New Modification of Homotopy Perturbation Method and the Fourth - Order Parabolic Equations with Variable Coefficients

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Abstract: In this paper, the exact solution of the fourth - order parabolic equations with variable coefficients is obtained by using a new homotopy perturbation method (NHPM), theoretical consideration are discussed. Finally, three examples are illustrated to show the validity and applicability of the proposed method.

Keywords: New Homotopy Perturbation Method (NHPM), Fourth - Order Parabolic Equations

1. Introduction

In this paper, we consider the fourth – order parabolic partial differential equations with the variable coefficients of the form;

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \mu(x, y, z) \frac{\partial^4 u}{\partial x^4} + \lambda(x, y, z) \frac{\partial^4 u}{\partial y^4} \\ + \eta(x, y, z) \frac{\partial^4 u}{\partial z^4} = g(x, y, z), \\ a < x, y, z < b, t > 0 \end{aligned} \quad (1)$$

Where, $\mu(x, y, z)$, $\lambda(x, y, z)$ and $\eta(x, y, z)$ are positive, Subject to the initial conditions;

$$u(x, y, z, 0) = f_0(x, y, z), u_t(x, y, z, 0) = f_1(x, y, z) \quad (2)$$

And the boundary conditions;

$$u(a, y, z, t) = g_0(y, z, t), \quad u(b, y, z, t) = g_1(y, z, t)$$

$$u(x, a, z, t) = k_0(x, z, t), \quad u(x, b, z, t) = k_1(x, z, t)$$

$$u(x, y, a, t) = h_0(x, y, t), \quad u(x, y, b, t) = h_1(x, y, t)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(a, y, z, t) &= \tilde{g}_0(y, z, t), \\ \frac{\partial^2 u}{\partial x^2}(b, y, z, t) &= \tilde{g}_1(y, z, t) \end{aligned} \quad (3)$$

$$\frac{\partial^2 u}{\partial y^2}(x, a, z, t) = \tilde{k}_0(x, z, t),$$

$$\frac{\partial^2 u}{\partial y^2}(x, b, z, t) = \tilde{k}_1(x, z, t)$$

$$\frac{\partial^2 u}{\partial x^2}(a, y, z, t) = \tilde{h}_0(x, y, t),$$

$$\frac{\partial^2 u}{\partial x^2}(x, y, b, t) = \tilde{h}_1(x, y, t)$$

Where the functions f_i , g_i , k_i , h_i , $\tilde{g}_i$, $\tilde{k}_i$, $\tilde{h}_i$, $i = 0, 1$ are continuous functions. it is worth mentioning that the problem equation (1 - 3) arise in the study of the transverse vibrations problem [1]. numerical computations of the transverse vibrations have been carried out by a number of authors for one dimensional space. the main focus of researchers was to obtain numerical solution by using several techniques, such as explicit and implicit finite - difference schemes [2, 3, 4, 5], Evans [6] who expressed the fourth order equation in two space variables as system of two

second - order equations to be solved by finite difference method in [7] Khaliq and Twizell solved the variable fourth order parabolic equations by using a family of second order method, Evans etc all [8] investigated the fourth order parabolic equation with constant coefficient by using the AGE method, recently Wazwaz [9, 10] Approached the variable coefficient fourth - order parabolic Equation directly by application Adomian decomposition method. Biazar and Ghazrini [11] applied He's variational iteration method and Deniz Agirseven and Turgut Özis [12] obtained the exact solution of the problem by using homotopy perturbation method. He's homotopy perturbation method [13 - 17] has been used by many researcher in physics and engineering to solve various problem [18 - 22]. In this paper we extend the modification of homotopy perturbation method (NHMP) [23 - 26] to obtain the exact solution of variables coefficients fourth order parabolic equations.

2. Basic Ideas of the (HPM)

To solve equation (1) with initial condition (2) by NHPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial^2 v}{\partial t^2} - u_0\right) + p\left(\frac{\partial^2 v}{\partial t^2} + \mu(x, y, z)\frac{\partial^2 v}{\partial x^2} + \lambda(x, y, z)\frac{\partial^2 v}{\partial y^2} + \eta(x, y, z)\frac{\partial^2 v}{\partial z^2} - g(x, y, z)\right) = 0 \quad (3)$$

Or:

$$\frac{\partial^2 v}{\partial t^2} = u_0 - p\left(u_0 + \mu(x, y, z)\frac{\partial^2 v}{\partial x^2} + \lambda(x, y, z)\frac{\partial^2 v}{\partial y^2} + \eta(x, y, z)\frac{\partial^2 v}{\partial z^2} - g(x, y, z)\right) \quad (4)$$

Applying the inverse operator, $L^{-1} = \int_0^t \int_0^t (\circ) dt dt$, to both sides of Eq. (4) we obtain:

$$v(x, y, z, t) = v(x, y, z, 0) + tv_t(x, y, z, 0) + \int_0^t \int_0^t u_0 dt dt - p \int_0^t \int_0^t \left(u_0 + \mu(x, y, z)\frac{\partial^2 v}{\partial x^2} + \lambda(x, y, z)\frac{\partial^2 v}{\partial y^2} + \eta(x, y, z)\frac{\partial^2 v}{\partial z^2} - g(x, y, z) \right) dt dt \quad (5)$$

Where, $v(x, y, z, 0) = f_0(x, y, z)$ and $v_t(x, y, z, 0) = f_1(x, y, z)$.

Assume the solution of Eq. (4) has the form:

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (6)$$

Suppose the initial approximation to the solution $u_0(x, y, z, t)$ has the form:

$$u_0(x, y, z, t) = \sum_{n=0}^{\infty} a_n(x, y, z) \rho_n(t) \quad (7)$$

Where, $a_0, a_1, a_2, \dots$ are unknown coefficients and are specific functions depending on problem. Substituting Eq. (6) into Eq (5) and equating terms of like power p , and equating each coefficients of p to zero we get:

$$\begin{aligned} p^0 : v_0(x, y, z, t) &= v_0(x, y, z, 0) + tv_{0t}(x, y, z, 0) + \int_0^t \int_0^t u_0 dt dt. \\ p^1 : v_1(x, y, z, t) &= - \int_0^t \int_0^t \left(u_0 + \mu(x, y, z)\frac{\partial^4 v_0}{\partial x^4} + \lambda(x, y, z)\frac{\partial^4 v_0}{\partial y^4} + \eta(x, y, z)\frac{\partial^4 v_0}{\partial z^4} - g(x, y, z) \right) dt dt \\ p^2 : v_2(x, y, z, t) &= - \int_0^t \int_0^t \left(\mu(x, y, z)\frac{\partial^4 v_1}{\partial x^4} + \lambda(x, y, z)\frac{\partial^4 v_1}{\partial y^4} + \eta(x, y, z)\frac{\partial^4 v_1}{\partial z^4} \right) dt dt \end{aligned} \quad (8)$$

By solving these equation in such a way that $v_1(x, y, z) = 0$ then Eq. (7) result in $v_j(x, y, z, t) = 0, j = 1, 2, 3, \dots$, therefore the exact solution may be obtained as follows:

$$u(x, y, z, t) = v_0(x, y, z, t) = f_0(x, y, z) + tf_1(x, y, z) + \sum_{n=0}^{\infty} a_n(x, y, z) \int_0^t \int_0^t \rho_n(t) dt dt$$

It is worth mentioning that if $g(x, y, z, t)$ and $u_0(x, y, z, t)$ are analytic at $t = t_0$ then their Taylor series written as;

$$\begin{aligned} u_0(x, y, z, t) &= \sum_{n=0}^{\infty} a_n(x, y, z)(t - t_0)^n \\ g(x, y, z, t) &= \sum_{n=0}^{\infty} a_n^*(x, y, z)(t - t_0)^n \end{aligned}$$

Can be used in Eq. (8), where $a_0, a_1, a_2, \dots$ are unknown coefficients and $a_0^*, a_1^*, a_2^*, \dots$ are known ones, which must be computed.

3. Application

In this section, we present three examples to illustrate our shames for variable coefficient parabolic equation.

Example 1.

Consider the one dimensional, variable coefficients fourth order parabolic equation [1];

$$\frac{\partial^2 u}{\partial t^2} + (1+x) \frac{\partial^4 u}{\partial x^4} = \left(x^4 + x^3 - \frac{6}{7!} \right) \cos t, \quad 0 < x < 1, \quad t > 0 \quad (9)$$

With the initial conditions;

$$u(x, 0) = \frac{6}{7!} x^7, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad (10)$$

And the boundary conditions;

$$u(0, t) = 0, \quad u(1, t) = \frac{6}{7!} x^7 \cos t$$

$$\frac{\partial^2 u}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1, t) = \frac{1}{20} \cos t \quad (11)$$

To solve Eq. (9) by (NHMP), we construct the following homotopy:

$$\frac{\partial^2 v}{\partial t^2} = u_0 - p \left[u_0 + (1+x) \frac{\partial^4 v}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!} \right) \cos t \right] \quad (12)$$

By integration of Eq. (12) we have:

$$v(x, t) = v(x, 0) + t v_0(x, 0) + \int_0^t \int_0^t u_0 dt dt$$

$$- p \int_0^t \int_0^t \left(u_0 + (1+x) \frac{\partial^4 v}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!} \right) \cos t \right) dt dt \quad (13)$$

Assume the solution of Eq. (13) has the following form:

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (14)$$

Substituting Eq. (14) in to Eq. (13) and equating the coefficients of like powers of p , results are:

$$p^0 : v_0(x, t) = v_0(x, 0) + t v_{0t}(x, 0) + \int_0^t \int_0^t u_0 dt dt.$$

$$p^1 : v_1(x, t) = - \int_0^t \int_0^t \left(u_0 + (1+x) \frac{\partial^4 v_0}{\partial x^4} - \left(x^4 + x^3 - \frac{6}{7!} \right) \cos t \right) dt dt$$

$$p^2 : v_2(x, t) = - \int_0^t \int_0^t (1+x) \frac{\partial^4 v_1}{\partial x^4} dt dt.$$

$$\text{Assuming } u_0(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n, \quad v(x, 0) = u(x, 0),$$

$$v_t(x, 0) = u_t(x, 0) \text{ and } v_1(x, t) = 0$$

Then we have:

$$v_1(x, t) = \left(-\frac{a_0}{2} - \frac{1}{2} \frac{6x^7}{7!} \right) t^2 + \left(-\frac{a_1}{6} \right) t^3$$

$$+ \left(-\frac{a_{0xxxx}}{24} - \frac{a_{0xxxx} x}{24} - \frac{x^4}{24} - \frac{x^3}{24} - \frac{1}{24} \frac{6x^7}{7!} - \frac{a_2}{12} \right) t^4$$

$$+ \left(-\frac{a_{3xxxx}}{20} \right) t^5 + \dots$$

It can be easily to shown that:

$$a_0 = -\frac{6}{7!} x^7, \quad a_1 = 0, \quad a_2 = \frac{1}{2} \frac{6}{7!} x^7, \quad a_3 = 0, \dots$$

Therefore we obtain the solution of Eq. (9);

$$u(x, t) = \frac{6x^7}{7!} + a_0(x) \frac{t^2}{2} + a_1(x) \frac{t^3}{6}$$

$$+ a_2(x) \frac{t^4}{12} + a_3(x) \frac{t^5}{20} + \dots = \frac{6}{7!} x^7 \cos t$$

This is an exact solution.

Example 2.

We next consider the fourth - order parabolic equation in two variables;

$$\frac{\partial^2 u}{\partial t^2} + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0$$

$$\frac{1}{2} < x, y < 1, \quad t > 0 \quad (15)$$

With the initial conditions:

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \quad (16)$$

And the boundary conditions:

$$u\left(\frac{1}{2}, y, t\right) = \left(2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!} \right) \sin t,$$

$$u(1, y, t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!} \right) \sin t$$

$$\frac{\partial^2 u}{\partial x^2} \left(\frac{1}{2}, y, t \right) = \frac{(0.5)^6}{24} \sin t, \quad \frac{\partial^2 u}{\partial x^2} (1, y, t) = \frac{1}{24} \sin t \quad (17)$$

$$\frac{\partial^2 u}{\partial x^2} \left(x, \frac{1}{2}, t \right) = \frac{(0.5)^6}{24} \sin t, \quad \frac{\partial^2 u}{\partial x^2} (x, 1, t) = \frac{1}{24} \sin t$$

To solve Eq. (15) by (NHMP), we construct the following homotopy:

$$\frac{\partial^2 v}{\partial t^2} = u_0 - p \left(u_0 + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 v}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 v}{\partial y^4} \right) \quad (18)$$

Applying the inverse operator, $L^{-1} = \int_0^t \int_0^t (\odot) dt dt$, to both sides of Eq. (18) we obtain:

$$v(x, y, t) = v(x, y, 0) + tv_t(x, y, 0) + \int_0^t \int_0^t u_0 dt dt - p \int_0^t \int_0^t \left(u_0 + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 v}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 v}{\partial y^4} \right) dt dt \quad (19)$$

Suppose the solution of Eq. (18) have the form Eq. (14) substituting Eq. (14) into Eq. (19) equating the coefficients of like powers of p we get:

$$p^0 : v_0(x, y, t) = v_0(x, y, 0) + tv_{0t}(x, y, 0) + \int_0^t \int_0^t u_0(x, y, t) dt dt.$$

$$p^1 : v_1(x, y, t) = - \int_0^t \int_0^t \left(u_0 + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 v_0}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 v_0}{\partial y^4} \right) dt dt.$$

$$p^2 : v_2(x, y, t) = - \int_0^t \int_0^t \left(2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 v_1}{\partial x^4} + 2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 v_1}{\partial y^4} \right) dt dt.$$

$$\text{Assume } u_0(x, y, t) = \sum_{n=0}^{\infty} a_n(x, y)t^n, \quad v(x, y, 0) = u(x, y, 0),$$

$v_t(x, y, 0) = u_t(x, y, 0)$ and solving the above equation for $v_1(x, y, t)$ leads to;

$$v_1(x, t) = \left(-\frac{a_0}{2} \right) t^2 + \left(-\frac{a_1}{6} - \frac{1}{6} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \right) t^3 + \left(\frac{a_2}{12} - \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{a_{0xxxx}}{12} - \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{a_{0yyyy}}{12} \right) t^4 + \left(-\frac{a_5}{20} - \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{a_{1xxxx}}{120} - \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{a_{0yyyy}}{120} \right) t^5 + \dots$$

By taking $v_1(x, y, t) = 0$, coefficients $a_n, n = 1, 2, 3, \dots$ can be determined as the following:

$$a_0 = 0, \quad a_1 = - \left(2 + \frac{x^6}{6!} + \frac{y^4}{6!} \right), \quad a_2 = 0,$$

$$a_3 = \frac{1}{6} \left(2 + \frac{x^6}{6!} + \frac{y^4}{6!} \right), \dots$$

Therefore, an exact solution of the Eq. (15) can be expressed as:

$$u(x, y, t) = v_0(x, y, t) = \left(2 + \frac{x^6}{6!} + \frac{y^4}{6!} \right) t + a_0(x, y) \frac{t^2}{2} + a_1(x, y) \frac{t^3}{6} + a_2(x, y) \frac{t^4}{12} + a_3(x, y) \frac{t^5}{20} + \dots = \left(2 + \frac{x^6}{6!} + \frac{y^4}{6!} \right) \sin t$$

Example 3.

We finally consider the following three - dimensional fourth - order parabolic Equation;

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{4!z} \right) \frac{\partial^4 u}{\partial x^4} + \left(\frac{1}{4!x} \right) \frac{\partial^4 u}{\partial y^4} + \left(\frac{1}{4!y} \right) \frac{\partial^4 u}{\partial z^4} = - \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - \frac{1}{x^5} - \frac{1}{y^5} - \frac{1}{z^5} \right] \cos t, \quad \frac{1}{2} < x, y, z < 1, \quad t > 0 \quad (20)$$

With the initial conditions:

$$u(x, y, z, 0) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \quad \frac{\partial u}{\partial t}(x, y, z, 0) = 0 \quad (21)$$

And the boundary conditions:

$$u\left(\frac{1}{2}, y, z, t\right) = \left(\frac{1}{2y} + \frac{y}{z} + 2z\right) \cos t, \quad u(1, y, z, t) = \left(\frac{1}{y} + \frac{y}{z} + z\right) \cos t, \quad u\left(x, \frac{1}{2}, z, t\right) = \left(2x + \frac{1}{z} + \frac{z}{x}\right) \cos t, \quad u(x, 1, z, t) = \left(x + \frac{1}{z} + \frac{z}{x}\right) \cos t$$

$$u\left(x, y, \frac{1}{2}, t\right) = \left(\frac{x}{y} + 2y + \frac{1}{2x}\right) \cos t, \quad u(x, y, 1, t) = \left(\frac{x}{y} + y + \frac{1}{x}\right) \cos t, \quad (22)$$

$$\frac{\partial u}{\partial x}\left(\frac{1}{2}, y, z, t\right) = \left(\frac{1}{y} - 4z\right) \cos t, \quad \frac{\partial u}{\partial x}(1, y, z, t) = \left(\frac{1}{y} - z\right) \cos t$$

$$\frac{\partial u}{\partial y}\left(x, \frac{1}{2}, z, t\right) = \left(-4x + \frac{1}{z}\right) \cos t$$

$$\frac{\partial u}{\partial y}(x, 1, z, t) = \left(-x + \frac{1}{z}\right) \cos t$$

$$\frac{\partial u}{\partial z}\left(x, y, \frac{1}{2}, t\right) = \left(-4y + \frac{1}{x}\right) \cos t,$$

$$\frac{\partial u}{\partial z}(x, y, 1, t) = \left(-y + \frac{1}{x}\right) \cos t$$

To solve Eq. (20) by (NHMP), Eq. (5), in this equation will be in conical form as the following:

$$\begin{aligned} v(x, y, z, t) &= v(x, y, z, 0) + tv_t(x, y, z, 0) + \int_0^t \int_0^t u_0 dt dt \\ &- p \int_0^t \int_0^t \left(u_0 + \left(\frac{1}{4!z}\right) \frac{\partial^4 v}{\partial x^4} + \left(\frac{1}{4!x}\right) \frac{\partial^4 v}{\partial y^4} + \left(\frac{1}{4!y}\right) \frac{\partial^4 v}{\partial z^4} \right. \\ &\left. + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - \frac{1}{x^5} - \frac{1}{y^5} - \frac{1}{z^5} \right] \cos t \right) dt dt \end{aligned} \quad (23)$$

Suppose the solution of Eq. (23) has the following form:

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (24)$$

Substituting Eq. (24) into Eq. (23) and equating the coefficients of like powers of p , to obtain:

$$p^0 : v_0(x, y, z, t) = v_0(x, y, z, 0) + tv_{0t}(x, y, z, 0)$$

$$+ \int_0^t \int_0^t u_0(x, y, z, t) dt dt.$$

$$\begin{aligned} p^1 : v_1(x, y, z, t) &= \int_0^t \int_0^t \left(u_0 + \left(\frac{1}{4!z}\right) \frac{\partial^4 v_0}{\partial x^4} \right. \\ &+ \left(\frac{1}{4!x}\right) \frac{\partial^4 v_0}{\partial y^4} + \left(\frac{1}{4!y}\right) \frac{\partial^4 v_0}{\partial z^4} \\ &\left. + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - \frac{1}{x^5} - \frac{1}{y^5} - \frac{1}{z^5} \right] \cos t \right) dt dt \end{aligned}$$

$$p^2 : v_2(x, y, z, t) = - \int_0^t \int_0^t \left(\left(\frac{1}{4!z}\right) \frac{\partial^4 v_1}{\partial x^4} + \left(\frac{1}{4!x}\right) \frac{\partial^4 v_1}{\partial y^4} + \left(\frac{1}{4!y}\right) \frac{\partial^4 v_1}{\partial z^4} \right) dt dt$$

$$\text{By assuming } u_0(x, y, z, t) = \sum_{n=0}^{\infty} a_n(x, y, z) t^n,$$

$$v(x, y, z, 0) = u(x, y, z, 0),$$

$v_t(x, y, z, 0) = u_t(x, y, z, 0)$, And solving $v_1(x, y, t)$, leads to the following result:

$$\begin{aligned} v_1(x, t) &= \left(-\frac{a_0}{2} - \frac{1}{2} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \right) t^2 \\ &+ \left(-\frac{a_1}{6} \right) t^3 + \left(\frac{a_2}{12} \frac{1}{24} \left(\frac{a_{0xxxx}}{4!z} + \frac{a_{0yyyy}}{4!x} + \frac{a_{0zzzz}}{4!y} \right) \right. \\ &+ \frac{1}{24} \left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right) - \frac{1}{24} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \Big) t^4 \\ &+ \left(\left(\frac{a}{20} \frac{1}{120} \left(\frac{a_{0xxxx}}{4!z} + \frac{a_{0yyyy}}{4!x} + \frac{a_{0zzzz}}{4!y} \right) \right. \right. \\ &\left. \left. - \frac{1}{120} \left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right) + \frac{1}{120} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \right) \right) t^5 + \dots \end{aligned}$$

By taking $v_1(x, y, z, t) = 0$, coefficients $a_n, n = 1, 2, 3, \dots$ can be determined as the following:

$$a_0 = - \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right), \quad a_1 = 0, \quad a_2 = \frac{1}{2} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right), \quad a_3 = 0, \dots$$

Therefore, the exact solution of the Eq. (20) can be expressed as:

$$\begin{aligned} u(x, y, z, t) &= v_0(x, y, z, t) \\ &= \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + a_0(x, y, z) \frac{t^2}{2} + a_1(x, y, z) \frac{t^3}{6} \\ &+ a_2(x, y, z) \frac{t^4}{12} + a_3(x, y, z) \frac{t^5}{20} + \dots \\ &= \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \cos t. \end{aligned}$$

4. Conclusions

In this article, NHMP, has been introduced for solving variable coefficients fourth - order parabolic partial differential equations, this method give the exact solution of the problems. This result reveals the method explicit, effective and easy to use.

References

- [1] A. Q. M. Khaliq, E. H. Twizell, A family of second order methods for variable coefficient fourth order parabolic partial differential equations, *Internat. J. Comput. Math.* 23 (1987) 63–76.
- [2] D. J. Gorman, *Free Vibrations Analysis of Beams and Shafts*, Wiley, New York, 1975.
- [3] C. Andrade, S. McKee, High frequency A. D. I. methods for fourth order parabolic equations with variable coefficients, *Internat. J. Comput. Appl. Math* 3 (1977) 11–14.
- [4] S. D. Conte, A stable implicit difference approximation to a fourth order parabolic equation, *J. ACM* 4 (1957) 210–212.

- [5] S. D. Conte, W. C. Royster, Convergence of finite difference solution to a solution of the equation of a vibrating beam, *Proc. Amer. Math. Soc.* 7 (1956) 742–749.
- [6] D. J. Evans, A stable explicit method for the finite difference solution of fourth order parabolic partial differential equations, *Comput. J.* 8 (1965) 280–287.
- [7] A. Q. M. Khaliq and E. H. Twizell, A family of second - order methods for variable coefficient fourth - order parabolic partial differential equations, *Int. J. Comput. Math* 23 (1987), pp. 63–76.
- [8] D. J. Evans, W. S. Yousef, A note on solving the fourth order parabolic equation by the AGE method, *Internat. J. Comput. Math.* 40 (1991) 93–97.
- [9] A. M. Wazwaz, Analytic treatment for variable coefficient fourth - order parabolic partial differential equations, *Appl. Math. Comput.* 123 (2001), pp. 219–227.
- [10] A. M. Wazwaz, Exact solutions for variable coefficients fourth - order parabolic partial differential equations in higher - dimensional spaces, *Appl. Mathe. Comput.* 130 (2002), pp. 415–424.
- [11] J. Biazar and H. Ghazvini, He's variational iteration method for fourth - order parabolic equations, *Comput. Math. Appl.* 54 (2007), pp. 1047–1054.
- [12] Deniz Ağırseven Turgut Öziş, He's homotopy perturbation method for fourth - order parabolic equations, *Int. J. Comput. Math. Jon* (2009).
- [13] J. H. He, Homotopy perturbation method: a new nonlinear analytical technique, *Applied Mathematics and Computation* 135 (2003) 73 - 79.
- [14] J. H. He, Comparison of Homotopy perturbation method and Homotopy analysis method, *Applied Mathematics and Computation* 156 (2004) 527 - 539.
- [15] J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation* 151 (2004) 287 - 292.
- [16] J. H. He, Homotopy perturbation method for bifurcation of nonlinear problems, *International Journal of Nonlinear Sciences and Numerical Simulation* 6 (2005) 207 - 208.
- [17] J. H. He, Some asymptotic method for strongly nonlinear equations, *International Journal of Modern Physics* 20 (2006) 1144 - 1199.
- [18] J. Biazar, M. Eslami, A new technique for non - linear two - dimensional wave equation, *Scientia Iranica, Transactions B: Mechanical Engineering*, 2013. *Mechanics*, 35 (1), pp.37 - 43 (2000).
- [19] S. T. Mohyud - Din, A. Yildirim, Homotopy perturbation method for advection problems, *Nonlinear Science Letters A* 1 (2010) 307 - 312.
- [20] A. Beléndez, T. Beléndez, A. Markuez, and C. Neipp, Application of He's homotopy perturbation method to conservative truly nonlinear oscillators, *Chaos Solitons and Fractals* (2006), doi: 10.1016/j.chaos. 2006. 09. 070.
- [21] A. Beléndez, T. Beléndez, C. Neipp, A. Hernandez, and M. L. Alvarez, Approximate solutions of a non linear oscillator typified as a mass attached to a stretched elastic wire by the homotopy perturbation method, *Chaos Solitons and Fractals* (2007): 10.1016/j.chaos, 2007. 01. 089.
- [22] A. Belendez, A. Hernandez, T. Belendez, E. Fernandez, M. L. Alvarez, and C. Neip, Application of He's homotopy perturbation method to the Duffing - harmonic oscillator, *Int. J. Nonlinear Sci. Numer. Simul.* 8 (1) (2007), pp. 79–88.
- [23] J. Biazar and M. Eslami, "A new homotopy perturbation method for solving systems of partial differential equations," *Comput. Math Appl.*, in Press.
- [24] J. Biazar and M. Eslami, "A new method for solving the hyperbolic telegraph equation," *Comput. Math. Model.*, in Press.
- [25] Mostafa Eslami1 and Jafar Biazar 2, analytical solution of the klein - gordon equation by a new Homotopy Perturbation Method *Computational Mathematics and Modeling*, Vol. 25, No. 1, March, 2014.
- [26] Mohammed ELbadri, A New Homotopy Perturbation Method for Solving Laplace Equation *Advances in Theoretical and Applied Mathematics* ISSN 0973 - 4554 Volume 8, Number 4 (2013), pp. 237 - 242.