On the Finding the Other Eigenvalues and Eigen Functions and Ortogonal Basis with a Nonlocal Parity Condition of the Third Kind

Naser Abbasi^{1,*}, Hamid Mottaghi Golshan², Mahmood Shakori¹

¹Department of Mathematics, Lorestan University, Khoramabad, Iran

²Department of Mathematics, Islamic Azad University, Ashtian Branch, Ashtian, Iran

Email address:

Naserabbasi_persia@yahoo.com (N. Abbasi), motgolham@gmail.com (H. M. Golshan), mahmoodshakoori@gmail.com (M. Shakori)

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Abstract: In the present paper, we find out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal parity condition, the completeness and the basis property in the elliptic part of the third kind of a domain in $L^2(0, \frac{\pi}{2})$. We also consider a new boundaries condition and analyze the orthogonal basis of the eigenfunctions depending on parameters of the problem.

Keywords: Frankl Problem, Lebesgue Integral, Holder Inequality, Bessel Equation

1. Introduction

The classical Frankl problem was considered in [3]. The problem was further developed in [2, pp.339-345], [12, pp.235-252]. Several authors have also have investigated this issue (see [1-13]). On the solution of the Frankl problem in a special domain in [12]. About spectrum of the gasdynamic problem of Frankl for the model equation of mixed type in [10]. About construction of the gasdynamic problem of Frankl in [11]. Basis property of eigen -functions of the generalized problem of Frankl with a nonlocal parity condition and with the discontinuity of the gradient of solution in [9].

In the present paper, we consider boundaries conditions of the third kind on the intervals $(-1,0)$ and $(0,1)$ of the axis OY for which the derivatives of functions with respect to x on these intervals are related by linear dependence. We show that if the dependence coefficient exceeds -1 (the coefficient cannot be zero, since, otherwise, the problem will degenerate), then the systems of eigenfunctions of the problem forms a Riesz basis in the elliptic part of the domain.

2. Statement of the Modified Frankl Problem

Definition 1. Find a solution

$$u(x, y) \in C^0(\overline{D_+ \cup D_{-1} \cup D_{-2}}) \cap C^2(D_{-1}) \cap C^2(D_{-2}),$$

of the modified generalized Frankl problem

$$u_{xx} + \operatorname{sgn}(y)u_{yy} + \mu^2 \operatorname{sgn}(x+y)u = 0. \quad (1)$$

in $D_+ \cup D_-$ and the boundary conditions

$$u(1, \theta) = 0, \theta \in [0, \frac{\pi}{2}], \quad (2)$$

in the polar coordinate system

$$u(0, y) = 0, y \in [-1, 1] \quad (3)$$

$$k \frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(0, -y), y \in (0, 1), k \neq 0 \text{ is a constant} \quad (4)$$

$$k \lim_{y \rightarrow +0} u(x, y) = \lim_{y \rightarrow -0} u(x, y), x \in (0, 1). \quad (5)$$

here D_+ is the domain in the top half-plane bounded by a circle

$$\gamma = \{(x, y), x^2 + y^2 = 1\}$$

and the segment $[0, 1]$ of the axis OY, and $D_- = D_{-1} \cup D_{-2}$ is the domain in the bottom half space, where D_{-1} is bounded by the characteristic $y = x - 1$ and $y = -x$ and the segment $[0, 1]$ of the axis OX and D_{-2} is bounded by the characteristic $y = x - 1$ and $y = -x$ and the segment $[-1, 0]$ of the axis OY.

Definition 2. System $\{x_n\}_{n \in N} \subset X$ is called complete in X if $\overline{L[\{x_n\}_{n \in N}]} = X$.

Definition 3. System $\{x_n\}_{n \in N} \subset X$ is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n \in N}]} \forall k \in N$.

Remark 4. If the system is minimal in $L_p(I)$, then it is also minimal in $L_p(J)$, for $J \supset I$, and if it is complete in $L_p(J)$ for $J \subset I$.

Theorem 5 ([1,5]). The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series.

In the first series, the eigenvalues $\lambda = \mu_{nk}^2$ are found from the equation

$$J_{4n}(\mu_{nk}) = 0,$$

where $\mu_{nk}, n, k = 1, 2, \dots$, are roots of the Bessel equation (6), $J_\alpha(z)$, is the Bessel function [6,7, Russian translation], and the eigenfunctions are given by the formulas

$$\begin{aligned} \tilde{u}_{nk}(r, \theta) &= A_{nk} J_{4n}(\mu_{nk} r) \sin(4n(\frac{\pi}{2} - \theta)), \text{ in } D_{+1}, \\ \tilde{u}_{nk}(r, \theta) &= k A_{nk} J_{4n}(\mu_{nk} \rho) \sinh(4n\psi), \text{ in } D_{-1}, \\ \tilde{u}_{nk}(r, \theta) &= k A_{nk} J_{4n}(\mu_{nk} R) \sinh(4n\varphi), \text{ in } D_{-2}, \end{aligned} \quad (6)$$

where

$$x = r \cos \theta, y = r \sin \theta \quad 0 \leq \theta \leq \frac{\pi}{2}, r^2 = x^2 + y^2 \text{ in } D_+,$$

$0 < \rho < 1, -\infty < \psi < 0, \rho^2 = x^2 - y^2$, in D_{-1} , and, $x = R \sinh \varphi, y = -R \cosh \varphi, 0 < \varphi < +\infty, R^2 = y^2 - x^2$ in D_{-2} .

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ are found from the equation

$$J_{\alpha_n}(\tilde{\mu}_{nk}) = 0.$$

$$\text{where } \Delta = \frac{1}{\pi} \arcsin \frac{\kappa}{\sqrt{1 + \kappa^2}}, \Delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\},$$

$\tilde{\alpha}_n = 2 + 4(n - \Delta)\{x_n\}_{n \in N} \subset X$ and the corresponding eigenfunctions have the form

$$\begin{aligned} \tilde{u}_{nk}(r, \theta) &= \tilde{A}_{nk} J_{\alpha_n}(\tilde{\mu}_{nk} r) \sin(\tilde{\alpha}_n(\frac{\pi}{2} - \theta)), \text{ in } D_{+1}, \\ \tilde{u}_{nk}(r, \theta) &= k \tilde{A}_{nk} J_{\alpha_n}(\tilde{\mu}_{nk} \rho) \sinh(\frac{2x}{x^2 + 1} \cosh(\tilde{\alpha}_n \psi) \\ &\quad - x \frac{x^2 - 1}{x^2 + 1} \sinh(\tilde{\alpha}_n \psi)), \text{ in } D_{-1}, \\ \tilde{u}_{nk}(r, \theta) &= k \tilde{A}_{nk} J_{\alpha_n}(\tilde{\mu}_{nk} R) \sinh(\tilde{\alpha}_n \varphi), \text{ in } D_{-2}. \end{aligned} \quad (7)$$

Theorem 6 ([5]). The function system

$$\{\cos(4n)(\frac{\pi}{2} - \theta)\}_{n=0}^\infty, \{\cos 4(n + \Delta)(\frac{\pi}{2} - \theta)\}_{n=1}^\infty, \quad (8)$$

is complete and a Riesz basis in $L^2(0, \frac{\pi}{2})$, provided that

$$\Delta \in (\frac{-1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}).$$

3. Main Results

Consider Frankl problem (1)-(5) with the new boundary condition

$$\frac{\partial u}{\partial x}(0, y) = 0, y \in (-1, 0) \cup (0, 1), \quad (9)$$

$$ku(0, y) = u(0, -y), y \in [0, 1], ku(0, 0^+) = u(0, 0^-), \quad (10)$$

$$k \frac{\partial u}{\partial y}(x, 0^+) = \frac{\partial u}{\partial y}(x, 0^-), -\infty < k < +\infty, 0 < x < 1. \quad (11)$$

Theorem 7. The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series.

In the first series, the eigenvalues $\lambda = \mu_{nk}^2$ are found from the equation

$$J_{4n+\Delta}(\mu_{nk}) = 0, \quad (12)$$

where $\mu_{nk}, n, k = 1, 2, \dots$, are roots of the Bessel equation (6), $J_\alpha(z)$, is the Bessel function [4], and the eigenfunctions are given by the formula

$$u_{nk} = \begin{cases} A_{nk} J_{4n}(\mu_{nk} r) \cos(4n)(\frac{\pi}{2} - \theta), & \text{in } D^+; \\ k A_{nk} J_{4n}(\mu_{nk} \rho) \cosh(4n\psi), & \text{in } D_{-1}; \\ k A_{nk} J_{4n}(\mu_{nk} R) \cosh(4n\varphi), & \text{in } D_{-2}, \end{cases} \quad (13)$$

where

$$x = r \cos \theta, y = r \sin \theta, \text{ for } 0 \leq \theta \leq \frac{\pi}{2}, r^2 = x^2 + y^2 \text{ in } D_+$$

$$\begin{aligned} D_+, \quad x &= \rho \cosh \psi, y = \rho \sinh \psi, \quad \text{for,} \\ 0 < \rho < 1, -\infty < \psi < 0, \rho^2 &= x^2 - y^2, \quad \text{in } D_{-1}, \quad \text{and,} \\ x &= R \sinh \varphi, y = -R \cosh \varphi, \quad \text{for,} \\ 0 < \varphi < +\infty, R^2 &= y^2 - x^2 \text{ in } D_{-2}. \end{aligned}$$

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ are found from the equation

$$J_{4(n+\Delta)}(\tilde{\mu}_{nk}) = 0. \quad (14)$$

where $n, k = 1, 2, \dots$ and the $(\tilde{\mu}_{nk})$ are the roots of the Bessel equation (8).

$$u_{nk} = \begin{cases} \tilde{A}_{nk} J_{4(n+\Delta)}(\tilde{\mu}_{nk} r) \cos 4(n+\Delta)(\frac{\pi}{2} - \theta), & \text{in } D^+; \\ \tilde{A}_{nk} J_{4(n+\Delta)}(\tilde{\mu}_{nk} \rho). & \\ [\cosh 4(n+\Delta)\varphi \cos 4(n+\Delta)\frac{\pi}{2} & \text{in } D_{-1}; \\ + \kappa \sinh 4(n+\Delta)\psi \cos 4(n+\Delta)], & \\ k \tilde{A}_{nk} J_{4(n+\Delta)}(\tilde{\mu}_{nk} R). & \\ \cosh(4(n+\Delta)\varphi). & \text{in } D_{-2}, \\ [\cos 4(n+\Delta)\frac{\pi}{2} - \sin 4(n+\Delta)\frac{\pi}{2}], & \end{cases}$$

where $\Delta = \frac{1}{\pi} \arcsin \frac{\kappa}{\sqrt{1+\kappa^2}}, \Delta \in (0, \frac{1}{2})$, and

$$A_{nk}^2 \int_0^1 J_{4n}^2(\mu_{nk} r) r dr = 1,$$

$$\tilde{A}_{nk}^2 \int_0^1 J_{4n+\Delta}^2(\tilde{\mu}_{nk} r) r dr = 1,$$

$$A_{nk} > 0 \text{ and } \tilde{A}_{nk} > 0.$$

Theorem 8. The function system

$$\{\cos(4n)(\frac{\pi}{2} - \theta)\}_{n=0}^\infty, \{\cos 4(n+\Delta)(\frac{\pi}{2} - \theta)\}_{n=1}^\infty, \quad (15)$$

is complete and a Riesz basis in $L^2(0, \frac{\pi}{2})$, provided that

$$\Delta \in (\frac{-1}{4}, \frac{1}{2}).$$

Proof. In order to prove this theorem we use the method in [1, 6] by considering convergence function

$$f(\theta) = \sum_{n=0}^\infty A_n \cos 4n(\frac{\pi}{2} - \theta) + \sum_{n=1}^\infty B_n \cos 4(n+\Delta)(\frac{\pi}{2} - \theta),$$

in $L^2(0, \frac{\pi}{2})$ and Riesz basis the system $(\sin 4(n+\Delta)(\frac{\pi}{2} - \theta))$ for $\Delta \in (\frac{-1}{4}, \frac{3}{4})$.

Remark 9. For $\Delta < \frac{-1}{4}$ the system (10) is not complete but is minimal, for $\Delta > \frac{3}{4}$ is complete but is not minimal, and if $\Delta = \frac{-1}{4}$, is complete and minimal.

Theorem 10. The system of eigenfunctions

$$u_{nk}(r, \theta) = A_{nk} J_{4n}(\mu_{nk} r) \cos(4n)(\frac{\pi}{2} - \theta),$$

$$\tilde{u}_{nk}(r, \theta) = \tilde{A}_{nk} J_{4(n+\Delta)}(\tilde{\mu}_{nk} r) [\cosh 4(n+\Delta)\varphi \cos 4(n+\Delta)\frac{\pi}{2}],$$

is complete and basis in the space $L^2(0, \frac{\pi}{2})$, therefore

$$\int_0^{\frac{\pi}{2}} \int_0^1 f(r, \theta) u_{nk}(r, \theta) r dr d\theta = 0,$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 f(r, \theta) \tilde{u}_{nk}(r, \theta) r dr d\theta = 0,$$

and $f \in L^2(0, \frac{\pi}{2})$ then $f = 0$ in $(0, \frac{\pi}{2})$.

Proof. Using Fubini theorem for any $n, k = 1, 2, \dots$ we have

$$0 = \int_0^{\frac{\pi}{2}} \int_0^1 f(r, \theta) u_{nk}(r, \theta) r dr d\theta$$

$$\int_0^1 (r J_{4n}(\mu_{nk} r) \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta) dr,$$

again since $f \in L^2\left(0, \frac{\pi}{2}\right)$ so;

$$\int_0^1 \int_0^{\frac{\pi}{2}} |f(r, \theta)|^2 d\theta dr < \infty.$$

and since the system $\{\sqrt{r} J_{4n}(\mu_{nk} r)\}_{k=1}^\infty$ in $L^2(0, 1)$ is orthogonal and complete, it is enough to prove:

$$\sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta \in L^2(0, 1).$$

Using the Holder inequality

$$\begin{aligned} |\sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta|^2 &< \frac{1}{2} r \int_0^{\frac{\pi}{2}} |f^2(r, \theta)| d\theta \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{\pi}{4} r \int_0^{\frac{\pi}{2}} |f(r, \theta)|^2 d\theta = \frac{\pi}{4} r \int_0^{\frac{\pi}{2}} |f(r, \theta)|^2 d\theta, \end{aligned}$$

with the integration interval $(0, 1)$.

$$\begin{aligned} \int_0^1 |\sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta|^2 dr &? \\ &< \frac{\pi}{4} \int_0^1 \int_0^{\frac{\pi}{2}} r |f(r, \theta)|^2 dr d\theta < \infty. \end{aligned}$$

This inequality is equivalent to

$$\left\{ \int_0^1 \sqrt{r} \left| \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta \right|^2 dr \right\}^{\frac{1}{2}} < \infty.$$

Also system $\{\sqrt{r} J_{4n}(\mu_{nk} r)\}_{k=1}^{\infty}$ is orthogonal and complete in $L^2(0, \frac{\pi}{2})$ of relation

$$\int_0^1 (\sqrt{r} J_{4n}(\mu_{nk} r) \sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta) dr = 0,$$

imply that

$$\sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta = 0.$$

According to theorem 6, we conclude that $f(r, \theta) = 0$ in $L^2(0, 1)$. Similarly, if we consider the above calculations for sequence $\{\cos 4(n + \Delta)(\frac{\pi}{2} - \theta)\}_{n=1}^{\infty}$, we have

$$\sqrt{r} \int_0^{\frac{\pi}{2}} f(r, \theta) \cos 4(n + \Delta)(\frac{\pi}{2} - \theta) d\theta = 0.$$

Because completeness

$$\{\cos 4(n + \Delta)(\frac{\pi}{2} - \theta)\}_{n=0}^{\infty}, f(r, \theta) = 0 \text{ in } L^2(0, 1).$$

The proof of the theorem is complete.

Remark 11. If $\Delta = 0$ then the system becomes the system $\{\sin(2n\theta)\}_{n=1}^{\infty}$ which is basis in the space $L^p(0, \frac{\pi}{2})$ and an orthogonal basis in the space $L^2(0, \frac{\pi}{2})$.

The proof of remark 11 results from theorem 10.

Remark 12. In case $\Delta > \frac{3}{4}$ and $\Delta \neq \frac{1}{2} + k, k \in \mathbb{N}$ then the system is complete but is not minimal.

In case $\Delta < \frac{-1}{4}$ and $\Delta \neq -\frac{1}{4} - k, k \in \mathbb{N}$ then the system is not complete but is minimal.

In case $\Delta = \frac{1}{2} + k, k \in \mathbb{Z}$ then the system is complete but is not minimal in the space $L^2(0, \frac{\pi}{4})$ and is not complete in the space $L^2(0, \frac{\pi}{2})$.

The proof of remark 12 results from theorem 8.

4. Conclusion

Consider Frankl problem (1)-(5) with the new boundary condition (9)-(10) and (11), so we find out the eigenvalues of the problem with a nonlocal parity condition, the completeness and the basis property in the elliptic part of the third kind of a domain in $L^2(0, \frac{\pi}{2})$.

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References

- [1] N. Abbasi, Basis property and completeness of the eigenfunctions of the Frankl problem. (Russian) Dokl. Akad. Nauk 425 (2009), no. 3, 295-298; translation in Dokl. Math. 79 (2009), no. 2, 193-196.
- [2] A. V. Bitsadze, Nekotorye klassy uravneni v chastnykh proizvodnykh. (Russian) [Some classes of partial differential equations] "Nauka", Moscow, 1981. 448pp.
- [3] F. Frankl, On the problems of Chaplygin for mixed sub- and supersonic flows. (Russian) Bull. Acad. Sci. URSS. Sr. Math. [Izvestia Akad. Nauk SSSR] 9, (1945), 121-143.
- [4] H. van Haeringen, L.P. Kok, Higher transcendental functions, Vol. II [McGraw-Hill, New York, 1953; MR 15, 419] by A. Erdlyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. Math. Comp. 41 (1983), no. 164, 778.
- [5] E.I. Moiseev, The basis property for systems of sines and cosines. (Russian) Dokl. Akad. Nauk SSSR 275 (1984), no. 4, 794-798.
- [6] N. Abbasi, E.I. Moiseev, Basis property of eigenfunctions of the Generalized Gaseodynamic problem of Frankl with a nonlocal oddness conditions, integral transforms and special Functions. (England) 21 (2010), no. 4, 286-294.
- [7] E. I. Moiseev, N. Abbasi, The basis property of an eigenfunction of the Frankl problem with a nonlocal parity condition in the space Sobolev ($W_1^p(0, \pi)$). Integral Transforms Spec. Funct. (England) 22 (2011), no. 6, 415-421.
- [8] E. I. Moiseev, On the solution of the Frankl problem in a special domain, Differ. Uravn. 8(4) (1992), pp. 721-723.

- [9] E. I. Moiseev and N. Abbasi, Basis property of eigen functions of the generalized problem of Frankl with a nonlocal parity condition and with the discontinuity of the gradient of solution, Differ.Uravn.44(10)(2009),pp.155 -160.
- [10] K. B. Sabitov, About spectrum of the gasdynamic problem of Frankl for the model equation of mixed type, Differ. Uravn. 39(11) (2003), pp.1568-1570.
- [11] K. B. Sabitov and V. V. Tikhomirov, About construction of the gasdynamic problem of Frankl, Differ.Uravn.23 (10) (1990), pp.100-109.
- [12] M.M. Smirnov, Uravneniyasmeshannogotipa. (Russian) [Equations of mixed type] Izdat. "Nauka", Moscow 1970 295pp. New York, 1955.vii+329. pp.
- [13] A. Zygmund, Trigonometrical series. Dover Publications, New York, 1955.vii+329. pp.