_1))\}\ ,\\ \min\{\beta(d((x_2*y_2)*z_2)),\beta(d(y_2))\}\end{Bmatrix}$$
$$=\min\begin{Bmatrix}\min\{\mu(d((x_1*y_1)*z_1)),\beta(d((x_2*y_2)*z_2))\}\ ,\\ \min\{\mu(d(y_1)),\beta(d(y_2))\}\end{Bmatrix}$$
$$=\min\begin{Bmatrix}(\mu\times\beta)((d((x_1*y_1)*z_1)),(d((x_2*y_2)*z_2)))\ ,\\ (\mu\times\beta)(d(y_1),d(y_2))\end{Bmatrix}.$$

Hence $\mu\times\beta$ is a fuzzy derivations BCC-ideal of $X\times X$.

Analogous to theorem 2.2[17], we have a similar result for fuzzy derivations BCC-ideal, which can be proved in similar manner, we state the result without proof.

Theorem 5.12 Let μ and β be a fuzzy derivations subset of BCC-algebra X, Such that $\mu\times\beta$ is a fuzzy derivations BCC-ideal of $X\times X$, then

(i) either $\mu(d(x))\le\mu(d(0))$ or $\beta(d(x))\le\beta(d(0)) \quad \forall\ x\in X$,

(ii) if $\mu(d(x))\le\mu(d(0)) \quad \forall\ x\in X$, then either $\mu(d(x))\le\beta(d(0))$ or $\beta(d(x))\le\beta(d(0))$,

(iii) if $\beta(d(x))\le\beta(d(0)) \quad \forall\ x\in X$, then either $\mu(d(x))\le\mu(d(0))$ or $\beta(d(x))\le\mu(d(0))$,

(iv) either μ or β is a fuzzy derivations BCC-ideal of X.

Theorem 5.13 Let β be a fuzzy derivations subset of BCC-algebra X and let μ_β be the strongest fuzzy derivations relation on then β is a fuzzy derivations BCC-ideal of X if and only if μ_β is a fuzzy derivations BCC-ideal of $X\times X$.

Proof. Let β be a fuzzy derivations BCC-ideal of X, 1. From (F_1), we get

$$\mu_\beta(0,0)=\min\{\beta(d(0)),\beta(d(0))\}=\min\{\beta(0),\beta(0)\}\ge$$
$$\min\{\beta(d(x)),\beta(d(y))\}=\mu_\beta(d(x),d(y))$$

2. $\forall\ (x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$, we have from ($F_2$)

$$\mu_\beta(d(x_1*z_1),d(x_2*z_2))=\min\{\beta(d(x_1*z_1)),\beta(d(x_2*z_2))\}$$
$$\ge\min\begin{Bmatrix}\min\{\beta(d((x_1*y_1)*z_1)),\beta(d(y_1))\}\ ,\\ \min\{\beta(d((x_2*y_2)*z_2)),\beta(d(y_2))\}\end{Bmatrix}$$
$$=\min\begin{Bmatrix}\min\{\beta(d((x_1*y_1)*z_1)),\beta(d((x_2*y_2)*z_2))\}\ ,\\ \min\{\beta(d(y_1)),\beta(d(y_2))\}\end{Bmatrix}$$
$$=\min\begin{Bmatrix}\mu_\beta((d((x_1*y_1)*z_1)),(d((x_2*y_2)*z_2)))\ ,\\ \mu_\beta(d(y_1),d(y_2))\end{Bmatrix}.$$

Hence μ_β is a fuzzy derivations BCC-ideal of $X\times X$.

Conversely, let μ_β be a fuzzy derivations BCC-ideal of $X\times X$,

1. $\forall\ (x,y)\in X\times X$, we have

$$\min\{\beta(0),\beta(0)\}=\mu_\beta(x,y)=\min\{\beta(x),\beta(y)\}.$$

It follows that $\beta(0)\ge\beta(x) \quad \forall\ x\in X$, which prove ($F_1$).

2. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X,$ then

$\min\left\{\beta(d(x_1*z_1)),\beta(d(x_2*z_2))\right\}$

$\mu_\beta(d(x_1*z_1),d(x_2*z_2))$

$\geq \min\left\{\mu_\beta(d((x_1,x_2)*(y_1,y_2))*(z_1,z_2)),\mu_\beta(d(y_1),d(y_2))\right\}$

$$= \min\begin{Bmatrix}\mu_\beta(d((x_1*y_1)*z_1),d((x_2*y_2)*z_2)),\\ \mu_\beta(d(y_1),d(y_2))\end{Bmatrix}$$

$$= \min\begin{Bmatrix}\min\{\beta(d((x_1*y_1)*z_1)),\beta(d((x_2*y_2)*z_2))\},\\ \min\{\beta(d(y_1)),\beta(d(y_2))\}\end{Bmatrix}$$

$$= \min\begin{Bmatrix}\min\{\beta(d((x_1*y_1)*z_1)),\beta(d(y_1))\},\\ \min\{\beta(d(x_2*y_2)*z_2),\beta(d(y_2))\}\end{Bmatrix}$$

In particular, if we take $x_2=y_2=z_2=0,$ then $\beta(d(x_1*z_1))\geq \min\left\{\beta(d(x_1*y_1)*z_1),\beta(d(y_1))\right\}$. This prove ($F_2$)

Hence β be a fuzzy derivations BCC-ideal of X.

6. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, the notion of fuzzy left and right derivations BCC-ideal in BCC-algebra are introduced and investigated the useful properties of fuzzy left and right derivations BCC-ideals in BCC-algebras.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCI-algebra, BCH-algebra, Hilbert algebra, BF-algebra, J-algebra, WS-algebra, CI-algebra, SU-algebra, BCL-algebra, BP-algebra, Coxeter algebra, BO-algebra, PU- algebras and so forth.

The main purpose of our future work is to investigate:

(1) The interval value, bipolar and intuitionist fuzzy left and right derivations BCC-ideal in BCC-algebra.

(2) To consider the cubic structure left and right derivations BCC- ideal in BCC-algebra.

We hope the fuzzy left and right derivations BCC-ideals in BCC-algebras, have applications in different branches of theoretical physics and computer science.

Algorithm for BCC-algebras

Input (X : set, $*$: binary operation)
Output (“ X is a BCC-algebra or not”)
Begin
If $X=\varphi$ then go to (1.);
End If
If $0\notin X$ then go to (1.);
End If
Stop: =false;
$i:=1$;
While $i\leq |X|$ and not (Stop) do
If $x_i*x_i\neq 0$ then
Stop: = true;
End If
$j:=1$
$k:=1$
While $j,k\leq |X|$ and not (Stop) do
If $((x_i*y_j)*(z_k*y_j))*(x_i*z_k))\neq 0$ then
Stop: = true;
End If
End While
End While
If Stop then
Output (“ X is not a BCC-algebra”)
Else
Output (“ X is a BCC-algebra”)
End If
End.

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