Abstract: George and Veeramani [1] modify the concept of fuzzy metric spaces introduced by Kramosil and Michalek [4], Aamri and Moutawakil [8] generalized the notion of non-compatible mapping in metric space by E.A. property. Continuing the above concept we prove some common fixed point theorem for a pair of weakly compatible maps under E.A. Like property.

Keywords: Fuzzy Metric Space, E. A. Property, E. A. Like Property, Weakly Compatible Maps

1. Introduction

Fuzzy set theory has various applications in different area. When the notion of fuzzy set was introduced, then it was the turning point in the development of mathematics. It was introduced by Zadeh [7]. Fuzzy set theory has various application in applied science such as neural network theory, stability theory, mathematical programming, modelling theory, engineering science, medical science etc. George and Veeramani [1] modify the concept of fuzzy metric spaces introduced by Kramosil and Michalek [4], with a view to obtain a Hausdorff topology on fuzzy metric spaces, continuously, many authors gives very important results a Sessa [14], Vasuki [12] etc. Aamri and Moutawakil [8] generalized the notion of non-compatible mapping in metric space by E.A. property. It was pointed out in [9] that property E.A. buys containment of ranges without any continuity requirement besides minimizes the commutatively at their point of coincidence.

In this paper, we establish some new results in common fixed point theorems in fuzzy metric spaces under E. A. Like [6].

2. Definition

Definition 2.1 [2] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norms if * satisfying conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous; if and only if
3. $a * 1 = a$ for all $a \in [0, 1]$
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Example 2.2 $a * b = \min\{a, b\}, a * b = a * b$.

Definition 2.3 [1] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions, $\forall x, y, z \in X, s, t > 0$:

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1$ if and only if $x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
5. $M(x, y, t; : (0, \infty) \to (0, 1]$ is continuous.

Then $M$ is called a fuzzy metric on $X$. Then $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

Example 2.4 [1] Let $(X, d)$ be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let $M_d$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$M_d(x, y, t) = \frac{t}{t + d(x, y)}$. Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric $d$ as the standard intuitionistic fuzzy metric.

Definition 2.5 Two self-mappings $f$ and $g$ of a fuzzy metric space $(X, M, *)$ are called compatible if
$\lim_{n \to \infty} M(fg x_n, fg x_n, t) = 1$ whenever $\{x_n\}$ is a sequence in $X$ such that

$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x$ for some $x \in X$.

Lemma 2.6 Let $(X, M, \ast)$ be fuzzy metric space. If there exists $q \in (0,1)$ such that

$M(x, y, z) \geq q M(y, x, z)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 2.7 Let $X$ be a set, $f$ and $g$ self maps of $X$. A point $x \in X$ is called a coincidence point of $f$ and $g$ iff $f x = g x$. We shall call $w = f x = g x$ a point of coincidence of $f$ and $g$.

Definition 2.8 [3] A pair of maps $f$ and $g$ satisfy the property $E. A.$ if there exists a sequence $\{x_n\}$ such that

$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z$ for some $z \in X$.

Definition 2.10 Let $f$ and $g$ be two self-maps of a fuzzy metric space $(X, M, \ast)$. We say that $f$ and $g$ satisfy the property $E. A.$ like property if there exists a sequence $\{x_n\}$ such that

$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z$ for some $z \in X$.

Definition 2.11 Let $X = [0, 2)$ be a fuzzy metric space. If there exists

$\{x, y, t\} = \left\{\frac{t}{1+4(x+y)}\right\}$

for all $x, y \in X$ then $(X, M, \ast)$ is a fuzzy metric space.

We define $x_n = 0.5 + \frac{1}{n}$ and $y_n = 1 + \frac{1}{n}$

We have $A(x) = \{0.25\} \cup \{0.26, 1\}$

$S(X) = [0.25] \cup (0.35, 1.75] \cup T(X) = (0.15, 0.5]$

And $B(X) = (0.25] \cup (0.20, 1.25]$

Also $\lim n \to \infty Ax_n = \lim n \to \infty \left\{0.5 \left\lfloor \frac{1}{n} \right\rfloor \right\} = 0.25 \in S(X)$

$\lim n \to \infty Sx_n = \lim n \to \infty \left\{0.5 \left\lfloor \frac{1}{n} \right\rfloor \right\} = 0.25 \in A(X)$

$\lim n \to \infty Ty_n = \lim n \to \infty \left\{0.5 \left\lfloor \frac{1}{n} \right\rfloor \right\} = 0.25 \in B(X)$

And $\lim n \to \infty By_n = \lim n \to \infty \left\{0.5 \left\lfloor \frac{1}{n} \right\rfloor \right\} = 0.25 \in T(X)$

Definition 2.12 (Common E. A. Property) Let $A, B, S, T: X \to X$ where $X$ is a fuzzy metric space, then the pair $(A, S)$ and $(B, T)$ said to satisfy common E. A. property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = \lim_{n \to \infty} B y_n = z$ for some $z \in X$.

Definition 2.13 (Common E. A. Like Property) Let $A, B, S$ and $T$ be self maps of a fuzzy metric space $(X, M, \ast)$, then the pairs $(A, S)$ and $(B, T)$ said to satisfy common E. A. Like property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = \lim_{n \to \infty} B y_n = z$,

where $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Role of E.A. property in proving common fixed point theorems can be concluded by following;

(1) It buys containment of ranges without any continuity requirements.

(2) It minimizes the commutativity conditions of the maps to the commutativity at their points of coincidence.

(3) It allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

Of course, if two mappings satisfy E. A. like property then they satisfy E. A. property also, but, on the other hand, E. A. like property relaxes the condition of containment of ranges and closeness of the ranges to prove common fixed point theorems, which are necessary with E. A. property.

3. Main Results

Theorem-(3.1) - Let $f$ and $g$ be self-maps of a fuzzy metric space $(X, M, \ast)$, satisfying $M(x, y, t) > 0$ for all $x, y \in X$ and $t > 0$, such that following condition holds-(1)

$M(f x, g y, t) \geq \max\left\{M(f x, g x, t), M(g y, g y, t)\right\},$

for all $x, y \in X$.

(II) $f$ and $g$ satisfy the E.A. Like property.

Where $r: [0,1] \to [0,1]$ is a continuous function such that

$r(t) > t$ for each $0 < t < 1$, $r(0) = 0$ and $r(1) = 1$.

Then there exist a unique common fixed point of $f$ and $g$.

Proof – Since $f$ and $g$ satisfy E. A. Like property. Therefore there exists a sequence $\{x_n\}$ in $X$

Such that

$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \in f(x)$ or $g(x).$ Suppose that

$\lim_{n \to \infty} f x_n = z \in g(x),$

Therefore $z = g u$ for some $u \in X$. Now we show that

$f u = g u$, from $(I)$, we have

$M(f u, f x_n, t) \geq \max\left\{M(f u, g x_n, t), M(g u, f x_n, t)\right\},$

Taking $l \lim n \to \infty$, we get.

$M(f u, z, t) \geq \max\left\{M(f u, z t), M(g u, z t)\right\}$

$= \max\left\{M(f u, z t), M(z, z t)\right\}$

$= r(1) = 1$

This implies that $f u = z = g u$. i.e $u$ is coincidence point of $f$ and $g$.

Since $f$ and $g$ are weakly compatible. Therefore $f z = g u = f u = g z$

Now we show that $f z = z$. If not from $(I)$, we have

$M(f z, f x_n, t) \geq \max\left\{M(f z, g x_n, t), M(g z, f x_n, t)\right\},$

Taking $l \lim n \to \infty$, we get.

$M(f z, z, t) \geq \max\left\{M(f z, z t), M(g z, z t)\right\}$

$= r(1) = 1$

$M(f z, z, t) \geq \max\left\{M(f z, z t), M(g z, z t)\right\}$

$= r(1) = 1$
which is a contradiction. Hence $fz = gz = z.$ Hence $z$ is a common fixed point of $f$ and $g.$

Uniqueness – Let $z_1$ be another fixed point of $f$ and $g$, such that $z_1 \neq z$, then from (I), we have

$$M(fz, fz, t) \geq r \left( \max \left\{ M(fz, gz, t), M(gz, fz, t) \right\} \right)$$

$$M(z, z_1, t) \geq r(\max(M(z, z_1, t), M(z, z_1, t), M(z, z_1, t)))$$

$$M(z, z_1, t) \geq r(M(z, z_t, t)) > M(z, z_1, t)$$

Which is a contradiction. Hence $z = 1 = z_1$.

Hence $z$ is a common fixed point of $f$ and $g$.

Uniqueness – Let $C$ be another fixed point of $f$ and $g$, such that $C \neq C$, then from (I), we have

$$M(C, C, t) \geq r \left( \max \left\{ M(C, C, t), M(C, C, t) \right\} \right)$$

$$M(C, C, t) \geq r(\max(M(C, C, t), M(C, C, t), M(C, C, t)))$$

$$M(C, C, t) \geq r(M(C, C, t)) > M(C, C, t)$$

Which is a contradiction. Hence $C = C$.

Theorem 3.2. Let $A, B, S, T$ be self-maps of a fuzzy metric space $(X, M, t)$ satisfying the following condition-

(I) \(M(Ax, By, t), M(Sx, At, t), M(By, Sx, t), a.M(Ax, By, t) + b.M(Ax, Ty, t), \)

$$c.M(Sx, By, t) + d.M(Sx, Ty, t))$$

where $a, b, c, d \geq 0$

Then $A, B, S, T$ have a unique common fixed point.

Proof – Since (A,S) and (B,T) satisfy common E. A. property for all $x, y$ in $X$ and $t > 0$, where $a$ and $b$ are not simultaneous 0, and $a, b, c, d \geq 0$.

Then $A, B, S, T$ have a unique common fixed point.

Now we claim that $A = S$ and $B = T$.

Hence $Bv = z = Tu$.

Since the pair $(A, S)$ is weakly compatible, therefore $Az = ASu = SAu = S$. A
\[ M(Az, z, t) \geq r(1) = 1 \]

Hence \( Az = z = Sz \)  

Since the pair \((B,T)\) is weakly compatible, therefore \( Bz = BTv = Tbv = Tz \).

Now we show that \( Bz = z \),

\[ M(Ax_n, Bz, t) \geq r(\max(M(Sx_n, Tz, t), M(Bz, Sx_n, t), M(Ax_n, Tz, t)), \]

\[ a, M(Ax_n, Bz, t) + b, M(Ax_n, Tz, t) \]

\[ a, M(Sx_n, Bz, t) + b, M(Sx_n, Tz, t) \]

\[ c, M(Sx_n, Bz, t) + d, M(Sx_n, Tz, t) \]

\[ M(Bz, Tz, t) + d \]

\[ \lim n \to \infty, \text{we get} \]

\[ M(z, Bz, t) \geq r(\max(M(z, Bz, t), M(Bz, z, t), M(z, Bz, t), \]

\[ a, M(z, Bz, t) + b, M(z, Bz, t) \]

\[ a, M(Bz, Tz, t) + b, M(Bz, Tz, t) \]

\[ c, M(z, Bz, t) + d, M(z, Bz, t) \]

\[ M(Bz, Tz, t) + d \]

\[ M(z, Bz, t) \geq r(1) = 1 \]

Hence \( Bz = z = Tz \).

Thus \( z \) is common fixed point of \( A, B, S \) and \( T \).

Uniqueness – Suppose that \( z_1 \) is another common fixed point of \( A, B, S \) and \( T \).

such that \( z_1 \neq z \), then from (1)

\[ M(Az, Bz_1, t) \geq r(\max(M(Sz, Tz_1, t), M(Bz_1, Sz, t), \]

\[ M(Az, Tz_1, t), M(Sz, Az, t), \]

\[ a, M(Az, Bz_1, t) + b, M(Az, Tz_1, t) \]

\[ a, M(Sz, Bz_1, t) + b, M(Sz, Tz_1, t) \]

\[ c, M(Sz, Bz_1, t) + d, M(Sz, Tz_1, t) \]

\[ M(Bz_1, Tz_1, t) + d \]

\[ M(z, z_1, t) \geq r(\max(M(z, z_1, t), M(z_1, z, t), M(z, z_1, t), \]

\[ M(z, z_1, t), a, M(z, z_1, t) + b, M(z, z_1, t) \]

\[ c, M(z, z_1, t) + d, M(z, z_1, t) \]

\[ M(z, z_1, t) \geq r(\max(M(z, z_1, t), M(z_1, z, t), \]

\[ M(z_1, z, t), 1, M(z, z_1, t) \]

\[ M(z_1, z, t) \geq r(1) = 1 \]

Hence \( z = z_1 \).

Theorem 3.3 Let \( A, B, S \) and \( T \) be self maps of a fuzzy metric space \((X, M, *)\) satisfying the following conditions:

(i) \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \),

(II) \( M(Ax, By, t) \geq \max(r \left( M(Sx, Ty, t), M(Ax, Sz, t), \right) \)

for all \( x, y \in X \) and \( t > 0 \).

(III) Pairs \((A, S)\) or \((B, T)\) satisfy E. A. property,

(IV) Pairs \((A, S)\) and \((B, T)\) are weakly compatible.

If the range of one of \( A, B, S \) and \( T \) is a closed subset of \( X \), then \( A, B, S \) and \( T \) have a common fixed point in \( X \).

Proof- Proof as above.

References


