Fuzzy Derivations BCC-Ideals on BCC-Algebras

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Abstract: In the theory of rings, the properties of derivations are important. In [15], Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some properties of its. In this manuscript, the concept of fuzzy left (right) derivations BCC-ideals in BCC-algebras is introduced and then investigate their basic properties. In connection with the notion of homomorphism, the authors study how the image and the pre-image of fuzzy left (right) derivations BCC-ideals under homomorphism of BCC-algebras become fuzzy left (right) derivations BCC-ideals. Furthermore, the Cartesian product of fuzzy left (right) derivations BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

Keywords: BCC-Ideals, Fuzzy Left (Right)-Derivations, the Cartesian Product of Fuzzy Derivations

1. Introduction

In 1966 Iami and Iseki [13, 14] introduced the notion of BCK-algebras. Iseki [11, 12] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. A BCC-algebra is an important class of logical algebras introduced by Y. Komori [16] and was extensively investigated by many researcher’s see [1, 3, 4, 5, 6, 7, 8]. The concept of fuzzy sets was introduced by Zadeh [21]. O. G. Xi [20] applied the concept of fuzzy sets to BCK-algebras. In the theory of rings, the properties of derivations are important. In [15], Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some of its properties, defined a d-derivation ideal and gave conditions for an ideal to be d-derivation. Two years later, Hamza and Al-Shehri [9, 10] studied derivation in BCK-algebras, a left derivation in BCI-algebras and investigated a regular left derivation of BCI-algebras. C. Prabpayak, U. Leerawat [18] applied the notion of a regular derivation to BCC-algebras and investigated some related properties.

In this paper, the authors consider the the concept of fuzzy left (right) derivations BCC-ideals in BCC-algebras and investigate some properties of it. Moreover, the concepts of the image and the pre-image of fuzzy left (right) derivations BCC-ideals under homomorphism of BCC-algebras is given and studies some its properties. The Cartesian product of fuzzy left (right) derivations BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1 [16] A BCC-algebra \((X, *, 0)\) is a non-empty set \(X\) with a constant \(0\) and a binary operation \(*\) such that for all \(x, y, z \in X\) satisfying the following axioms:

\[(\text{BCC-1}) \quad ((x * y) * (z * y)) * (x * z) = 0.\]

\[(\text{BCC-2}) \quad x * 0 = x.\]

\[(\text{BCC-3}) \quad x * x = 0.\]

\[(\text{BBC-4}) \quad 0 * x = 0.\]

\[(\text{BCC-5}) \quad x * y = y * x = 0 \implies x = y.\]

Definition 2.2 [5] Let \((X, *, 0)\) be a BCC-algebra, we can define a binary relation \(\leq\) on \(X\) as, \(x \leq y\) if and only if \(x \leq y\)
\( * \ y = 0, \) this makes \((X, \leq)\) as a partially ordered set.

**Proposition 2.3** [8] Let \((X, *, 0)\) be a BCC-algebra. Then the following hold \(\forall \ x, y, z \in X\):
1. \((x * y) * z = x\).
2. \(x \leq y \) implies \(x * z \leq y * z\).
3. \(x \leq y \) implies \(z * y \leq z * x\).
4. \((x * y) * (z * y) \leq x * z\).

For elements \(x\) and \(y\) of a BCC-algebra \(X = (X, *, 0)\) denote \(x \land y = y * (y * x)\).

**Lemma 2.4** [8] Let \((X, *, 0)\) be a BCC-algebra. Then the following hold \(\forall \ x, y \in X\):
1. \(0 \land x = x\).
2. \(x \land y \leq y\).

**Example 2.5** [8] Let \(X = \{0, 1, 2, 3\}\) be a set in which the operation \(*\) is defined as follows:

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Then \((X, *, 0)\) is a BCC-algebra.

**Definition 2.6** [8] Let \((X, *, 0)\) be a BCC-algebra and \(S\) be a non-empty subset of \(X\), then \(S\) is called subalgebra of \(X\) if \(x * y \in S \ \forall x, y \in S\).

**Definition 2.7** [8] Let \((X, *, 0)\) be a BCC-algebra and \(A\) be a non-empty subset of \(X\), then \(A\) is called ideal of \(X\) if it satisfied the following conditions:
1. \(0 \in A\).
2. \(x * y \in A, \ y \in A \) implies \(x \in A \ \forall x, y \in X\).

**Definition 2.8** [8] Let \((X, *, 0)\) be a BCC-algebra and \(A\) be a non-empty subset of \(X\), then \(A\) is called BCC-ideal of \(X\) if it satisfied the following conditions:
1. \(0 \in A\).
2. \(\forall x, y, z \in X\).

**Definition 2.9** [6] Let \((X, *, 0)\) be a BCC-algebra, a fuzzy set \(\mu\) in \(X\) is called a fuzzy subalgebra if \(\mu(x * y) \geq \min \{\mu(x), \mu(y)\} \ \forall x, y \in X\).

**Definition 2.10** [6] Let \((X, *, 0)\) be a BCC-algebra, a fuzzy set \(\mu\) in \(X\) is called a fuzzy BCC-ideal of \(X\) if it satisfied the following conditions:
1. \((F_1)\) \(\mu(0) \geq \mu(x)\).
2. \((F_2)\) \(\mu(x * z) \geq \min \{\mu(x), \mu(y)\} \ \forall x, y, z \in X\).

**Definition 2.11** [8] Let \((X, *, 0)\) be a BCC-algebra, \(x, y \in X\), we denote \(x \land y = y * (y * x)\).

**Definition 2.12** [18] Let \((X, *, 0)\) be a BCC-algebra. A map \(d: X \to X\) is called a left- right derivation (briefly \((l, r)\)-derivation) of \(X\) if\
\[d(x * y) = (d(x) * y) \land (x * d(y)) \ \forall x, y \in X\].

Similarly, a map \(d: X \to X\) is called a right- left derivation (briefly \((r, l)\)-derivation) of \(X\) if\
\[d(x * y) = (x * d(y)) \land (d(x) * y) \ \forall x, y \in X\].

A map \(d: X \to X\) is called a derivation of \(X\) if \(d\) is both a \((l, r)\)-derivation and a \((r, l)\)-derivation of \(X\).

**Example 2.13** [18] Let \(X = \{0, 1, 2, 3\}\) be a BCC-algebra, in which the operation \(*\) is defined as follows:

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Define a map \(d: X \to X\) by\
\[d(x) = \begin{cases} 
0 & \text{if } x = 0, 1, 3 \\
2 & \text{if } x = 2.
\end{cases}\]

Then it is clear that \(d\) is a derivation of \(X\).

**Example 2.14** Let \(X = \{0, 1, 2, 3\}\) be a BCC-algebra, in which the operation \(*\) is defined as follows:

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Define a map \(d: X \to X\) by\
\[d(x) = \begin{cases} 
x & \text{if } x = 1, 3 \\
0 & \text{if } x = 0, 2.
\end{cases}\]

Then \(d\) is a \((r, l)\)-derivation of \(X\) but is not a \((l, r)\)-derivation of \(X\).

**Definition 2.15** [18] Let \(X\) be a BCC-algebra and \(d: X \to X\) be a map of a QS-algebra \(X\), then \(d\) is called regular if \(d(0) = 0\).

**Lemma 2.16** [18] A derivation \(d\) of BCC-algebra \(X\) is regular.

**Proposition 2.17** [18] Let \((X, *, 0)\) be a BCC-algebra with partial order \(\leq\), and let \(d\) be a derivation of \(X\). Then the following hold for all \(x, y \in X\):
1. \(d(x) \leq x\).
2. \(d(x * y) \leq d(x) * y\).
3. \(d(x * y) \leq x * d(y)\).
4. \(d(d(x)) \leq x\).
5. \(d(x * d(x)) = 0\).
6. \(d^{-1}(0) = \{x \in X: d(x) = 0\}\) is a sub-algebra of \(X\).

**Definition 2.18** Let \(X\) be a BCC-algebra and be a
3. Fuzzy Derivations BCC-Ideals on BCC-Algebras

In this section, we will discuss and investigate a new notion called fuzzy left (right) derivations BCC-ideals on BCC-algebras and study several basic properties which are related to fuzzy left (right) derivations BCC-ideals.

Definition 3.1 Let \((X,*,0)\) be a BCC-algebra and \(d: X \to X\) be a self map. A non-empty subset \(A\) of a BCC-algebra \(X\) is called left derivations BCC-ideal of \(X\) if it satisfies the following conditions:
1. \(0 \in A\),
2. \((x*y)*z \in A, d(y) \in A \ implies \ (x*z) \in A\)

Definition 3.2 Let \((X,*,0)\) be a BCC-algebra and \(d: X \to X\) be a self map. A fuzzy set \(A\) is defined as follows:
1. \(0 \in A\),
2. \((x*y)*d(z) \in A, d(y) \in A \ implies \ (x*z) \in A\)

Definition 3.3 Let \((X,*,0)\) be a BCC-algebra and \(d: X \to X\) be a self map. A non-empty subset \(A\) of a BCC-algebra \(X\) is called right derivations BCC-ideal of \(X\) if it satisfies the following conditions:
1. \(0 \in A\),
2. \((x*y)*z \in A, d(y) \in A \ implies \ (x*z) \in A\)

Definition 3.4 Let \((X,*,0)\) be a BCC-algebra and \(d: X \to X\) be a self map. A fuzzy set \(A\) is defined as follows:
1. \(0 \in A\),
2. \((x*y)*d(z) \in A, d(y) \in A \ implies \ (x*z) \in A\)

Definition 3.5 Let \((X,*,0)\) be a BCC-algebra and \(d: X \to X\) be a self map. A fuzzy set \(A\) is called fuzzy left (right) derivations BCC-ideal of \(X\) if it satisfies the following conditions:
1. \(\mu(d(x*z)) \geq \min\{\mu((x*y)*d(z)), \mu(d(y))\}\)
2. \(\forall x, y, z \in X\)

Remark 3.7
1. If \(d\) is fixed, the definitions (3.1., 3.2., 3.3.) gives the definition fuzzy left (right)-derivations BCC-ideal of \(X\) if it satisfies the following conditions:
3. \((F_1) \ \mu(0) \geq \mu(d(x)) \ \forall x \in X\)
4. \((F_2) \ \mu(d(x*z)) \geq \min\{\mu((x*y)*d(z)), \mu(d(y))\}\)

Example 3.8 Let \(X = \{0,1,2,3\}\) be a BCC-algebra, in which the operation \(*\) is defined as follows:

Define a map \(d: X \to X\) by
\[
d(x) = \begin{cases} 0 & \text{if } x = 0, 1, 3 \\ 2 & \text{if } x = 2 \end{cases}
\]

Define a fuzzy set \(\mu: X \to [0,1]\) by \(\mu(d(0)) = t_0, \mu(d(1)) = t_1, \mu(d(2)) = \mu(d(3)) = t_2\), where \(t_0 < t_1 < t_2\). Routine calculations give that \(\mu\) is not fuzzy left (right)-derivations BCC-ideal of BCC-algebra.

Example 3.9 Let \(X = \{0,1,2,3,4,5\}\) be a BCC-algebra, in which the operation \(*\) is defined as follows:

Define a map \(d: X \to X\) by
\[
d(x) = \begin{cases} 0 & \text{if } x = 0, 1, 2, 3, 4, 5 \\ 5 & \text{if } x = 5 \end{cases}
\]

Define a fuzzy set \(\mu: X \to [0,1]\) by \(\mu(d(0)) = t_0, \mu(d(1)) = t_1, \mu(d(2)) = \mu(d(3)) = t_2, \mu(d(4)) = \mu(d(5)) = t_2\), where \(t_0, t_1, t_2 \in [0,1]\) with
If \( t_0 > t_1 > t_2 \), routine calculations give that \( \mu \) is fuzzy left (right)-derivations BCC-ideal of BCC-algebra.

**Theorem 3.10** Let \( \mu \) be a fuzzy left derivations BCC-ideal of BCC-algebra \( X \).

1. If \( x \leq d(y) \), then \( \mu(d(x)) \geq \mu(d(y)) \)
2. If \( x \ast y \leq d(x) \), then \( \mu(d(x \ast y)) \geq \mu(d(x)) \)
3. If \( (x \ast y) \ast (z \ast y) \leq d(x \ast z) \), then
\[
\mu(d((x \ast y) \ast (z \ast y))) \geq \mu(d(x \ast z))
\]
4. If \( \mu(d(x \ast y)) = \mu(d(0)) \), then \( \mu(d(x)) \geq \mu(d(y)) \).

**Proof.** 1. Let \( x \leq d(y) \) and since \( d(y) \leq y \), hence \( x \leq y \), i.e. \( x \ast y = 0 \), then
\[
\mu(d(x)) = \mu(d(x \ast 0)) \geq \min \{ \mu(d(x \ast y) \ast 0), \mu(d(0)) \}
\]
\[
\geq \min \{ \mu(d(x \ast y)), \mu(d(y)) \}
\]
\[
\geq \min \{ \mu(d(0)), \mu(d(y)) \}
\]
\[
\geq \min \{ \mu(0), \mu(d(y)) \} = \mu(d(y)).
\]
2. Let \( x \ast y \leq d(x) \), then by Theorem 3.10.1, we get
\[
\mu(d(x \ast y)) \geq \mu(d(x)).
\]
3. Let \( (x \ast y) \ast (z \ast y) \leq d(x \ast z) \), then by Theorem 3.10.1, we get
\[
\mu(d((x \ast y) \ast (z \ast y))) \geq \mu(d(x \ast z)).
\]
4. Let \( \mu(d(x \ast y)) = \mu(d(0)) \), then
\[
\mu(d(x)) = \mu(d(x \ast 0))
\]
\[
\geq \min \{ \mu(d(x \ast y) \ast 0), \mu(d(0)) \}
\]
\[
\geq \min \{ \mu(d(0)), \mu(d(y)) \}
\]
\[
\geq \min \{ \mu(0), \mu(d(y)) \} = \mu(d(y)).
\]

**Proposition 3.11** The intersection of any set of fuzzy left derivations BCC-ideals of BCC-algebra \( X \) is also fuzzy left derivations BCC-ideal.

**Proof.** Let \( \{ \mu_i \} \) be a family of fuzzy left derivations BCC-ideals of BCC-algebra \( X \), then \( \forall x, y, z \in X \),
\[
(\inf \{ \mu_i(0) \}) = \inf \{ \mu_i(d(x)) \} = (\inf \{ \mu_i \}) (d(x)) \text{ and }
\]
\[
(\sup \{ \mu_i \}) (d(x \ast z)) = \sup \{ \mu_i(d(x \ast z)) \}
\]
\[
\geq \inf \{ \mu_i(d(x \ast y) \ast z), \mu_i(d(y)) \}
\]
\[
\geq \inf \{ \mu_i(d(x \ast y) \ast z), \inf \mu_i(d(y)) \}
\]
\[
= \min \{ (\inf \mu_i)(d(x \ast y) \ast z),(\inf \mu_i)(d(y)) \}.
\]

**Theorem 3.13** Let \( \mu \) be a fuzzy set in \( X \), then \( \mu \) is a fuzzy left derivations BCC-ideal of \( X \) if and only if it satisfies:
\[
\forall \alpha \in [0, 1], U(\mu, \alpha) \neq \emptyset \implies U(\mu, \alpha) \text{ is BCC-ideal of } X \} \}
\]
\[
\text{where } U(\mu, \alpha) = \{ x \in X / \mu(d(x)) \geq \alpha \}.
\]

**Proof.** Assume that \( \mu \) is a fuzzy left derivations BCC-ideal of \( X \), let \( \alpha \in [0, 1] \) be such that \( U(\mu, \alpha) \neq \emptyset \) and \( x, y \in X \) such that \( x \in U(\mu, \alpha) \), then \( \mu(d(x)) \geq \alpha \) and so by (FL2),
\[
\mu(d(0)) = \mu(d(0 \ast y)) \geq \min \{ \mu(d(0 \ast x \ast y)), \mu(d(x)) \}
\]
\[
= \min \{ \mu(d(0 \ast y)), \mu(d(x)) \}
\]
\[
= \min \{ \mu(0 \ast y), \mu(d(x)) \} = \min \{ \mu(0), \mu(d(x)) \} = \alpha,
\]
\[
\text{hence } 0 \in U(\mu, \alpha).
\]
\[
\text{Let } d(x \ast y) \ast z \in U(\mu, \alpha) \text{ and } d(y) \in U(\mu, \alpha),
\]
\[
\text{it follows from (FL2)} \text{ that }
\]
\[
\mu(d(x \ast z)) \geq \min \{ \mu(d(x \ast y) \ast z), \mu(d(y)) \} = \alpha,
\]
\[
\text{so that } x \ast z \in U(\mu, \alpha). \text{ Hence } U(\mu, \alpha) \text{ is }
\]
\[
\text{BCC-ideal of } X.
\]
\[
\text{Conversely, suppose that } \mu \text{ satisfies (A), let } x, y, z \in X \text{ be such that}
\]
\[
\mu(d(x \ast z)) < \min \{ \mu(d(x \ast y) \ast z), \mu(d(y)) \},
\]
\[
\text{taking } \beta_0 = 1 / 2 \left\{ \mu(d(x \ast z)) + \min \{ \mu(d(x \ast y) \ast z), \mu(d(y)) \} \right\},
\]
\[
\text{we have } \beta_0 \in [0, 1] \text{ and }
\]
\[
\mu(d(x \ast z)) < \beta_0 < \min \{ \mu(d(x \ast y) \ast z), \mu(d(y)) \},
\]
\[
\text{it follows that } d(x \ast y) \ast z \in U(\mu, \beta_0) \text{ and } U(\mu, \beta_0), \text{ this is a contradiction and therefore } \mu \text{ is a fuzzy left derivations }
\]
\[
\text{BCC-ideal of } X.
\]

**Theorem 3.14** Let \( \mu \) be a fuzzy set in \( X \), then \( \mu \) is a fuzzy right derivations BCC-ideal of \( X \) if and only if it satisfies:
\[
\forall \alpha \in [0, 1], U(\mu, \alpha) \neq \emptyset \implies U(\mu, \alpha) \text{ is BCC-ideal of } X \} \}
\]
\[
\text{where } U(\mu, \alpha) = \{ x \in X / \mu(d(x)) \geq \alpha \}.
\]

**Proof.** Clear

**Definition 3.15** Let \( \mu \) be a fuzzy derivations BCC-ideal of BCC-algebra \( X \), the BCC-ideals \( \mu_t \) \( t \in [0, 1] \) are called level BCC-ideal of \( \mu \).

## 4. Image (Pre-image) of Fuzzy Derivations BCC-Ideals Under Homomorphism

In this section, we introduce the concepts of the image and the pre-image of fuzzy left and right derivations BCC-ideals in BCC-algebras under homomorphism of BCC-algebras.

**Definition 4.1** Let \( f \) be a mapping from the set \( X \) to a set.
If $\mu$ is a fuzzy subset of $X$, then the fuzzy subset $\beta$ of $Y$ is defined by
\[
\beta(f(x)) = \begin{cases} 
\sup_{y \in f^{-1}(d(y))} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset \\
0, & \text{otherwise}
\end{cases}
\]
said to be the image of $\mu$ under $f$.

Similarly if $\beta$ is a fuzzy subset of $Y$, then the fuzzy subset $\mu = \beta \circ f$ in $X$ (i.e. the fuzzy subset is defined by $\mu(x) = \beta(f(x)) \quad \forall \ x \in X$) is called the preimage of $\beta$ under $f$.

**Theorem 4.2** An onto homomorphic preimage of a fuzzy right derivations BCC-ideal is also a fuzzy right derivations BCC-ideal under homomorphism of BCC-algebras.

**Proof.** Let $f : X \to X'$ be an onto homomorphism of BCC-algebras, $\beta$ a fuzzy right derivations BCC-ideal of $X'$ and $\mu$ the preimage of $\beta$ under $f$, then $\beta(f(d(x))) = \mu(d(x)), \forall x \in X$. Let $x, y \in X$, we have
\[
\mu(d(0)) = \beta(f(d(0))) \geq \beta(d(f(x))) = \mu(d(x)).
\]

Now let $x, y, z \in X$, then
\[
\mu(d(x \ast z)) = \beta(f(d(x \ast z))) \geq \min \{ \beta(f(x) \ast f(y)) \ast f(d(z)), \beta(f(d(y))) \} = \min \{ \mu(x \ast y) \ast d(z), \mu(d(y)) \}.
\]

The proof is completed.

**Theorem 4.3** An onto homomorphic preimage of a fuzzy left derivations BCC-ideal is also a fuzzy left derivations BCC-ideal.

**Proof.** Clear.

**Definition 4.4** A fuzzy subset $\mu$ of $X$ has sup property if for any subset $T$ of $X$,

there exist $t_{0} \in T$ such that, $\mu(t_{0}) = \sup_{t \in T} \mu(t)$.

**Theorem 4.5** Let $f : X \to Y$ be a homomorphism between BCC-algebras $X$ and $Y$. For every fuzzy left derivations BCC-ideal $\mu$ in $X$, $f(\mu)$ is a fuzzy left derivations BCC-ideal of $Y$.

**Proof.** By definition
\[
\beta(d(y')) = f(\mu)(d(y')) = \sup_{d(x) \in f^{-1}(d(y'))} \mu(d(x))
\]

$\forall \ y' \in Y$ and $\sup \varphi = 0$ .We have to prove that $\beta(d(x' \ast z')) \geq \min \{ \beta(d(x' \ast y') \ast z'), \beta(d(y')) \}, \forall x', y', z' \in Y$.

Let $f : X \to Y$ be an onto a homomorphism of BCC-algebras, $\mu$ a fuzzy left derivations BCC-ideal of $X$ with sup property and $\beta$ the image of $\mu$ under $f$, since $\mu$ is a fuzzy left derivations BCC-ideal of $X$, we have $\mu(d(0)) \geq \mu(d(x)) \quad \forall \ x \in X$. Note that $0 \in f^{-1}(0')$, where 0, 0' are the zero of $X$ and $Y$ respectively. Thus,
\[
\beta(d(0')) = \sup_{d(t) \in f^{-1}(d(0'))} \mu(d(t)) = \mu(d(0)) = \mu(0) \geq \mu(d(x)), \forall x \in X, \text{ which implies that}
\]
\[
\beta(d(0')) \geq \sup_{d(t) \in f^{-1}(d(x'))} \mu(d(t)) = \beta(d(x')), \forall x' \in Y, \forall x', y', z' \in Y, \ let
\]
\[
d(x_{0}) \in f^{-1}(d(x')), \ d(y_{0}) \in f^{-1}(d(y')) , \ d(z_{0}) \in f^{-1}(d(z'))
\]

be such that
\[
\mu(d(x_{0} \ast z_{0})) = \sup_{d(t) \in f^{-1}(d(x_{0} \ast z_{0}))} \mu(d(t), \mu(y_{0})
\]
\[
\sup_{d(t) \in f^{-1}(d(x_{0} \ast y_{0} \ast z_{0}))} \mu(d(t))
\]

\[
\beta(d(x' \ast y') \ast z') = \sup_{d(t) \in f^{-1}(d(x' \ast y') \ast z')} \mu(d(x' \ast y') \ast z') = \sup_{d(t) \in f^{-1}(d(x' \ast y') \ast z')} \mu(d(t)).
\]

Then
\[
\beta(d(x' \ast z')) = \sup_{d(t) \in f^{-1}(d(x' \ast z'))} \mu(d(t)) = \mu(d(x_{0} \ast z_{0}))
\]
\[
\geq \min \{ \mu(d(x_{0} \ast y_{0} \ast z_{0}), \mu(d(y_{0})))
\]

\[
\min \{ \sup_{d(t) \in f^{-1}(d(x' \ast y') \ast z')} \mu(d(t)), \sup_{d(t) \in f^{-1}(d(y'))} \mu(d(t)) \} = \min \{ \beta(d(x' \ast y') \ast z'), \beta(d(y')) \}.
\]

Hence $\beta$ is a fuzzy left derivations BCC-ideal of $Y$.

**Theorem 4.6** Let $f : X \to Y$ be a homomorphism between BCC-algebras $X$ and $Y$. For every fuzzy right derivations BCC-ideal $\mu$ in $X$, $f(\mu)$ is a fuzzy right derivations BCC-ideal of $Y$.

**Proof.** Clear.

### 5. Cartesian Product of Fuzzy Left Derivations BCC-ideals

**Definition 5.1** A fuzzy $\mu$ is called a fuzzy relation on any set $S$, if $\mu$ is a fuzzy subset $\mu : S \times S \to [0,1]$.

**Definition 5.2** If $\mu$ is a fuzzy relation on a set $S$ and is a fuzzy subset of $S$, then $\mu$ is a fuzzy relation on $\beta$ if $\mu(x, y) \leq \min \{ \beta(x), \beta(y) \}, \forall x, y \in S$.

**Definition 5.3** Let $\mu$ and $\beta$ be a fuzzy subset of a set $S$, the Cartesian product of $\mu$ and $\beta$ is defined by $(\mu \times \beta)(x, y) = \min \{ \mu(x), \beta(y) \}, \forall x, y \in S$.

**Lemma 5.4** Let $\mu$ and $\beta$ be a fuzzy subset of a set $S$, then
(i) $\mu \times \beta$ is a fuzzy relation on $S$.

(ii) $(\mu \times \beta)_t = \mu_t \times \beta$, \quad $\forall \, t \in [0,1]$.  

**Definition 5.5** If $\mu$ is a fuzzy derivations relation on a set $S$ and $\beta$ is a fuzzy derivations subset of $S$, then $\mu$ is a fuzzy derivations relation on $S$ if 

$$\mu(d(x,y)) \leq \min \{\beta(d(x)), \beta(d(y))\}, \quad \forall \, x,y \in S.$$  

**Definition 5.6** Let $\mu$ and $\beta$ be a fuzzy derivations subset of a set $S$, the Cartesian product of $\mu$ and $\beta$ is defined by 

$$(\mu \times \beta)(d(x,y)) = \min \{\mu(d(x)), \beta(d(y))\}, \quad \forall \, x,y \in S.$$  

**Definition 5.7** If $\beta$ is a fuzzy derivations subset of a set $S$, the strongest fuzzy derivations relation on $S$, that is a fuzzy derivations relation on $\beta$ is $\mu_\beta$ given by 

$$\mu_\beta(d(x,y)) = \min \{\beta(d(x)), \beta(d(y))\}, \quad \forall \, x,y \in S.$$  

**Lemma 5.8** [2] For a given fuzzy derivations subset of a set $S$, let $\mu_\beta$ be the strongest fuzzy derivations relation on $S$, then for $t \in [0,1]$, we have $(\mu_\beta)_t = \beta_t \times \beta_t$.

**Proposition 5.9** For a given fuzzy derivations subset $\beta$ of BCC-algebra $X$, let $\mu_\beta$ be the strongest fuzzy derivations relation on $X$. If $\mu_\beta$ is a fuzzy derivations BCC-ideal of $X \times X$, then $\beta(d(x)) \leq \beta(d(0)) = \beta(0) \forall \, x \in X$.

**Proof.** Since $\mu_\beta$ is a fuzzy derivations BCC-ideal of $X \times X$, it follows from (F.1) that 

$$\mu_\beta(x,x) = \min \{\beta(d(x)), \beta(d(x))\} \leq \beta(d(0),0) = \beta(0),$$  

where $(0,0) \in X \times X$, then $\beta(d(x)) \leq \beta(d(0)) = \beta(0)$.

**Remark 5.10** Let $X$ and $Y$ be BCC-algebras, we define $*$ on $X \times Y$ by 

$$(x,y)*(u,v) = (x*u,y*v) \quad \forall \, (x,y),(u,v) \in X \times Y,$$ 

then clearly $(X \times Y,*,(0,0))$ is a BCC-algebra.

**Theorem 5.11** Let $\mu$ and $\beta$ be a fuzzy derivations BCC-ideals of BCC-algebra $X$, then $\mu \times \beta$ is a fuzzy derivations BCC-ideal of $X \times X$.

**Proof.** 1. 

$$(\mu \times \beta)(d(0,0)) = \min \{\mu(0), \beta(0)\} \geq \min \{\mu(d(x)), \beta(d(x))\} = (\mu \times \beta)(d(x,y)) \quad \forall \, (x,y) \in X \times X.$$

2. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then 

$$(\mu \times \beta)(d(x_1, z_1) \times y_1, z_1) \times y_1 = \min \{\mu(d(x_1, z_1)), \beta(d(x_1, z_1))\} \geq \min \{\mu(d(x_1, z_1)), \beta(d(y_1, z_2))\} \geq \min \{\mu(d((x_1 \times y_1) \times z_1)), \mu(d(y_1, z_2))\}.$$  

Hence $\mu_\beta$ is a fuzzy derivations BCC-ideal of $X \times X$.

Conversely, let $\mu_\beta$ be a fuzzy derivations BCC-ideal of $X \times X$, 

1. $\forall \, (x,y) \in X \times X$, we have 

$$\min \{\beta(0), \beta(0)\} = \mu_\beta(x,y) = \mu_\beta(\beta(x), \beta(y)),$$  

It follows that $\beta(0) \geq \beta(x) \forall \, x \in X$, which prove (F.1).
2. Let \((x_i, y_j, z_k) \in X \times X \times X\), then
\[
\min \left\{ \beta(d(x_i \ast z_j), \beta(d(x_j \ast z_k))) \right\} \\
\mu_{\beta}(d(x_i \ast z_j), d(x_j \ast z_k)) \\
\geq \min \left\{ \mu_{\beta}(d((x_i \ast y_j) \ast (y_j \ast x_j) \ast (x_j \ast y_k)), d((x_j \ast y_k) \ast (x_j \ast y_k))) \right\} \\
= \min \left\{ \mu_{\beta}(d(x_i \ast y_j), d(x_j \ast y_k)) \right\} \\
= \min \left\{ \beta(d(x_i \ast y_j), \beta(d(x_j \ast y_k))) \right\} \\
= \min \left\{ \beta(d(x_i \ast y_j), \beta(d(x_j \ast y_k))) \right\} \\
\text{In particular, if we take } x_i = y_j = z_k = 0, \text{ then} \\
\beta(d(x_i \ast z_j)) \geq \min \left\{ \beta(d(x_i \ast y_j), \beta(d(y_j))) \right\} . \text{ This prove (F_2)}.
\]
Hence \(\beta\) be a fuzzy derivations BCC-ideal of \(X\).

6. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, the notion of fuzzy left and right derivations BCC-ideal in BCC-algebra are introduced and investigated the useful properties of fuzzy left and right derivations BCC-ideals in BCC-algebras.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCI-algebra, BCH-algebra, Hilbert algebra, BF-algebra, J-algebra, WS-algebra, CI-algebra, SU-algebra, BCL-algebra, BP-algebra, Coxeter algebra, BO-algebras and so forth.

The main purpose of our future work is to investigate:
1. The interval value, bipolar and intuitionist fuzzy left and right derivations BCC-ideal in BCC-algebra.
2. To consider the cubic structure left and right derivations BCC-ideal in BCC-algebra.

We hope the fuzzy left and right derivations BCC-ideals in BCC-algebras, have applications in different branches of theoretical physics and computer science.

Algorithm for BCC-algebras

Input \((X: \text{set, } \ast: \text{binary operation})\)

Output (“\(X\) is a BCC-algebra or not”)

Begin
If \(X = \emptyset\) then go to (1.);
End If
If \(0 \notin X\) then go to (1.);
End If
Stop: =false;
\(i := 1\);

While \(i \leq |X|\) and not (Stop) do
If \(x_i \ast x_j \neq 0\) then
Stop: = true;
End If
\(j := 1\);
\(k := 1\);
While \(j, k \leq |X|\) and not (Stop) do
If \(\beta((x_i \ast y_j) \ast (x_i \ast z_k)) \neq 0\) then
Stop: = true;
End If
End While
End If
Else
Output ("\(X\) is a BCC-algebra")
End If
End.

References


