The Study of the Concept of Q*Compact Spaces

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Abstract: The aim of this research is to extend the new type of compact spaces called Q* compact spaces, study its properties and generate new results of the space. It investigates the Q*-compactness of topological spaces with separable, Q*-metrizable, Q*-Hausdorff, homeomorphic, connected and finite intersection properties. The closed interval [0, 1] is Q*-compact. So, it is deduced that the closed interval [0, 1] is Q*-compact. For example, if

Q* closed in (X, τ) if S is closed and Int (S) = φ. Its compliment S' is therefore Q* open [9, 10]. If every open cover of X has a finite sub cover then X is called a compact space. (X, τ) is said to be separable if it has a countable dense subset. Let X be a set and ℑ a family of subsets of X. Then ℑ is said to have Finite Intersection Property if for any finite number F₁, F₂, ..., Fₙ of members of ℑ, F₁ ∩ F₂ ∩ ... ∩ Fₙ ≠ φ [9].

2. Preliminaries

This section gives an overview of the basic definitions of a compact space, Q*-compact which is the new type of a compact space.

Definition: A subset A of a topological space (X, τ) is said to be compact if every open covering of A has a finite subcovering. If the compact subset A equals X, then (X, τ) is said to be a compact space.

Definition: Let (X, τ) be a topological space. Then it is said to be connected if the only clopen subsets of X are X and φ.
Definition: A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( Q^*O \) compact if every \( \tau - Q^* \) open cover of \( X \) has a finite sub cover.

Definition: Let \( (X, \tau) \) be a topological space. Then it is said to be \( Q^* \)-connected if the only \( Q^* \) open subsets of \( X \) are \( X \) and \( \varnothing \).

3. Results on Generalization of \( Q^*O \) Compact Space

Theorem: The closed interval \([0, 1]\) is \( Q^*\)-compact.

Proof: Let \( G_\alpha, \alpha \in \Lambda \) be any open covering of \([0, 1]\). Then for each \( x \in [0,1] \), there is a \( G_\alpha \) such that \( x \in G_\alpha \). As \( G_\alpha \) is open in \( X \), there exist \( U_x \), open in \([0,1]\), such that \( x \in U_x \subseteq G_\alpha \).

Now define a subset \( S \subseteq [0,1] \) as follows:

\[
S = \{ x \in [0,1] : \exists \{ U_x \} \text{ such that } x \in U_x \subseteq G_\alpha \}
\]

Then for each \( x \in [0,1] \), \( U_x \) is an interval containing \( x \), and \( U_x \) is open. So \( x \in S \) if and only if there exists an \( \alpha \in \Lambda \) such that \( x \in U_x \). Hence \( S \) is \( Q^* \)-compact.

Example: Suppose \( \tau = \{ \mathcal{X}, \mathcal{E}, \mathcal{F}, \mathcal{G} \} \). Let \( S = \{ e, f, g \} \). Then \( S \) is \( Q^* \)-compact if and only if \( \tau = \{ \mathcal{X}, \mathcal{E}, \mathcal{F}, \mathcal{G} \} \) is not \( Q^* \)-open.

Remark: Every \( Q^*O \) compact space is compact, but the converse is not necessarily true.

Theorem: A subset \( S \) of \( \mathbb{R} \) is \( Q^* \)-compact if and only if \( S \) is closed and bounded.

Proof: First suppose that \( S \) is \( Q^* \)-compact. To see that \( S \) is bounded, suppose not. Then there is some point \( p \in cl(S) \) such that \( p \notin S \). For each \( n \), define the neighborhood around \( p \) of radius \( 1/n \), \( N_n = (p, 1/n) \). Take the complement of the closure of \( N_n U_n = R \setminus cl(N_n) \) is open (since its complement is closed), and we have \( \bigcap_{n=1}^{\infty} U_n = \varnothing \).

Hence, \( S \) is closed and bounded. Therefore \( S \) is \( Q^* \)-compact.

Conversely, there is need to show that if \( S \) is closed and bounded, then \( S \) is \( Q^* \)-compact. Let \( \mathcal{X} \) be an open cover for \( S \). For each \( x \in \mathcal{X} \), define the set

\[
S_x = S \cap (-\infty, x],
\]

and let

\[
B = \{ x : S_x \text{ is covered by a finite subcover of } \mathcal{X} \}.
\]

Since \( S \) is closed and bounded, hypothesis tells us that \( S \) has both a maximum and a minimum. Let \( d = \min S \). Then \( S_x = \{ d \} \) and this is certainly covered by a finite subcover of \( \mathcal{X} \). Therefore, \( d \in B \) and \( B \) is nonempty. If it is shown that \( B \) is not bounded above, then it will contain a number \( p \) greater than \(\min S \). But then, \( S_p = S \) so we can conclude that \( S \) is covered by a finite subcover, and is therefore \( Q^*\)-compact.
compact.

Toward this end, suppose that $B$ is bounded above and let $m = \sup B$. We shall show that $m \in S$ and $m \notin S$ both lead to contradictions.

If $m \in S$, then since $S$ is an open cover of $S$, there exists $F_0 \in S$ such that $m \in F_0$. Since $F_0$ is open, there exists an interval $[x_1, x_2]$ in $F_0$ such that $x_1 < m < x_2$. Since $x_1 < m$ and $m = \sup B$, there exists $F_1, \ldots, F_n$ in $S$ that cover $S_n$. But then $F_0, F_1, \ldots, F_n$ cover $S_n$, so that $x_2 \in B$. But this contradicts $m = \sup B$.

If $m \notin S$, then since $S$ is closed there exists $\epsilon > 0$ such that $N(m, \epsilon) \cap S = \emptyset$. But then $S_{m-\epsilon} = S_{m+\epsilon}$.

Since $m - \epsilon \notin B$ then $m + \epsilon \in B$, which again contradicts $m = \sup B$.

Therefore, either way, if $B$ is bounded above, we get a contradiction. We conclude that $B$ is not bounded above, and $S$ must be Q*-compact.

Theorem: Let $(X, \tau)$ be a Q*-compact metrizable space. Then $(X, \tau)$ is separable.

Proof: Let $d$ be a metric space on $X$ which induces the topology $\tau$. For each positive integer $n$, let $S_n$ be the family of all open balls having centres in $X$ and radius $\frac{1}{n}$. Then $S_n$ is an open covering of $X$ and so there is a finite subcovering $\mu_n = \{U_{n,1}, U_{n,2}, \ldots, U_{n,k}\}$, for some $k \in \mathbb{R}$. Let $y_{n,j}$ be the centre of $U_{n,j}$, $j = 1, \ldots, K$, and $Y_n = \{y_{n,1}, y_{n,2}, \ldots, y_{n,k}\}$.

Put $Y = \bigcup_{n=1}^{\infty} Y_n$. Then $Y$ is a countable subset of $X$. Now showing that $Y$ is dense in $(X, \tau)$.

If $V$ is any non-empty open set in $(X, \tau)$, then for any $v \in V$, $V$ contains an open ball, $B_v$, of radius $\frac{1}{n}$, about $v$, for some $n \in \mathbb{R}$. As $\mu_n$ is an open cover of $X$, $v \in U_{n,j}$, for some $j$. Thus $d(v, y_{n,j}) < \frac{1}{n}$ and so $y_{n,j} \in B \subseteq V$. Hence, $V \cap Y \neq \emptyset$, and so $Y$ is dense in $X$.

Theorem: Let $(X, \tau)$ be a topological space. Then $(X, \tau)$ is Q*-compact if and only if every family $S$ of closed subsets of $X$ with the finite intersection property satisfies $\bigcap F \neq \emptyset$.

Proof: Assume that every family $S$ of closed subsets of $X$ with the finite intersection property satisfies $\bigcap F \neq \emptyset$.

Let $\mu$ be any open covering of $X$. Put $S$ equal to the family of complements of members of $\mu$. So each $F \in S$ is closed in $(X, \tau)$. AS $\mu$ is an open covering in $X$, $\bigcap F \neq \emptyset$. By our assumption, then $S$ does not have finite intersection property. So, for some $F_1, F_2, \ldots, F_n$ in $S$, $F_1 \cap F_2 \cap \cdots \cap F_n \neq \emptyset$. Thus $U_1 \cup U_2 \cup \cdots \cup U_n = X$, where $U_i = X \setminus F_i$, $i = 1, \ldots, n$. So $\mu$ has a finite subcovering. Hence, $(X, \tau)$ is Q*-compact.

The converse statement is proved similarly.

Theorem: Let $f$ be a continuous mapping of a Q*-compact metric space $(X, d)$ onto a Q*-Hausdorff space $(Y, \tau)$. Then $(Y, \tau)$ is Q*-compact and metrizable.

Proof: Since every Q*-continuous image of a compact space is compact (Padma 2015), the space $(Y, \tau)$ is certainly compact and metrizable. As the map $f$ is surjective, define the metric $d_1$ on $Y$ as follows:

$$d_1(y_1, y_2) = \inf \left\{ d(a, b) : a \in f^{-1}\{y_1\} \text{ and } b \in f^{-1}\{y_2\} \right\},$$

for all $y_1$ and $y_2$ in $Y$.

To show that $d_1$ is indeed a metric. Since $\{y_1\}$ and $\{y_2\}$ are closed in the Q*-Hausdorff space $(Y, \tau)$, $f^{-1}\{y_1\}$ and $f^{-1}\{y_2\}$ are Q*-compact. So, the product $f^{-1}\{y_1\} \times f^{-1}\{y_2\}$, which is a subspace of the product space $(X, \tau) \times (X, \tau)$, is Q*-compact, where $\tau$ is the topology induced by the metric $d$.

Observing that $d : (X, \tau) \times (X, \tau) \to \mathbb{R}$ is a continuous mapping, then $d\left(f^{-1}\{y_1\} \times f^{-1}\{y_2\}\right)$ has a least element.

So there exist an element $x_1 \in f^{-1}\{y_1\}$ and an element $x_2 \in f^{-1}\{y_2\}$ such that

$$d(x_1, x_2) = \inf \left\{ d(a, b) : a \in f^{-1}\{y_1\}, b \in f^{-1}\{y_2\} \right\} = d_1(y_1, y_2).$$

Clearly if $y_1 \neq y_2$, then $f^{-1}\{y_1\} \cap f^{-1}\{y_2\} = \emptyset$. Thus $x_1 \neq x_2$ and hence $d(x_1, x_2) > 0$, that is $d_1(y_1, y_2) > 0$.

It is easily verified that $d_1$ has the other properties required of a metric, and so a metric on $Y$.

Let $\tau_2$ be the topology induced on $Y$ by $d_1$. To show that $\tau_1 = \tau_2$.

Firstly, by the definition of $d_1$, $f : (X, \tau) \to (X, \tau_2)$ is certainly continuous.

Observe that for a subset $C$ of $Y$, $C$ is a closed subset of $(Y, \tau_2)$

$\Rightarrow f^{-1}(C)$ is a closed subset of $(X, \tau)$

$\Rightarrow f^{-1}(C)$ is a Q*-compact subset of $(X, \tau)$

$\Rightarrow f\left(f^{-1}(C)\right)$ is a Q*-compact subset of $(X, \tau_2)$.
SO $\tau_1 \subseteq \tau_2$

Similarly, we have $\tau_2 \subseteq \tau_1$, and thus $\tau_1 = \tau_2$

Theorem: Let $(X, \tau)$ be a Q*-compact space and $f: (X, \tau) \rightarrow \mathbb{R}$ a continuous mapping. Then $f(X)$ has a greatest element and a least element.

Theorem: If $(X_1, \tau_1), (X_2, \tau_2), ..., (X_n, \tau_n)$ are Q*- compact spaces, then $\prod_{i=1}^{n} (X_i, \tau_i)$ is a Q*- compact space.

Proof: The first part of this proof is to show that the product of any two Q* compact topological spaces is Q*- compact.

By our inductive hypothesis $(X_1, \tau_1) \times \cdots \times (X_N, \tau_N)$ is Q*- compact, so the right-hand side is the product of two Q*- compact spaces and thus is Q*- compact. Therefore, the left-hand side is also Q*- compact.

Theorem: Let $\{(X_i, \tau_i) : i \in I\}$ be any family of topological spaces. Then $\prod_{i \in I} (X_i, \tau_i)$ is Q*-compact if and only if each $(X_i, \tau_i)$ is Q*-compact.

Theorem: If $X$ is not Q*- compact, then $X$ is homeomorphic to an open dense set in $\mathcal{X}$. (Where $\mathcal{X}$ is not too larger than $X$.)

Proof: Suppose we ensure that $\mathcal{X}$ is not “too large”, that is, not “too much larger” than $X$.

First show that $X$ is homeomorphic to the set $\{X\} \subset \mathcal{X}$.

Construct a function that sends each point of $X$ to the corresponding point in $\{X\}$. This function is obviously one-to-one and onto, and it is continuous (and so is its inverse) because the open sets in $\{X\}$ are exactly the open sets in $X$.

The set $\{X\}$ is open in $\mathcal{X}$, because it does not contain $\infty$ and it is open in $X$. To show that $\{X\}$ is dense, we can simply show that it is not closed, or that $\infty$ is not open. (If that’s the case, then $\{X\}$ is not its own closure, and the only other option is that its closure is $\mathcal{X}$). If $\infty$ is open, then its complement, $\{X\}$, must be closed. But this would imply that $X$ is Q*- compact, contradicting our earlier assumption. So $\infty$ cannot be open, meaning $\{X\}$ must be dense.

Theorem: If none of the components of $X$ is Q*- compact, then $\mathcal{X}$ is connected.

Proof: Assume that $\mathcal{X}$ is not connected, i.e. there is some set $U$ in $\mathcal{X}$ that is open and closed, but is not $\varnothing$ or $\mathcal{X}$. Its complement, $\mathcal{V}$, is also open and closed without being $\varnothing$ or $\mathcal{X}$. Either $U$ or $\mathcal{V}$ contains $\infty$; take the one that does not, and call it $W$. $W$ is Q*-compact because its complement is open and contains $\infty$.

First let us consider the case that $X$ is connected. We have already established that $W$ is not $\varnothing$. It cannot be all of $X$ either, because $W$ is Q*- compact and $X$ is not. $W$ is open in $X$ because it is open in $\mathcal{X}$ and does not contain $\infty$. It is closed in $X$ because its complement (either $U \cap X$ or $\mathcal{V} \cap X$) is open in $X$. So, $W$ is open, closed, not $\varnothing$, and not $X$, which implies that $X$ is not connected. This contradicts our assumption, so $\mathcal{X}$ must be connected.

4. Conclusion

But what if $X$ is not connected? In this case, we look at the connected components of $X$. Any open set including $\infty$ must also contain points in each of the components of $X$ (because the complement of the open set is Q*- compact, and if the complement included an entire connected component, then that component would need to be Q*-compact, but it is not). So $W$ contains some points in each of the components. But this would imply that the connected components are not connected, which is our contradiction. So again, $\mathcal{X}$ must be connected.

It is also true that every Q*O compact space is a Q*- Lindelof space. Every Q*O - compact topological space is Q*- compactly countable. Since the space is Q*O - compact, every $\tau$ - Q* open covering of $X$ has a finite subcover. Hence, every countable $\tau$- Q* open covering of $X$ has a finite subcover and therefore it is countably compact.

References


