Pentacyclic Harmonic Graph

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Abstract: Let $G$ be a graph on $n$ vertices $v_1, v_2, \ldots, v_n$ and let $d(v_i)$ be the degree of vertex $v_i$. A graph $G$ is defined to be harmonic if $(d(v_1), d(v_2), \ldots, d(v_n))$ is an eigenvector of the $(0,1)$-adjacency matrix of $G$. We now show that there are 4 regular and 45 non-regular connected pentacyclic harmonic graphs and determine their structure. In the end we conclude that all of c-cyclic harmonic graphs for $1 \leq c \leq 5$ are planar graphs.

Keywords: Harmonic Graph, Eigenvalue, Spectra

1. Introduction

Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$ vertices $v_1, v_2, \ldots, v_n$ and $|E(G)| = m$ edges. We say that $G$ is c-cyclic, whenever $c = m - n + p$, which $p$ is the number components of $G$. In [1] B. Borovicanin and et al, studied the c-cycle graphs for $c = 1, 2, 3, 4$. All harmonic trees were constructed in [8] and the number of walks counted on some harmonic graph in [4, 5]. In [9, 10, 11] founded some result on harmonic graphs.

In this paper we study the c-cycle graphs for $c = 5$. If the graph $G$ is connected and $c = 0$ then $G$ is a tree.

The following elementary properties of harmonic graphs obtain of the spectra properties of graphs [2, 3, 6, 7]. Let $d(v_i)$ be the degree of vertex $v_i$ for $1 \leq i \leq n$, that is the number of the first neighbors of $v_i$. Vertex of degree k is called a k-vertex. Vertex of degree zero is called pendent. The column-vector $(d(v_1), d(v_2), \ldots, d(v_n))'$ is denoted by $d(G)$. The number of k-vertex denoted by $n_k$ and we have

$$n_1 = 1$$

$$n_0 = 0$$

Definition 1. The adjacency matrix $A(G) = [a_{ij}]$ is the $n \times n$ matrix for which $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. Eigenvalues and eigenvectors of matrix $A(G)$ is called eigenvalues and eigenvectors of graph $G$.

Definition 2. A graph $G$ is said to be harmonic if there exists a constant $\lambda$, such that

$$\sum_{v \in V(G)} d(v) = \lambda d(G).$$

Thus, graph $G$ is harmonic if and only if $d(G)$ is one of its eigenvectors, theses graphs are called $\lambda$-harmonic.

Equation (3) result that $\lambda$ is a rational number and equation (4) implies that $\lambda$ is not proper fraction, they follows that $\lambda$ must be an integer.

Example 1. A $\lambda$-regular graph is a $\lambda$-harmonic graph.

By summing the expressions (3) over all $i = 1, 2, \ldots, n$ we have

$$\sum_{v \in V(G)} d(v)(d(v) - \lambda) = 0$$

equivalently

$$\sum_{k \geq 1} k(k-\lambda)n_k = 0.$$
2. Some Auxiliary Results

We have the follow results of [1].

Lemma 1.

i. Let H be a graph obtained from G by adding to it an arbitrary number of isolated vertices, then H is harmonic if and only if G is harmonic.

ii. Any graph without isolated vertices is $\lambda$-harmonic if and only if all its components are $\lambda$-harmonic.

iii. Let G be a connected $\lambda$-harmonic graph. Then $\lambda$ is greatest eigenvalue of G and its multiplicity is one. Also if $m > 0$ then $\lambda \geq 1$ and equality occurs if and only if $G = K_2$.

From Lemma 2.1., it is enough to restrict our considerations to connected non-regular graphs. In [8], shown that for any positive integer $\lambda$ there is a unique connected $\lambda$-harmonic that is a tree and denoted by $T_\lambda$.

Let $G$ be a connected $\lambda$-harmonic graph. Then $\lambda$ is the unique connected non-regular $\lambda$-harmonic graph, depicted in Figures 1-17.

Theorem 1.

There are exactly 45 non-regular connected pentacyclic harmonic graphs, depicted in Figures 1-17.

Proof:

Because of Lemma 6, if $\lambda = 5$ then $\lambda$ cannot be greater than 4. Since $\lambda = 5$, therefore, $m = n + 4$, on the other hand, Lemmas 2 and 5, result that $\lambda$ cannot equal to 1 and 2, hence $\lambda = 3$ or $\lambda = 4$. At the first, suppose that $\lambda = 3$.

By the Lemma 2.4 if $\Delta$ is the maximal degree in a pentacyclic harmonic graph, then $\Delta \leq 6$ and in case $\lambda = 4$ by Lemma 4 we have $\Delta \leq 12$. From Lemma 2 we the conclude that only the following 11 cases need to be examined:

Case 1: $\lambda = 3$, $\Delta = 6$
Case 2: $\lambda = 3$, $\Delta = 5$
Case 3: $\lambda = 3$, $\Delta = 4$
Case 4: $\lambda = 4$, $\Delta = 12$
Case 5: $\lambda = 4$, $\Delta = 11$
Case 6: $\lambda = 4$, $\Delta = 10$
Case 7: $\lambda = 4$, $\Delta = 9$
Case 8: $\lambda = 4$, $\Delta = 8$
Case 9: $\lambda = 4$, $\Delta = 7$
Case 10: $\lambda = 4$, $\Delta = 6$
Case 11: $\lambda = 4$, $\Delta = 5$

Case 1: Lemma 5 implies that $n_1 - n_2 \geq 0$. By means of relation (7), for $\lambda = 5$, we have

$$-n_1 + n_3 + 2n_4 + 3n_5 + 4n_6 = 8$$

from which

$$2n_4 + 3n_5 + 4n_6 - 8 = n_1 - n_2 \leq 0$$

and we can conclude that

$$1 \leq n_6 \leq 2, n_5 \leq 1, n_4 \leq 2$$

from equation (1.6) we get

$$-2n_4 - 2n_5 + 4n_6 + 10n_7 + 18n_8 = 0.$$  

According to Lemma 7, the 5 and 6-vertices are adjacent only to 3-vertices. Since (3) the two neighbors of every 3-vertex, adjacent to a 6-vertex, must be a 1 and 2-vertex. Therefore $n_6 \geq 6$, $n_5 \geq 3$, and consequently, $n_1 + n_2 \geq 9$. In what follows we distinguish between 12 subcases:

Subcase 1:

$$n_1 = 0, n_2 = 5, n_3 = 1, n_4 = 9, n_5 = 1, n_6 = 4$$  

In this subcase, we have, $n_1 = 6$, $n_2 = 3$, $n_3 = 10$, $n_4 = 0$, $n_5 = 0$, $n_6 = 1$. Each of the three 2-vertices must be adjacent to two 3-vertices, and exactly of 4, 3-vertices remains. Therefore there cannot exist a 3-harmonic satisfies the condition (12).
Subcase 2:

\[ n_4 = 1, n_5 = 0, n_6 = 1, n_1 + n_2 = 11, n_i = n_i + 2 \]  (13)

The 4 and 6-vertices are adjacent only to 3-vertices and therefore the number of 3-vertices is greater than or equal to 10. Because of \( n_i \geq 3 \) we now have \( n_i = n_i - 2 \geq 8 \). Then, in this subcase we get

\[ n_1 = 8, n_2 = 3, n_i = 10, n_4 = 0, n_6 = 1 \]  (14)

the neighbors 3-vertex adjacent to 4-vertex has a 2-vertex, a 3-vertex, so we need at least 5, 2-vertices. This subcase also impossible.

Subcase 3:

\[ n_4 = 2, n_5 = 0, n_6 = 1, n_1 + n_2 = 13, n_3 = n_3 \]  (15)

Similar arguments subcase 2, we have

\[ n_1 = 10, n_2 = 3, n_i = 10, n_4 = 0, n_6 = 1 \]  (16)

this graph is nonconnected, then this subcase is impossible.

Subcase 4:

\[ n_4 = 0, n_5 = 1, n_6 = 1, n_1 + n_2 = 14, n_3 = n_i + 1 \]  (17)

The 5 and 6-vertices are adjacent only to 3-vertices and therefore the number of 3-vertices is greater than or equal to 11. Because of \( n_i \geq 3 \) we now have \( n_i = n_i - 1 \geq 10 \). Then, in this subcase we get

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( n_i \) & \( n_1 \) & \( n_2 \) & \( n_3 \) & \( n_4 \) & \( n_5 \) & \( n_6 \) \\
\hline
(a) & 10 & 4 & 11 & 0 & 1 & 1 \\
(b) & 11 & 3 & 12 & 0 & 1 & 1 \\
\hline
\end{tabular}
\caption{Cases of \( n_i = 0, n_1 = 1, n_6 = 1, n_1 + n_2 = 14, n_i = n_i + 1 \).}
\end{table}

the only case (a) can occurs and its graph is as follows.

Figure 1. First member of a family of 3-harmonic graphs which \( c=5 \) and \( \Delta=6 \).

Subcase 5:

\[ n_4 = 1, n_5 = 1, n_6 = 1, n_1 + n_2 = 16, n_i = n_i - 1 \]  (18)

Since 3-vertices adjacent to 4-vertex, 5-vertex, 6-vertex, are distinct, so we need at least 15, 3-vertices, then \( n_i \geq 16 \). It result that \( n_1 + n_2 \geq 19 \) that impossible.

Subcase 6:

\[ n_4 = 2, n_5 = 1, n_6 = 1, n_1 + n_2 = 18, n_i = n_i - 3 \]  (19)

Similar arguments Subcase 1.5 this subcase is impossible.

Subcase 7:

\[ n_4 = 0, n_5 = 0, n_6 = 2, n_1 + n_2 = 16, n_i = n_i \]  (20)

Since 3-vertices adjacent to 6-vertices are distinct, so we need at least 12, 3-vertices and 6, 2-vertices, then \( n_i \geq 12 \). It result that \( n_1 + n_2 = n_1 + n_2 \geq 18 \) that impossible.

Subcase 8:

\[ n_4 = 1, n_5 = 0, n_6 = 2, n_1 + n_2 = 20, n_i = n_i - 2 \]  (21)

In this subcase we have \( n_i \geq 16 \) then \( n_i \geq 18 \) and

\[ n_1 + n_2 \geq 21 \] that impossible.

Subcase 9:

\[ n_4 = 2, n_5 = 0, n_6 = 2, n_1 + n_2 = 22, n_i = n_i - 4 \]  (22)

In this subcase we have \( n_i \geq 16 \) then \( n_i \geq 20 \) and

\[ n_1 + n_2 \geq 23 \] that impossible.

Subcase 10:

\[ n_4 = 0, n_5 = 1, n_6 = 2, n_1 + n_2 = 23, n_i = n_i - 3 \]  (23)

In this subcase we have \( n_i \geq 17 \) therefore \( n_i \geq 20 \) and

\[ n_1 + n_2 \geq 26 \] that this is a contradiction.

Subcase 11:

\[ n_4 = 1, n_5 = 1, n_6 = 2, n_1 + n_2 = 25, n_i = n_i - 5 \]  (24)

In this subcase we have \( n_i \geq 21 \) thus \( n_i \geq 26 \) and

\[ n_1 + n_2 \geq 32 \] that this cannot happen.

Subcase 12:

\[ n_4 = 2, n_5 = 1, n_6 = 2, n_1 + n_2 = 27, n_i = n_i - 7 \]  (25)

In this subcase we get \( n_i \geq 21 \) hence \( n_i \geq 28 \) and

\[ n_1 + n_2 \geq 34 \] that impossible.

Case 2: \( \lambda = 3, \Delta = 5 \)

Lemma 5 follows that \( n_3 - n_i \geq 0 \). Equations (6) and (7) now became

\[ -2n_i - 2n_2 + 4n_3 + 10n_4 + 4n_6 = 0, \]  (26)

and

\[ -n_1 + n_3 + 2n_4 + 3n_5 = 8. \]  (27)

We have following subcases:

Subcase 13:

\[ n_4 = 0, n_5 = 2, n_1 + n_2 = 10, n_i = n_i + 2 \]  (28)

By the Lemma 7, every 5-vertex is adjacent only with 3-
vertices. Therefore $n_i \geq 10$ and $n_j \geq 8$, then we have

$$n_i \geq 10$$
$$n_j \geq 8$$

Therefore, $n_1 \geq 10$ and $n_2 \geq 8$.

### Table 2. Cases of $n_i = 0, n_j = 2, n_i + n_j = 10, n_1 = n_2 + 2$

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$n_j$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 8</td>
<td>2</td>
<td>10</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(b) 9</td>
<td>1</td>
<td>11</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(c) 10</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Case (b) does not hold, but for case (a) we have 3 harmonic graphs as follow:

![Figure 2](image1.png)

**Figure 2.** First members of a family of 3-harmonic graphs which $c=5$ and $\Delta=5$.

and for case (c) we have 8 harmonic graphs as follow:

![Figure 3](image2.png)

**Figure 3.** Second members of a family of 3-harmonic graphs which $c=5$ and $\Delta=5$.

**Subcase 14:**

$$n_i = 1, n_j = 2, n_i + n_j = 12, n_1 = n_1$$

(29)

In this subcase we have $n_1 \geq 14$ and $n_2 \geq 14$ that impossible.

**Subcase 15:**

$$n_i = 2, n_j = 2, n_i + n_j = 14, n_1 = n_2 - 2$$

(30)

Since $n_3 - n_i \geq 0$ then this subcase is impossible.

**Subcase 16:**

$$n_i = 0, n_j = 1, n_i + n_j = 5, n_1 = n_1 + 5$$

(31)

In this subcase we have $n_1 \geq 5$ then we have cases (b), (c) and (e) do not hold, but for case (a) we have 2 harmonic graphs as follow:

![Figure 4](image3.png)

**Figure 4.** Third members of a family of 3-harmonic graphs which $c=5$ and $\Delta=5$.

for case (d) we have 1 harmonic graphs as follow:

![Figure 5](image4.png)

**Figure 5.** Fourth members of a family of 3-harmonic graphs which $c=5$ and $\Delta=5$.

for case (f) we have 3 harmonic graphs as follow:

![Figure 6](image5.png)

**Figure 6.** Fifth members of a family of 3-harmonic graphs which $c=5$ and $\Delta=5$.

**Subcase 17:**

$$n_i = 1, n_j = 1, n_i + n_j = 7, n_1 = n_1 + 3$$

(32)

In this subcase we have $n_1 \geq 9$ then $n_1 \geq 6$ and

### Table 3. Cases of $n_i = 0, n_j = 1, n_i + n_j = 5, n_1 = n_1 + 5$

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$n_j$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 0</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(b) 1</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(c) 2</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(d) 3</td>
<td>2</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(e) 4</td>
<td>1</td>
<td>9</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(f) 5</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.** Cases of $n_i = 1, n_j = 1, n_i + n_j = 7, n_1 = n_1 + 3$.

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$n_j$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 6</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(b) 7</td>
<td>0</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
since the neighbors 3-vertex adjacent to 4-vertex has a 2-
vertex, a 3-vertex, so we need at least 2, 2-vertices, then this
subcase is impossible.

**Subcase 18:**

\[ n_4 = 2, n_5 = 1, n_1 + n_2 = 9, n_3 = n_7 + 1 \]

(33)

In this subcase we have \( n_3 \geq 9 \) hence \( n_1 \geq 8 \) and

Table 5. Cases of \( n_1 = 2, n_5 = 1, n_1 + n_2 = 9, n_3 = n_7 + 1 \).

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</thead>
<tbody>
<tr>
<td>(a)</td>
<td>8</td>
<td>1</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>(b)</td>
<td>9</td>
<td>0</td>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>

since the neighbors 3-vertex adjacent to 5-vertex has a
2-vertex, a 3-vertex, so we need at least 2, 2-vertices, then
this subcase is impossible.

**Case 3:** \( \lambda = 3, \Delta = 4 \)

Lemma 5 follows that \( n_3 - n_j \geq 0 \). Equations (6) and (7)
now became

\[ -2n_1 - 2n_2 + 4n_4 = 0, \]

and

\[ -n_1 + n_2 + 2n_4 = 8 \]

(35)

\( n_3 - n_j \geq 0 \) results that \( 1 \leq n_4 \leq 4 \) and the other hand
\( n_1 + n_2 = 2n_4 \), then we have following subcases:

**Subcase 19:**

\[ n_4 = 1, n_1 + n_2 = 2, n_3 = n_7 + 6 \]

(36)

Equivalently

Table 6. Cases of \( n_3 = 1, n_1 + n_2 = 2, n_3 = n_7 + 6 \).

<p>| | | | | |</p>
<table>
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<tr>
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</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>2</td>
<td>0</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>(c)</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

since the neighbors 3-vertex adjacent to 4-vertex has a 2-
vertex, a 3-vertex, so we need at least 2, 2-vertices, then
subcases (a) and (b) are impossible, but for case (c) we have
2 harmonic graphs as follow:

**Subcase 20:**

\[ n_4 = 2, n_1 + n_2 = 4, n_3 = n_7 + 4 \]

(37)

equivalently

Table 7. Cases of \( n_4 = 2, n_1 + n_2 = 4, n_3 = n_7 + 4 \).

<p>| | | | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>(b)</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>(c)</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>(d)</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>(e)</td>
<td>4</td>
<td>0</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

for case (a) we have 3 harmonic graphs as follow:

**Figure 8. Second members of a family of 3-harmonic graphs which c=5 and \( \Delta = 4 \).**

In subcase (b) we need at least 6, 3-vertices, then this
subcase is impossible, also subcases (d) and (e) are
impossible, but for case (c) we have 3 harmonic graphs as
follow:

**Figure 9. Third members of a family of 3-harmonic graphs which c=5 and \( \Delta = 4 \).**

**Subcase 21:**

\[ n_4 = 3, n_1 + n_2 = 6, n_3 = n_7 + 2 \]

(38)

If any two 4-vertices of three 4-vertices be adjacent, then
\( n_3 = 0 \), thus this manner cannot occurs. If just a 4-vertex
adjacent with the other 4-vertices, then we have 2 harmonic
graphs as follow:

**Figure 10. Fourth members of a family of 3-harmonic graphs which c=5 and \( \Delta = 4 \).**

if the only two 4-vertices of three 4-vertices be adjacent, then
we have 2 harmonic graphs as follow:
Figure 11. Fifth members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

Subcase 22:

\[ n_1 = 4, \ n_1 + n_2 = 8, \ n_3 = n_4 \]  
(39)

equivalently

Table 8. Cases of \( n_1 = 4, \ n_1 + n_2 = 8, \ n_3 = n_4 \).

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( n_3 )</th>
<th>( n_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 0</td>
<td>8</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(b) 1</td>
<td>7</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(c) 2</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(d) 3</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(e) 4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(f) 5</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>(g) 6</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>(h) 7</td>
<td>1</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>(k) 8</td>
<td>0</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

In above subcases we need to the even number of 3-vertices, then these subcases (b), (d), (e) and (h) are impossible. For case (a) we have 4 harmonic graphs as follow:

Figure 12. Sixth members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

for case (c) we have 3 harmonic graphs as follow:

Figure 13. Seventh members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

for case (e) we have 2 harmonic graphs as follow:

Figure 14. Eighth members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

for case (g) we have 2 harmonic graphs as follow:

Figure 15. Ninth members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

and also for case (k) we have 3 harmonic graphs as follow:

Figure 16. Tenth members of a family of 3-harmonic graphs which \(c=5\) and \(\Delta=4\).

Case 4: \( \lambda = 4, \ \Delta = 12 \)

Lemma 5 follows that \(2n_i - n_i \geq 0\). Equations (6) and (7) now became

\[-3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_5 + 21n_6 + 32n_7 + 45n_8 + 60n_9 + 77n_{10} + 96n_{11} = 0, \]  
(40)

and

\[-n_1 + n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 + 8n_8 + 9n_9 + 10n_{10} = 8. \]  
(41)

Since \( n_{12} \geq 1 \) and \( 2n_1 - n_i \geq 0 \) then this case is impossible.

Case 5: \( \lambda = 4, \ \Delta = 11 \)

Lemma 5 follows that \(2n_i - n_i \geq 0\). Equations (6) and (7) now became

\[-3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_5 + 21n_6 + 32n_7 + 45n_8 + 60n_9 + 77n_{10} = 0, \]  
(42)

and

\[-n_1 + n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 + 8n_8 + 9n_{11} = 8. \]  
(43)

Because of \( n_{12} \geq 1 \) and \( 2n_1 - n_i \geq 0 \) this cannot happen.

Case 6: \( \lambda = 4, \ \Delta = 10 \)

Lemma 5 follows that \(2n_i - n_i \geq 0\). Equations (6) and (7) now became

\[-3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_5 + 21n_6 + 32n_7 + 45n_8 + 60n_{10} = 0, \]  
(44)

and

\[-n_1 + n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 + 8n_{10} + 9n_{11} = 8. \]  
(45)

Since \( n_{10} \geq 1 \) and \( 2n_i - n_i \geq 0 \) then (45) implies that
\[ n_0 = 1, \ 2n_1 = n_1, \]
\[ n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8 = n_9 = 0 \]  
(46)

and (45) results that \( 3n_1 + 4n_2 = 60 \). Since vertices adjacent to 10-vertex are 4-vertex then \( n_1 \geq 10 \) and \( n_2 \geq 20 \). The equation \( 3n_1 + 4n_2 = 60 \) leads to \( n_1 = 20, \ n_2 = 0, \ n_4 = 10 \). This harmonic graph as follow:

Figure 17. First member of a family of 4-harmonic graphs which \( c=5 \) and \( \Delta=10 \).

Case 7: \( \lambda = 4, \ \Delta = 9 \)

From Lemma 5 we get \( 2n_1 - n_1 \geq 0 \). Equations (6) and (7) imply that

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 \\
\]

\[ + 21n_7 + 32n_8 + 45n_9 = 0, \]  
(47)

and

\[ -n_1 + n_2 + 2n_4 + 3n_5 + 4n_6 + 5n_7 + 6n_8 + 7n_9 = 8. \]  
(48)

Because of \( n_9 \geq 1 \) and \( 2n_1 - n_1 \geq 0 \) the relation (48) implies that

\[ n_9 = 1, \ 2n_1 = n_1, \ n_1 = 1, \ n_1 = n_6 = n_7 = n_8 = 0 \]  
(49)
or

\[ n_9 = 1, \ 2n_1 - 1 = n_1, \ n_1 = n_5 = n_6 = n_7 = n_8 = 0 \]  
(50)

and (47) results that \( 3n_1 + 4n_2 + 3n_3 = 45 \). Since vertices adjacent to 9-vertex are 4-vertex then \( n_1 \geq 9 \) and \( n_1 \geq 17 \). The equation \( 3n_1 + 4n_2 + 3n_3 = 45 \) results that this case is impossible.

Case 8: \( \lambda = 4, \ \Delta = 8 \)

Lemma 5 leads to \( 2n_1 - n_1 \geq 0 \). From equations (6) and (7) we get

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 \\
\]

\[ + 12n_7 + 21n_8 + 32n_9 = 0, \]  
(51)

and

\[ -n_1 + n_2 + 2n_4 + 3n_5 + 4n_6 + 5n_7 + 6n_8 + 7n_9 = 8. \]  
(52)

Since \( n_9 \geq 1 \) and \( 2n_1 - n_1 \geq 0 \) then (52) implies that

\[ n_9 = 1, \ 2n_1 - n_1 = 2, \ n_1 = n_6 = n_7 = n_8 = 0 \]  
(53)

and (51) implies that \( 3n_1 + 4n_2 + 3n_3 = 32 \). Since vertices adjacent to 8-vertex are 4-vertex so \( n_1 \geq 8 \) and \( n_1 \geq 14 \). The equation \( 3n_1 + 4n_2 + 3n_3 = 32 \) results that this case is impossible.

Case 9: \( \lambda = 4, \ \Delta = 7 \)

From Lemma 5 we have \( 2n_1 - n_1 \geq 0 \). Equations (6) and (7) lead to

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 + 21n_7 = 0, \]  
(54)

and

\[ -n_1 + n_2 + 2n_4 + 3n_5 + 4n_6 + 5n_7 = 8. \]  
(55)

Since \( n_9 \geq 1 \) and \( 2n_1 - n_1 \geq 0 \) then (55) implies that

\[ n_1 = 1, \ 2n_4 - n_1 + n_3 + 3n_5 = 3, \ n_6 = 0 \]  
(56)

and (54) results that \( 3n_1 + 4n_2 + 3n_3 = 21 \). If \( n_1 = 0 \) then vertices adjacent to 7-vertex are 4-vertex then \( n_1 \geq 7 \) and \( n_1 \geq 11 \). The equation \( 3n_1 + 4n_2 + 3n_3 = 21 \) results that this case is impossible. Also, if \( n_1 = 1 \) then \( n_1 = 0 \) and \( 2n_4 = n_1 \), then since some of vertices adjacent to 7-vertex are 4-vertex then \( n_4 \geq 5 \) and \( n_1 \geq 10 \). The equation \( 3n_1 + 4n_2 + 26 \) results that this case is impossible.

Case 10: \( \lambda = 4, \ \Delta = 6 \)

Lemma 5 follows that \( 2n_1 - n_1 \geq 0 \). Equations (6) and (7) now became

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 = 0, \]  
(57)

and

\[ -n_1 + n_2 + 2n_4 + 3n_5 + 4n_6 = 8. \]  
(58)

We have following subcases:

Subcase 23:

\[ n_9 = 1, \ 2n_1 = n_1, \ n_1 + 3n_5 = 4, \]

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 = 0 \]  
(59)

If \( n_1 = 0 \) then \( n_9 = 4 \) and \( n_1 = n_5 = n_6 = 0 \), therefore this manner not occurs. If \( n_1 = 1 \) then \( n_1 = 1 \) and \( 3n_1 + 4n_2 = 14 \).

On the other hand \( n_1 \geq 4 \) thus \( n_1 \geq 8 \) that this contradiction with \( 3n_1 + 4n_2 = 14 \). Hence this manner also cannot occurs.

Subcase 24:

\[ n_9 = 1, \ 2n_4 = n_1 + 1, \ n_1 + 3n_5 = 3, \]

\[ -3n_1 - 4n_2 - 3n_3 + 5n_4 + 12n_6 = 0 \]  
(60)

If \( n_1 = 0 \) then \( n_1 = 3 \) and \( n_1 = n_4 = 0 \) and \( n_4 = 1 \) therefore this manner is impossible. If \( n_1 = 1 \) then \( n_1 = 0 \) and \( 3n_1 + 4n_2 = 17 \). Also, \( n_9 \geq 4 \) thus \( n_1 \geq 8 \) that this contradiction with \( 3n_1 + 4n_2 = 17 \). Therefore this case also cannot occurs.
Subcase 25:
\[ n_1 = 1, 2n_2 = n_1 + 2, n_3 + 3n_4 = 2, \\
-3n_4 - 4n_5 - 3n_6 + 5n_7 + 12 = 0 \] (61)

In this subcases \( n_1 = 2, n_3 = 0, n_1 = 2, n_4 = 0, n_7 = 2 \) then this case is impossible.

Subcase 26:
\[ n_1 = 1, 2n_2 = n_1 + 3, n_4 + 3n_5 = 1, \\
-3n_4 - 4n_5 - 3n_6 + 5n_7 + 12 = 0 \] (62)

In this subcases \( n_1 = 1, n_2 = 0, n_3 = 3, n_4 = 0, n_7 = 3 \) then this case cannot happen.

Subcase 27:
\[ n_1 = 1, 2n_2 = n_1 + 4, n_4 + 3n_5 = 0, \\
-3n_4 - 4n_5 - 3n_6 + 5n_7 + 12 = 0 \] (63)

In this subcases \( n_1 = n_2 = 0, 3n_4 + 4n_5 = 12 \). Since vertices adjacent to 6-vertex are 4-vertex then \( n_3 \geq 6 \) and \( n_1 \geq 8 \). So this case is impossible.

Subcase 28:
\[ n_1 = 2, 2n_2 = n_1, n_3 = n_5 = 0, \\
3n_4 + 4n_2 = 24 \] (64)

Since some vertices adjacent to 6-vertex are 4-vertex then \( n_3 \geq 4 \) and \( n_2 \geq 8 \). That this contradiction with \( 3n_1 + 4n_2 = 24 \), then this case is impossible.

Case 11: \( \lambda = 4, \Delta = 5 \)

Lemma 5 follows that \( 2n_3 - n_4 \geq 0 \). Equations (6) and (7) now became
\[ -3n_1 - 4n_2 - 3n_4 + 5n_3 = 0, \] (65)

and
\[ -n_1 + n_3 + 2n_4 + 3n_5 = 8. \] (66)

Since vertices adjacent to 5-vertex are 4-vertex then \( n_3 = 5 \) and \( n_1 = 5 \) that this contradiction with \( 3n_1 + 4n_2 = 5 \). Hence this case is impossible.

Subcase 29:
\[ n_1 = 1, n_2 = 0, 2n_4 = n_1 + 5, 3n_1 + 4n_2 = 5 \] (67)

Since some vertices adjacent to 5-vertex are 4-vertex then \( n_2 = 5 \) and \( n_1 = 5 \) that this contradiction with \( 3n_1 + 4n_2 = 5 \). Hence this case is impossible.

Subcase 30:
\[ n_1 = 1, n_2 = 1, 2n_4 = n_1 + 4, 3n_1 + 4n_2 = 2 \] (68)

Since \( n_1 \geq 0 \) and \( n_2 \geq 0 \), hence this case cannot happen.

Subcase 31:
\[ n_1 = 2, n_2 = 0, 2n_4 = n_1 + 2, 3n_1 + 4n_2 = 10 \] (69)

Since some vertices adjacent to 5-vertex are 4-vertex then \( n_2 \geq 3 \) and \( n_4 \geq 4 \) that this contradiction with \( 3n_1 + 4n_2 = 10 \). Therefore this case is impossible.

Subcase 32:
\[ n_1 = 2, n_2 = 1, 2n_4 = n_1 + 1, 3n_1 + 4n_2 = 7 \] (70)

Some vertices adjacent to 5-vertex are 4-vertex then \( n_2 \geq 3 \) and \( n_4 \geq 5 \) that this contradiction with \( 3n_1 + 4n_2 = 7 \). Thus this case is impossible.

Subcase 33:
\[ n_1 = 2, n_2 = 2, 2n_4 = n_1, 3n_1 + 4n_2 = 4 \] (71)

Since some vertices adjacent to 5-vertex are 4-vertex then \( n_2 \geq 3 \) and \( n_4 \geq 6 \) that this contradiction with \( 3n_1 + 4n_2 = 4 \). Hence this case is impossible.

Definition 3: A graph is planar if it can be drawn in a plane without graph edges crossing.

Corollary 1: All of \( c \)-cyclic nonregular harmonic graphs for \( c \leq 5 \) are planar graphs.

4. Regular Harmonic Graphs

If a pentacyclic \( \lambda \)-harmonic graph be regular then we have \( n = \frac{8}{\lambda - 2} \) and \( n \geq \lambda + 1 \), therefore we have the only \( \lambda = 3, n = 8 \). In this case we have 4, 3-harmonic graphs as follows:

![Figure 18. Connected regular pentacyclic harmonic graphs.](image)

5. Conclusions

Let \( \#r(c) \) and \( \#nr(c) \) be denote the number of connected \( c \)-cyclic regular and nonregular harmonic graphs, respectively, for a fixed value \( c \). According to fined results, the number of harmonic graphs as follows.

<table>
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<th>( c )</th>
<th>( #r(c) )</th>
<th>( #nr(c) )</th>
<th>Remark</th>
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<td>( \infty )</td>
<td>( \lambda \geq 1 )</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>( \lambda = 2 )</td>
</tr>
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<td>0</td>
<td>0</td>
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<td>18</td>
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<td>5</td>
<td>4</td>
<td>45</td>
<td>( \lambda = 3, \lambda = 4 )</td>
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\( \geq 6 \) finite finite

Table 9. The number of harmonic graphs.
References


