On a Consumer Problem

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1. Introduction

Let at the moment \( t \) the production mapping \( a \) be given

\[
\begin{align*}
a(X) &= \{ \tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n) \in (R^m)^n | 0 \leq \sum_{i=1}^n \tilde{x}^i \leq 1 \} \\
\leq\sum_{i=1}^n B^k \cdot x^k + \left( F^1(x^1), \ldots, F^n(x^n) \right), x^k \\
&= (x^{k1}, \ldots, x^{kn}), k = 1, n \}
\end{align*}
\]

where \( X = (x^1, \ldots, x^n) \in (R^m)^n, B^k \) is a diagonal matrix the main diagonal of which has a form

\[
(v^{k1}, \ldots, v^{kn}), v^{ki} \in [0, 1], k = 1, n.
\]

\( F^j(x) \) are production functions of the branches:

\[
F^j(x) = \min_{i=1}^n \frac{x^i}{c_{ij}}, c_{ij} > 0, (i, j = 1, n).
\]

Production mapping \( a^k(x) \) of the branch \( k \) has a form:

\[
a^k(x) = (0, B^k x + (0, \ldots, 0, F^k(x), 0, \ldots, 0)) \in (R^n)^n.
\]

Note. The mapping \( a^k \) is completely defined by the set

\[
\{ v^{kj}, c^{ik}, (i, k = 1, n) \},
\]

where \( c^{ik} > 0, v^{ki} \in [0, 1], k = 1, n \).

Let \( I = \{ 1, 2, \ldots, n \}, \ell = (\ell^1, \ldots, \ell^n) \).

If consider (2) then the utility function of the \( k - th \) branch takes the form

\[
U^k(\ell, x^k) = \sum_{j=1}^m \ell^j \cdot v^{kj} \cdot x^k + \ell^k \cdot \min_{i=1}^n \frac{x^{ki}}{c_{ji}}
\]

where \( \ell = (\ell^1, \ell^2, \ldots, \ell^n) \) is a cost vector, the set \( I \) is defined by the formula (5).

By the definition the set \( (P, x^1, \ldots, x^n, y) \) is an equilibrium state if \( \sum_{k=1}^n x^k = y \) and \( x^k \) is a solution of the \( k - th \) consumer problem [5-7]

\[
U^k(\ell, x) \rightarrow \max, \text{ subject to } [P, x] \leq \lambda^k, x \in R^m, \ell \in I.
\]

where \( U^k \) is in the form (6), \( \lambda^k \) is a component of the budget vector \( \Psi = (\lambda_1, \lambda_2, \ldots, \lambda^n) \).

Let the vector \( \tilde{x}^k \) be a solution of the \( k - th \) consumer problem

\[
U^k(\ell, x) \rightarrow \max, x \in \tilde{P} = \{ x \geq 0, [P, x] \leq \lambda^k \} \in I.
\]

Then the equilibrium vector \( x^k \) has a form [4]

\[
x^k = \lambda^k \cdot \tilde{x}^k \in I.
\]

Note that from (4) and (9) follows that

\[
y = \sum_{k=1}^n x^k = \sum_{k=1}^n \lambda^k \cdot \tilde{x}^k.
\]
This means that
\[ y \in \text{cone}(\tilde{x}^k | k \in I). \]

In the future, we will be interested in the following problem. Given a mapping \( a^k \), i.e. \( v^k_i, c^u_i(k, i \in I) \) and \((P, x^1, ..., x^n, y)\) are a set of the vectors such that \( \sum_i x_i^k = y, x_i^k \leq 0 \). Determine whether there is model \( M \) with given \( v^k_i, c^u_i(k, i \in I) \) and, if so, to find it, that is, specify \( \ell_i \) and \( \tilde{\lambda}(i \in I) \) such that this set is a state of equilibrium in the model \( y, U(\ell, \Psi) \). From this problem, it follows that it is a problem with \( 2 \cdot n \) unknowns \( \ell_i, \tilde{\lambda}(i \in I) \).

### 2. Materials and Methods

Let \( x \in R^n \). Throughout the following notation will be used below

\[ I_1(x) = \{ i \in I | x_i^k = 0 \} , \]
\[ I_2(x) = \{ i \in I | x_i^k > 0 \} , \]
\[ R^k(x) = \{ i \in I | \frac{x_i^k}{c^u_i(k, i \in I)} = \min \frac{x_j^k}{c^u_j(k, i \in I)} \} (k \in I), \]
\[ Q^k(x) = \{ i \in I | R^k(x)(k \in I) \} . \]

Before talking about equilibrium, we examine the consumer problem. Along with set \( \tilde{V} \) in the consumer problem (8) we can consider the set

\[ V = \{ x \geq 0 | [P, x] = 1 \} , \]

Due to the homogeneity of the functions \( U^k(\ell, x) \) their maximums on the sets \( \tilde{V} \) and \( V \) coincide.

Let \( \tilde{x} \) be a maximum point in the \( k \)-th consumer problem (8). Then this point satisfies the necessary and sufficient conditions for the extremum differentiable on the direction function

\[ (U^k)(\tilde{x}, g) = 0 \quad g \in G_\tilde{x}(V), \]

where \( G_\tilde{x}(V) = \{ g \in R^n | \exists \alpha > 0: \tilde{x} + \alpha \cdot g \in V \forall \alpha \in (0, \alpha_o) \} \).

And the set \( V \) is defined by the formula (11).

Introduce the set \( G_\tilde{x}(V) \):

\[ G_\tilde{x}(V) = \{ g \in R^n | \exists \alpha > 0: [P, \tilde{x} + \alpha \cdot g] = 1, \tilde{x} + \alpha \cdot g \geq 0 \forall \alpha \in (0, \alpha_o) \} . \]

Since \([P, \tilde{x}] = 1\) then from \([P, \tilde{x} + \alpha \cdot g] = [P, \tilde{x}] + \alpha \cdot [P, g] = 1\) follows that \([P, g] = 0\). From the condition \( \tilde{x} + \alpha \cdot g \geq 0 \) or \( \tilde{x}^i + \alpha \cdot g^i \geq 0 \) for \( \forall i \in I \) follows that
a) If \( \tilde{x}^i = 0 \) then \( \alpha \cdot g^i \geq 0 \), consequently \( g^i \geq 0 \);

b) If \( \tilde{x}^i > 0 \) then \( \tilde{x}^i + \alpha \cdot g^i \geq 0 \) for \( \forall g^i \) small enough \( \alpha \).

From the foregoing, we find that the set \( G_\tilde{x}(V) \) can be written as

\[ G_\tilde{x}(V) = \{ g \in R^n | [P, g] = 0, g^i \geq 0 \forall i \in I_1(\tilde{x}) \} , \]

where the set \( I_1(\tilde{x}) \) is defined by the formula (10).

Let us solve the properties of the solution of the consumer problem. Particular attention is paid to how these properties are associated with the structure of the set \( I_1(\tilde{x}) \).

**Lemma 1.** Let \( \tilde{x} \) be a solution of the \( k \)-th consumer problem. Then if \( I_1(\tilde{x}) \neq \emptyset \) then \( \tilde{x} \in R^k(\tilde{x}) \).

**Proof.** Let \( \tilde{x} \) be a seeking solution of the \( k \)-th consumer problem and \( I_1(\tilde{x}) \neq \emptyset \). i.e. there exists an index \( i \in I \) such that \( x_i^k = 0 \). Then suppose that \( R^k(\tilde{x}) = 1 \). But since \( I_1(\tilde{x}) \neq \emptyset \) we get \( \tilde{x} = 0 \) that is impossible. If \( I_1(\tilde{x}) \subset R^k(\tilde{x}) \) then \( \tilde{x}^i < 0 \) for all \( i \in R^k(\tilde{x}) \) that is also impossible. Consequently, \( I_1(\tilde{x}) \subset R^k(\tilde{x}) \).

The lemma is proved.

**Consequence.** If \( R^k(\tilde{x}) = 1 \) then \( I_1(\tilde{x}) \neq \emptyset \).

Let us study in detail the \( k \)-th consumer problem. Let \( \tilde{x} \) be a solution of this problem and the vector \( P = (P^1, ..., P^n) \) be given where \( P^i \geq 0 \) for any \( i \in I \).

The utility function of the \( k \)-th branch in the point \( x \) has a form [2]

\[ U^k(\ell, x) = \sum \ell^j \cdot v^k_j \cdot x^j + \ell^k \cdot \min_{j \in I} \frac{x^j}{c^u_j(k \in I)} \]

where \( \ell = (\ell^1, ..., \ell^n) \) is a given cost vector.

Introduce the vector

\[ \ell^v = (\ell^1 \cdot v^1, ..., \ell^n \cdot v^n)(k \in I) . \]

Then (14) takes the form

\[ U^k(\tilde{x}, x) = U^k(\ell, x) = [\ell^v, x] + \ell^k \cdot \min_{j \in I} \frac{x^j}{c^u_j(k \in I)} \]

To investigate the \( k \)-th consumer problem, we apply the necessary and sufficient conditions for the extremum, according which the maximum is reached in the point \( \tilde{x} \) if and only if

\[ (U^k)(\tilde{x}, g) = 0 \quad g \in G_\tilde{x}(V)(k \in I), \]

where the cone \( G_\tilde{x}(V) \) is defined by the formula (13).

It is well known that \((U^k)'(\tilde{x}, g) = q^k(g)\), where

\[ q^k(g) = [\ell^v, g] + \ell^k \cdot \min_{j \in I} \frac{g^j}{c^u_j} \in R^n . \]

Introduce the denotations

\[ q^k(g) = \ell^k \cdot \min_{j \in I} \frac{g^j}{c^u_j} \in R^n . \]

Thus if in the point \( \tilde{x} \) the maximum is reached then

\[ q^k(g) = 0 \quad g \in G_\tilde{x}(V) = \{ g \in R^n | [P, g] = 0, g^i \geq 0 \forall i \in I_1(\tilde{x}) \} \]

where the set is defined \( I_1(\tilde{x}) \) by (10).

Consider some particular cases.

1. Let

\[ I_1(\tilde{x}) = \emptyset , \]

where \( \tilde{x} \) is a maximum pint in the \( k \)-th consumer problem and the set \( I_1(x) \) is defined by the formula (10).
In this case from (13) follows that $G_{\mathbf{X}}(V) = \Omega$, where
\[ \Omega = \{ g \in \mathbb{R}^n \mid [P, g] = 0 \}. \] (19)

Then the necessary and sufficient condition for the optimality $\bar{x}$ in the branch $k$ takes the form
\[ q^k(g) = [\ell^k_v, g] + \bar{q}^k(g) \leq 0, \forall g \in \Omega, \] (20)

where the function $\bar{q}^k(g)$ is defined by the formula (16).

Lemma 2. The following conditions are equivalent:
1) $q^k(g) \leq 0, \forall g \in \Omega$;
2) $\exists \mu^k > 0, \mu^k \cdot P \in \partial q^k$,

where $\partial q^k$ is a superdifferential of the function $q^k$.

Proof. The function $q^k(g)$ is concave. Let the inequality $q^k(g) \leq 0 \forall g \in \Omega$ take place.

Since $q^k(0) = 0$ then in the point $\bar{g} = 0$ the function $q^k$ reaches its maximum in the set $\Omega$. It is well known that necessary and sufficient conditions for the maximum of the concave function $q^k(g)$ in the point $\bar{g} = 0$ on the set $\{ g \in \mathbb{R}^n \mid [P, g] = 0 \}$ consist in the existing of the element $f^k \in \partial q^k(g)$ such that
\[ \max_{[P, g] = 0} \{ f^k, g \} = 0. \]

But it means that $[f^k, g] = 0$ in the same place where $[P, g] = 0$. It implies that for some $\mu^k$ takes place the following equality
\[ f^k = \mu^k \cdot P. \]

Since $f^k$ and $P$ are positive we get $\mu^k > 0$.

Citing the same arguments, but in reverse order, it is easy to show that from condition 2) of the lemma follows condition 1).

The proof is complete.

Lemma 3. Superdifferential of the function $q^k(g)(g \in R^n)$ defined by the formula (15) has a form [8-10]
\[ \partial q^k = \ell^k_v + \partial \bar{q}^k(k \in I), \]
where
\[ \ell^k_v = (\ell^1_v \cdot y^{k1}, ..., \ell^n_v \cdot y^{kn}), \]
\[ \bar{q}^k(g) = \ell^k_v \cdot \min_{i \in R^k(\bar{x})} \| g_i \| \]
moreover
\[ \partial \bar{q}^k = \left\{ f = \ell^k_v \cdot (f^1, ..., f^n) \mid \exists \alpha^i \geq 0: \sum_{i \in R^k(\bar{x})} \alpha^i = 1, f^i = \alpha^i \cdot \frac{\ell^i_v}{\| g_i \|}, \right\} \cap R^k(\bar{x}); \]
\[ f^i = 0, i \in Q^k(\bar{x})(k \in I). \] (21)

Proof. From (15), (16) we have
\[ \partial q^k = \ell^k_v + \partial \bar{q}^k(k \in I). \]

Define the vector $C^k_i = \frac{1}{\| \ell^i_v \|} e^i \in R^n$, where $e^i$ is $i$-th coordinate or $(i, k \in I)$. Then
\[ [C^k_i, g] = \ell^k_v \cdot g^i \in R^n. \]

From the definition of the superdifferential we obtain (21). The lemma is proved.

Lemma 4. The number $\mu^k(k \in I)$ defined in the Lemma 2. Is equal to
\[ \mu^k = \frac{\ell^k_v + \sum_{i \in R^k(\bar{x})} \alpha^i \cdot y^{ki} \cdot c^k_j}{\sum_{i \in R^k(\bar{x})} \alpha^i \cdot c^k_j} (k \in I) \] (22)

where $\ell^k_v = (\ell^1_v \cdot y^{k1}, \ell^2_v \cdot y^{k2}, ..., \ell^n_v \cdot y^{kn})$.

Proof. Let $\mu^k > 0$ be such that $\mu^k \cdot P \in \ell^k_v + \partial \bar{q}^k$, i.e. condition 2) of the Lemma 2 is satisfied. Using (21) one can obtain from this that for some $\alpha^i \geq 0, \sum_{i \in R^k(\bar{x})} \alpha^i = 1$ the following equality holds true
\[ \left\{ \begin{array}{l}
\mu^k \cdot p^1 = \ell^1_v \cdot y^{k1} + \frac{\alpha^1}{\| \ell^1_v \|} \cdot \ell^k_v, \\
\mu^k \cdot pr^r = \ell^r_v \cdot y^{kr} + \frac{\alpha^r}{\| \ell^r_v \|} \cdot \ell^k_v, \\
\mu^k \cdot pr^{r+1} = \ell^{r+1} \cdot y^{kr+1},
\end{array} \right. \] (23)

where $r = |R^k(\bar{x})|$. It follows from the last that
\[ \mu^k = \frac{\ell^1_v \cdot y^{k1} \cdot c^k_j}{\| \ell^1_v \cdot c^k_j \|} + \frac{\alpha^1 \cdot \ell^k_v}{\| \ell^1_v \|} = \ldots \frac{\ell^r_v \cdot y^{kr} \cdot c^k_j}{\| \ell^r_v \cdot c^k_j \|} + \frac{\alpha^r \cdot \ell^k_v}{\| \ell^r_v \|} = \ldots \frac{\ell^{r+1} \cdot y^{kr+1} \cdot c^k_j}{\| \ell^{r+1} \cdot c^k_j \|} = \ldots \frac{\alpha^{r+1} \cdot \ell^k_v \cdot c^k_j}{\| \ell^{r+1} \cdot c^k_j \|} \] (24)

Let’s fix the index $j \in R^k(\bar{x})$ and express all $\alpha^i(i \in R^k(\bar{x}) \setminus \{j\})$ through $\alpha^i$:
\[ \alpha^i = \frac{1}{\| \ell^i_v \cdot c^k_j \|} \left( \alpha^i \cdot \ell^k_v \cdot c^k_j + \ell^i_v \cdot y^{ki} \cdot c^k_j \right) \] \[ \forall i \in R^k(\bar{x}) \setminus \{j\}. \]

Due the conditions $\sum_{i \in R^k(\bar{x})} \alpha^i = 1$:
\[ \sum_{i \in R^k(\bar{x})} \alpha^i = \frac{\alpha^i}{\| \ell^i_v \cdot c^k_j \|} \left( \sum_{i \in R^k(\bar{x}) \setminus \{j\}} \| \ell^i_v \cdot c^k_j \| + \alpha^i \right) \]
\[ + \sum_{i \in R(k)} p_i \cdot c_i = \frac{1}{\overline{p}^k} \cdot \sum_{j \in R(k)(j)} \sum_{i \in R(k)} p_i \cdot c_i = \alpha^i \cdot \overline{p}^k \cdot c_i + \ell^i \cdot v^k_i \cdot c_i = \]

\[ = \frac{\alpha^i \cdot \overline{p}^k \cdot c_i + \ell^i \cdot v^k_i \cdot c_i}{\overline{p}^k \cdot p_j + c_j} = 1. \]

From this we obtain \( \alpha^i (j \in R(k)) \):

\[ \alpha^i = \frac{\overline{p}^k \cdot c_i + \ell^i \cdot v^k_i \cdot c_i}{\overline{p}^k \cdot p_j + c_j} (j \in R(k)). \tag{25} \]

Substituting the obtained values of \( \alpha^i (j \in R(k)) \) into the first equality (18), we get (22).

Theorem is proved.

Proposition 1. For any \( \overline{p}^k, \varphi^k, m \in Q^k(x) \) and \( \mu^k \) defined on \( R^k(x) \), the functions \( q_m^k(h) \) are superlinear.

Take

\[ I_1^m = I_1(x) \setminus \{m\}, T_m = \{g \in R^{n-1} | g^i \geq 0, i \in I_1^m\}. \]  \tag{29}

The cone \( T_m^* \) adjoint to the cone \( T_m \) has a form

\[ T_m^* = \{w \in R^{n-1} | w^i \geq 0, i \in I_1^m ; w^i = 0, i \in I_2^m\}. \]  \tag{30}

We'll consider the functional \( q_m^k(g) \) only on the cone \( T_m \).

For any \( m \in I \) the conditions

1. \( q^k_m(g) \leq 0 \) for all \( g \in G_m(V) \);
2. \( q^k_m(g) \leq 0 \) for all \( g \in T_m \);

are equivalent.

Proof. Let \( q^k(g) \leq 0 \) \( g \in G_m(V) \) i.e. (see (15))

\[ \ell^k + \overline{p}^k \cdot \min_{i \in R(k)(i)} \frac{\overline{p}^k_i \cdot c_i}{c_i} \leq 0, g \in G_m(V). \]  \tag{31}

Due to (12) e have \( [P, g] = 0 \); fixing the index \( m \in I \), expressing \( g^m \) and substituting it into the left hand side on the last inequality, after some eliminations we get (31).

Having the same argument in reverse order, we get the contradiction.

Proposition is proved.

Theorem 2. Let the strictly positive vector \( P = (P^1, ..., P^m) \) be given. If the vector \( x \) for which

\[ I_1(x) = \emptyset, \]  \tag{33}

is a maximum point in the \( k \)-th consumer problem (8), then for \( m \in Q^k(x) \) the following relation is satisfied.
be given and take the form that the inequality (37) would be satisfied and for the desired result.

Proof. Necessity. Let strictly positive vector \( P = (p^1, ..., p^n) \) be given and \( \tilde{x} \) is the point at which the utility function of the \( k \)-th branch of (8) satisfying (33) takes its maximum.

Then due the proposition 1 necessary and sufficient optimality conditions for the vector \( \tilde{x} \) in the \( k \)-th branch take the form

\[
q^k_{m}(\tilde{g}) \leq 0 \forall \tilde{g} \in T_m(m, k \in I),
\]

where the cone \( T_m \) is defined by the formula (29).

By the definition of the superdifferential we have

\[
q^k_{m}(\tilde{g}) \leq 0 \forall \tilde{g} \in t \in \tilde{m} \implies 0 \in \partial q^k_{m}(k \in I),
\]

where superdifferential \( \partial q^k_{m} \) has a form (32). Consequently

\[
-\ell^k \in \partial q^k_{m} + T_m(m, k \in I).
\]

Depending on the choice of the index \( m \in I \) the function \( q^k_{m}(\tilde{g}) \) may have various forms (see (28)).

1) Let \( m \in Q^k(\tilde{x}) \). Then from (28) we have

\[
q^k_{m}(\tilde{g}) = \ell^k \cdot \min_{i \in R^k(\tilde{x})} \frac{g^i}{c^i}.
\]

Superdifferential of this function has form (21).

Substituting (21), (30) and (28) into the relation (36) we get that there exist the numbers \( a^i \geq 0, w^i \geq 0 \) such that

\[
\sum_{i \in R^k(\tilde{x})} a^i = 1,
\]

\[
-\frac{\alpha^m}{c^m} \cdot \ell^k - w^i = p^i \left( \frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \right) \forall i \in R^k(\tilde{x}),
\]

\[0 = p^i \left( \frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \right) \forall j \in Q^k(\tilde{x}).
\]

From this immediately follows the inequality (34).

We claim the opposite. Suppose that the conditions of the theorem hold true in the case when \( m \in Q^k(\tilde{x}) \), i.e. we can choose the numbers \( a^i \geq 0 \) and \( w^i \geq 0 (i \in R^k(\tilde{x})) \) by such way that the inequality (37) would be satisfied and \( \sum_{i \in R^k(\tilde{x})} a^i = 1 \). It means that the relation (36) is satisfied or \( 0 \in \partial q^k_{m} \). Consequently, by the definition of the superdifferential the following condition takes place \( q^k_{m}(\tilde{g}) \leq 0 \forall \tilde{g} \in T_m(m \in Q^k(\tilde{x})) \), that indeed is necessary and sufficient condition for the optimality of the vector \( \tilde{x} \) in the \( k \)-th branch.

2) Consider the case when \( m \in R^k(\tilde{x}) \). Then from (28) we get

\[
q^k_{m}(\tilde{g}) = \ell^k \cdot \min_{i \in R^k(\tilde{x})} \left\{ \frac{g^i}{c^i} - \frac{1}{p^m} \cdot \frac{c^m}{c^{km}} \cdot \sum_{i \in A(m)} p^i \cdot g^i \right\},
\]

Superdifferential of which has a form [8, 9]

\[
\partial q^k_{m} = \ell^k \cdot \mathcal{C}_O \left\{ \frac{1}{c^i} \cdot e^i, \bar{\ell} \left| i \in R^k(\tilde{x}) \right\{m, \right\}
\]

\[
\bar{\ell} = -\frac{1}{p^m} \cdot \frac{c^m}{c^{km}} \cdot (p^1, ..., p^{m-1}, p^{m+1}, ..., p^n)
\]

\[
\left( \sum_{i \in R^k(\tilde{x})} a^i \cdot \frac{1}{c^i} \cdot e^i - \frac{a^m}{p^m} \cdot c^m \cdot (p^1, ..., p^{m-1}, p^{m+1}, ..., p^n) \right).
\]

The according to (28), (30) and (38) from (36) follows that there exists numbers \( a^i \geq 0, w^i \geq 0 (i \in R^k(\tilde{x})) \) such that

\[
\sum_{i \in R^k(\tilde{x})} a^i = 1,
\]

\[
\left( \frac{a^m}{p^m} \cdot c^m \cdot p^i - \frac{a^i}{c^i} \right) \cdot \ell^k - w^i = p^i \left( \frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \right) \forall i \in R^k(\tilde{x}),
\]

\[
\frac{p^i}{p^m} \cdot u^m \cdot \frac{c^m}{c^{km}} \cdot a^m = \frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \cdot p^j \forall j \in Q^k(\tilde{x}).
\]

These lead us to

\[
\frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \forall i \in R^k(\tilde{x}),
\]

\[
\frac{\ell^i}{p^i} \cdot \frac{v^i}{m} \cdot \frac{u^m}{p^m} \cdot u^{km} \forall j \in Q^k(\tilde{x}).
\]

Since the first inequality of the last system holds true for \( \forall i, m \in R^k(\tilde{x}) \), it turns to the equality. As a result, we obtain the desired result.

Let’s claim the opposite. Suppose that the conditions of the theorem in the case when \( m \in R^k(\tilde{x}) \), i.e. we can choose the
numbers $\alpha^i \geq 0$ and $w^i \geq 0 \ (i \in R^k(x))$ such that the inequality (39) would be satisfied or $0 \in \partial q_m^k$. This is equivalent to the condition $q_m^k(\bar{g}) \leq 0 \forall \bar{g} \in T_m$, that is indeed a necessary and sufficient condition for the optimality of the vector $\bar{x}$ in the $k$-th branch. Theorem is proved.

3. Conclusion

(1). The form of the superdifferential of the utility function is defined.

(2). Necessary and sufficient condition for the existence of the maximum of this function is derived.

(3). The maximum rate of the growth of the industries total wealth determined.

(4). A necessary and sufficient condition is obtained or optimality of the state vector of the branches.

References


