Weihgted Cesaro Sequence Space and Related Matrix Transformation

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To cite this article:

Md. Fazlur Rahman, A. B. M. Rezaul Karim. Weihgted Cesaro Sequence Space and Related Matrix Transformation. *Pure and Applied Mathematics Journal*. Vol. 4, No. 6, 2015, pp. 237-241. doi: 10.11648/j.pamj.20150406.12

Abstract: In this paper we define the weighted Cesaro sequence spaces $ces(p, q)$. We prove the space $ces(p, q)$ is a complete paranorm space. In section-2 we determine its Kothe-Toeplitz dual and continuous dual. In section-3 we establish necessary and sufficient condition for a matrix A to map $ces(p, q)$ to l_∞ and $ces(p, q)$ to c , where l_∞ is the space of all bounded sequences and c is the space of all convergent sequences. We also get some known and unknown interesting results as corollaries.

Keywords: Sequence Space, Kothe-Toeplitz Dual, Matrix Transformation

1. Introduction

Let ω be the space of all (real or complex) sequences and let l_∞ and c respectively the Banach spaces of bounded and convergent sequence $x = (x_n)$ endowed with the norm

$$\|x\| = \sup_{k \geq 1} |x_k|$$

In [8] Shiue introduce the Cesaro sequence space ces_p as

$$ces_p = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |x_k| \right)^p < \infty \right\} \text{ for } 1 < p < \infty$$

$$\text{and } ces_\infty = \left\{ x = (x_k) \in \omega : \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^n |x_k| < \infty \right\} \text{ for } p = \infty$$

In [4] Leibowitz studied some properties of this space and showed that it is a Banach space. Lim [9] defined this space in a different norm as

$$ces_p = \left\{ x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{k=1}^{2^r} |x_k| \right)^p < \infty \right\} \text{ for } 1 < p < \infty$$

and

$$ces_\infty = \left\{ x = (x_k) \in \omega : \sup_{r \geq 0} \frac{1}{2^r} \sum_{k=1}^{2^r} |x_k| < \infty \right\} \text{ for } p = \infty$$

where $\sum_r$ denotes a sum over the ranges $[2^r, 2^{r+1})$, determined its dual spaces and characterize some matrix classes. Later in [10] Lim extended this space ces_p to $ces(p)$ for the sequence $p = (p_r)$ with $\inf p_r > 0$ and defined as

$$ces(p) = \left\{ x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{k=1}^{2^r} |x_k| \right)^{p_r} < \infty \right\}.$$

For positive sequence of real numbers $(p_n), (q_n)$ and $Q_n = q_1 + q_2 + \dots + q_n$

Johnson and Mohapatra [11] defined the Cesaro sequence space $ces(p, q)$ as

$$ces(p, q) = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^n q_k |x_k| \right)^{p_n} < \infty \right\}$$

and studied some inclusion relations.

What amounts to the same thing defined by Khan and Rahman [3] as

$$ces(p, q) = \left\{ x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{k=1}^{2^r} q_k |x_k| \right)^{p_r} < \infty \right\}$$

For $p = (p_r)$ with $\inf p_r > 0$, $Q_{2^r} = q_{2^{r-1}} + q_{2^{r-1}+1} + \dots + q_{2^r}$ and $\sum_r$ denotes a sum over the ranges $[2^r, 2^{r+1})$.

They determined its Kothe -Toeplitz dual and characterized some matrix classes.

In this paper we define the Cesaro weighted sequence space $ces(p, q)$ in the following way.

Definition. If (q_n) is a bounded sequence of positive real numbers, then for $p = (p_r)$ with $\inf p_r > 0$, the Cesaro weighted sequence space $ces(p, q)$ is defined by

$$ces(p, q) = \left\{ x = (x_k) \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} < \infty \right\}$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \dots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$.

If $q_n = 1$ for all n , then $ces(p, q)$ reduces to $ces(p)$ studied by Lim[10]. Also, if $p_n = p$ for all n and $q_n = 1$ for all n , then $ces(p, q)$ reduces to ces_p studied also Lim[9]. Obviously, $l(p) \subset ces(p) \subset ces(p, q)$ for $p_r \geq 1$.

In their paper [1] Maji and Srivastava defined the weighted Cesaro sequence space $ces(p, q)$ with a different norms and studied on some operators and inclusion results.

The main purpose of this note is to define and investigate the weighted Cesaro sequence space $ces(p, q)$, determine its Kothe-Toeplitz dual and characterize the class of matrices $(ces(p, q), l_{\infty})$ and $(ces(p, q), c)$, where l_{∞} and c are respectively the spaces of bounded and convergent complex sequences. By specializing sequences (p_n) and (q_n) , we get the results of Lim[9], [10] as corollaries. Meanwhile, we also determine all continuous linear functional on $ces(p, q)$ for all $1 < p_r < \infty$.

With regard to notation, the dual space of $ces(p, q)$, i.e., the space of all continuous linear functionals on $ces(p, q)$, will be denoted by $ces^*(p, q)$. We write

$$A_r(n) = \max_r \left| \frac{a_{n,k}}{q_k} \right|$$

where for each n the maximum with respect to k in $[2^r, 2^{r+1})$.

Throughout the paper the following well-known inequality (see [6] or [7]) will be frequently used. For any $E > 0$ and any two complex numbers a and b we have

$$|ab| \leq E(|a|^t E^{-t} + |b|^t) \quad (1)$$

$$\text{Where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1$$

To begin with, we show that the space $ces(p, q)$ is paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M} \quad (2)$$

$$\text{Provided } H = \sup_r p_r < \infty \text{ and } M = \max\{1, H\}$$

Clearly

$$g(\theta) = 0$$

$$g(-x) = g(x),$$

where $\theta = (0, 0, 0, \dots)$

Since $p_r \leq M, M \geq 1$ so for any $x, y \in ces(p, q)$ we have by Minkowski's inequality

$$\begin{aligned} & \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k (x_k + y_k)| \right)^{p_r} \right)^{1/M} \\ & \leq \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r (|q_k x_k| + |q_k y_k|) \right)^{p_r} \right)^{1/M} \\ & \leq \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M} + \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k y_k| \right)^{p_r} \right)^{1/M} \end{aligned}$$

which shows that g is subadditive.

Finally we have to check the continuity of scalar multiplication. From the definition of $ces(p, q)$, we have $\inf p_r > 0$. So, we may assume that $\inf p_r \equiv \rho > 0$. Now for any complex λ with $||\lambda|| < 1$, we have

$$\begin{aligned} g(\lambda x) &= \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |\lambda q_k x_k| \right)^{p_r} \right)^{1/M} \\ &= |\lambda|^{p_r/M} \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right)^{1/M} \\ &\leq \sup_r ||\lambda||^{\frac{p_r}{M}} g(x) \\ &\leq ||\lambda||^{\frac{\rho}{M}} g(x) \rightarrow 0 \text{ as } \lambda \rightarrow 0 \end{aligned}$$

It is quite routine to show that $ces(p, q)$ is a metric space with the metric $d(x, y) = g(x - y)$ provided that $x, y \in ces(p, q)$, where g is defined by (2). And using a similar method to that in [3] one can show that $ces(p, q)$ is complete under the metric mentioned above.

2. Kothe-Toeplitz Duals

If X is a sequence space we define ([2], [5])

$$X^{+1} = X^{\alpha} = \left\{ a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty, \right. \\ \left. \text{for every } x \in X \right\}$$

$$X^{+} = X^{\beta}$$

$$= \left\{ a = (a_k) \right.$$

$$\left. \in \omega : \sum_k a_k x_k \text{ is convergent, for every } x \in X \right\}$$

Now we are going to give the following theorem by which the generalized Kothe-Toeplitz dual $ces^+(p, q)$ will be determined.

Theorem 1: If $1 < p_r \leq \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{t_r} = 1$, for $r = 0, 1, 2, \dots$ then

$$ces^+(p, q) = [ces(p, q)]^\beta$$

$$= \left\{ a = (a_k) : \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} < \infty, \right. \\ \left. \text{for some integer } E > 1 \right\}.$$

Proof: Let $1 < p_r \leq \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{t_r} = 1$, for $r = 0, 1, 2, \dots$. Define

$$\mu(t) = \left\{ a = (a_k) : \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} < \infty, \right. \\ \left. \text{for some integer } E > 1 \right\} \quad (3)$$

We want to show that $ces^+(p, q) = \mu(t)$.

Let $x \in ces(p, q)$ and $a \in \mu(t)$. Then using inequality (1) we get

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k| &= \sum_{r=0}^{\infty} \sum_r |a_k x_k| \\ &= \sum_{r=0}^{\infty} \sum_r \left| \frac{a_k}{q_k} q_k x_k \right| \\ &= \sum_{r=0}^{\infty} \sum_r \left| \frac{a_k}{q_k} \right| |q_k x_k| \\ &\leq \sum_{r=0}^{\infty} \max_r \left| \frac{a_k}{q_k} \right| \sum_r |q_k x_k| \\ &= \sum_{r=0}^{\infty} Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \frac{1}{Q_{2^r}} \sum_r |q_k x_k| \\ &\leq E \sum_{r=0}^{\infty} \left\{ \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} + \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right\} \\ &= E \left\{ \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right\} \\ &< \infty \end{aligned}$$

which implies that the series $\sum_{k=1}^{\infty} a_k x_k$ convergent.

Therefore,

$$a \in \text{dual of } ces(p, q) = ces^+(p, q)$$

This shows, $\mu(t) \subset ces^+(p, q)$

Conversely, suppose that $\sum a_k x_k$ is convergent for all $x \in ces(p, q)$ but $a \notin \mu(t)$. Then

$$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} = \infty, \text{ for every integer } E > 1$$

So we can define a sequence $0 = n(0) < n(1) < n(2) < \dots$, such that $\gamma = 0, 1, 2, \dots$, we have

$$M_\gamma = \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} (\gamma + 2)^{-t_r/p_r} > 1$$

Now we define a sequence $x = (x_k)$ in the following way:

$$x_{N(r)} = Q_{2^r}^{t_r} |a_{N(r)}|^{t_r-1} \operatorname{sgn} a_{N(r)} (\gamma + 2)^{-t_r} M_\gamma^{-1}$$

for $n(\gamma) \leq r \leq n(\gamma + 1) - 1, \gamma = 0, 1, 2, \dots$ and $x_k = 0$ for

$k \neq N(r)$, where $N(r)$ is such that $|a_{N(r)}| = \max_r \left| \frac{a_k}{q_k} \right|$, the maximum is taken with respect to k in $[2^r, 2^{r+1})$.

Therefore,

$$\begin{aligned} \sum_{k=2^{n(\gamma)}}^{2^{n(\gamma+1)-1}} a_k x_k &= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} (Q_{2^r} |a_{N(r)}|)^{t_r} (\gamma + 2)^{-t_r} M_\gamma^{-1} \\ &= M_\gamma^{-1} (\gamma + 2)^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} (Q_{2^r} |a_{N(r)}|)^{t_r} (\gamma + 2)^{-t_r/p_r} \\ &= M_\gamma^{-1} M_\gamma (\gamma + 2)^{-1} \\ &= (\gamma + 2)^{-1} \end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{\gamma=0}^{\infty} (\gamma + 2)^{-1} \text{ diverges.}$$

Moreover

$$\begin{aligned} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \\ &= \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{(t_r-1)p_r} |a_{N(r)}|^{(t_r-1)p_r} (\gamma + 2)^{-t_r/p_r} M_\gamma^{-p_r} \\ &\leq (\gamma + 2)^{-2} M_\gamma^{-1} \sum_{r=n(\gamma)}^{n(\gamma+1)-1} Q_{2^r}^{t_r} |a_{N(r)}|^{t_r} (\gamma + 2)^{-t_r/p_r} \\ &= (\gamma + 2)^{-2} M_\gamma^{-1} M_\gamma \\ &= (\gamma + 2)^{-2} \end{aligned}$$

Therefore

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \leq (\gamma + 2)^{-2} < \infty$$

That is, $x \in ces(p, q)$ which is a contradiction to our assumption. Hence

$$a \in \mu(t). \text{ That is, } \mu(t) \supset ces^+(p, q).$$

Then combining the two results, we get

$$ces^+(p, q) = \mu(t).$$

The continuous dual of $ces(p, q)$ is determined by the following theorem.

Theorem 2: Let $1 < p_r \leq \sup_r p_r < \infty$. Then continuous dual $ces^*(p, q)$ is isomorphic to $\mu(t)$, which is defined by (3)

Proof: It is easy to check that each $x \in ces(p, q)$ can be written in the form

$$x = \sum_{k=1}^{\infty} x_k e_k, \text{ where } e_k = (0, 0, 0, \dots, 0, 1, 0, \dots)$$

and the 1 appears at the k -th place. Then for any $f \in ces^*(p, q)$ we have

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k. \quad (4)$$

Where, $f(e_k) = a_k$. By theorem 1, the convergence of $\sum a_k x_k$ for every x in $ces(p, q)$ implies that $a \in \mu(t)$.

If $x \in ces(p, q)$ and if we take $a \in \mu(t)$, then by theorem 1, $\sum a_k x_k$ converges and clearly defines a linear functional on $ces(p, q)$. Using the same kind of argument as in theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \leq E \left(\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left| \frac{a_k}{q_k} \right| \right)^{t_r} E^{-t_r} + 1 \right) g(x)$$

whenever $g(x) \leq 1$, where $g(x)$ is defined by (2).

Hence $\sum a_k x_k$ defines an element of $ces^*(p, q)$.

Furthermore, it is easy to see that representation (4) is unique. Hence we can define a mapping

$$T: ces^*(p, q) \rightarrow \mu(t).$$

by $T(f) = (a_1, a_2, \dots)$ where the a_k appears in representation (4). It is evident that T is linear and bijective. Hence $ces^*(p, q)$ is isomorphic to $\mu(t)$.

3. Matrix Transformations

In the following theorems we shall characterize the matrix classes $(ces(p, q), l_{\infty})$ and $(ces(p, q), c)$. Let $A = (a_{n,k})$, $n, k = 1, 2, \dots$ be an infinite matrix of complex numbers and X, Y two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y and denote it by $A \in (X, Y)$ if for every sequence $x = (x_k) \in X$ the sequence $A(x) = A_n(x)$ is in Y , where

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$$

provided the series on the right is convergent.

Theorem 3: Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p, q), l_{\infty})$ if and only if there exists an integer $E > 1$, such that $U(E) < \infty$.

Where

$$U(E) = \sup_n \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} E^{-t_r} \text{ and } \frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \dots$$

Proof: *Sufficiency:* Suppose there exists an integer $E > 1$, such that $U(E) < \infty$. Then by inequality (1), we have

$$\begin{aligned} \sum_{k=1}^{\infty} |a_{n,k} x_k| &= \sum_{r=0}^{\infty} \sum_r \left| \frac{a_{n,k}}{q_k} q_k x_k \right| = \sum_{r=0}^{\infty} \sum_r \left| \frac{a_{n,k}}{q_k} \right| |q_k x_k| \\ &\leq \sum_{r=0}^{\infty} Q_{2^r} \max_r \left| \frac{a_{n,k}}{q_k} \right| \left| \frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right| \\ &\leq E \left(\sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} \right) \\ &< \infty \end{aligned}$$

Therefore, $A \in (ces(p, q), l_{\infty})$.

Necessity: Suppose that $A \in (ces(p, q), l_{\infty})$, but

$$\sup_n \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} E^{-t_r} = \infty \text{ for every integer } E > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k} x_k$ converges for every n and $x \in ces(p, q)$, whence $(a_{n,k})_{k=1,2,\dots} \in ces^+(p, q)$ for every n .

By theorem 1, it follows that each A_n defined by

$$A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$$

is an element of $ces^*(p, q)$. Since $ces(p, q)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $ces(p, q)$, by the uniform boundedness principle there exists a number L independent of n and a number $\delta < 1$, such that

$$|A_n(x)| \leq L \quad (5)$$

for every n and $x \in S[\theta, \delta]$ where $S[\theta, \delta]$ is the closed sphere in $ces(p, q)$ with centre at the origin θ and radius δ .

Now choose an integer $G > 1$, such that

$$G \delta^M > L.$$

Since

$$\sup_n \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} G^{-t_r} = \infty$$

there exists an integer $m_0 > 1$, such that

$$R = \sum_{r=0}^{m_0} (Q_{2^r} A_r(n))^{t_r} G^{-t_r} > 1 \quad (6)$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \text{ if } k \geq 2^{m_0+1}$$

$$x_{N(r)} = Q_{2^r} \delta^{M/p_r} (\text{sgn } a_{n,N(r)}) |a_{n,N(r)}|^{t_r-1} R^{-1} G^{-t_r/p_r}$$

and $x_k = 0$ if $k \neq N(r)$ for $0 \leq r \leq m_0$, where $N(r)$ is the smallest integer such that

$$|a_{n,N(r)}| = \max_r \left| \frac{a_{n,k}}{q_k} \right|$$

Then one can easily show that $g(x) \leq \delta$ but $|A_n(x)| > L$, which contradicts (5). This complete the proof of the theorem.

Corollary 3.1 (see [10]). Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p), l_{\infty})$ if and only if there exists an integer $E > 1$ such that $U(E) < \infty$, where

$$U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{t_r} \text{ and } \frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \dots$$

Proof: If $q_n = 1$ for every n in the above theorem, then we obtain the result.

Corollary 3.2 (see [9]). Let $1 < p < \infty$. Then $A \in (ces_p, l_{\infty})$ if and only if $\sup_n (\sum_{r=0}^{\infty} (2^r A_r(n))^t)^{1/t} < \infty$ where $\frac{1}{p} + \frac{1}{t} = 1$

Proof: If $q_n = 1$ and $p_n = p$ for all n in the above theorem, then we get the result.

Theorem 4. Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p, q), c)$ if and only if

(i) $a_{n,k} \rightarrow \alpha_k$ ($n \rightarrow \infty, k$ is fixed) and

(ii) there exists an integer $E > 1$, such that $U(E) < \infty$, where

$$U(E) = \sup_n \sum_{r=0}^{\infty} (Q_{2^r} A_r(n))^{t_r} E^{-t_r}$$

and $\frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \dots$

Proof: Necessity. Suppose $A \in (ces(p, q), c)$. Then $A_n(x)$ exists for each $n \geq 1$ and $\lim_{n \rightarrow \infty} A_n(x)$ exists for every $x \in ces(p, q)$. Therefore by an argument similar to that in theorem 3 we have condition (ii). Condition (i) is obtained by taking $x = e_k \in ces(p, q)$, where e_k is a sequence with 1 at the k -th place and zeros elsewhere.

Sufficiency. The conditions of the theorem imply that

$$\sum_{r=0}^{\infty} (Q_{2^r} \max_r \left| \frac{\alpha_k}{q_k} \right|)^{t_r} E^{-t_r} \leq U(E) < \infty \quad (7)$$

By (7) it is easy to check that $\sum_k \alpha_k x_k$ is absolutely convergent for each $x \in ces(p, q)$. For each $x \in ces(p, q)$ and $\varepsilon > 0$, we can choose an integer $m_0 > 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |q_k x_k| \right)^{p_r} < \varepsilon^M$$

Then by inequality (1), we have

$$\sum_{k=2^{m_0}}^{\infty} |a_{n,k} - \alpha_k| |x_k| \leq E \left(\sum_{r=m_0}^{\infty} (Q_{2^r} B_r(n))^{t_r} E^{-t_r} + 1 \right) (g_{m_0}(x))^{1/M}$$

$$< E(2U(E) + 1)\varepsilon,$$

where $B_r(n) = \max_r \left| \frac{a_{n,k} - \alpha_k}{q_k} \right|$ and

$$\sum_{r=m_0}^{\infty} (Q_{2^r} B_r(n))^{t_r} E^{-t_r} \leq 2U(E) < \infty$$

It follows immediately that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

This shows that $A \in (ces(p, q), c)$ which proved the theorem.

Corollary 4.1 (see [10]). Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p), c)$ if and only if

- (i) $a_{n,k} \rightarrow \alpha_k (n \rightarrow \infty, k \text{ is fixed}), \text{ and}$
- (ii) there exists an integer $E > 1$ such that $U(E) < \infty$, where

$$U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{t_r} E^{-t_r} \text{ and } \frac{1}{p_r} + \frac{1}{t_r} = 1, r = 0, 1, 2, \dots$$

Proof: If $q_n = 1$ for all n in the above theorem, then statements (i) and (ii) follow.

Corollary 4.2 (see [9]). Let $1 < p < \infty$. Then $A \in (ces_p, c)$ if and only if

- (i) $a_{n,k} \rightarrow \alpha_k (n \rightarrow \infty, k \text{ is fixed}), \text{ and}$
- (ii) $\sup_n (\sum_{r=0}^{\infty} 2^{rt} A_r^t(n))^{1/t} < \infty, \text{ where } \frac{1}{p} + \frac{1}{t} = 1.$

Proof: If $q_n = 1$ and $p_n = p$ for all n in the above theorem, then we get the results.

Corollary 4.3. Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p, q), c_0)$ if and only if

- the condition of Theorem 3 holds, and
- $a_{n,k} \rightarrow 0 (n \rightarrow \infty, k \text{ is fixed}), \text{ where } c_0 \text{ is the space of all null sequences.}$

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