

Terquem theorem with the spherical helix strip

Filiz Ertem Kaya

Department of Mathematics, Faculty of Science and Arts, Nigde University, Nigde, Turkey

Email address: fertem@nigde.edu.tr

To cite this article:

Abstract: The spherical helix and the strip are respectively proved firstly by Scofield and Sabuncuoglu and Hacısalihoglu. In this paper helix strip on sphere is investigated by using characteristics of spherical helix and strip. Firstly using strip after helix and finally spherical helix. So spherical helix strips are obtained. Furthermore Joachimsthal Theorem and Terquem Theorem are investigated when the strip and helix strips which lie on the sphere and given a characterization about spherical helix strips.

Keywords: Helix, Strip, Spherical Helix Strip, 2010 Mathematics Subject Classification. 53A04, 53A05

1. Introduction

Struikand Blaschke investigated that if a helix has constant curvature, then its projection on a plane perpendicular to its axis is a plane curve of constant curvature, hence circle and the spherical helix projects on a plane perpendicular to its axis in an arc of an epicycloid. Sabuncuoglu and Hacısalihoglu proved higher curvatures of curve-surface pair (we use the strip as curve-surface pair) and Keles showed some relations of curve-surface pair on two different surfaces that we know these as Joachimsthal theorems. In 3-dimensional Euclidean Space, a regular curve is described by its curvatures and a strip is described by its curvatures and . The relations between the curvatures of a strip and the curvatures of the curve can be seen in many differential books and papers. We know that a regular curve is called a general helix if its first and second curvatures and are not constant, but is constant ([2],[8]). Also if a helix lie on a sphere, it is called a spherical helix. The spherical helix provides being a helix condition and the sphericity condition by using the curvatures of helix and . Also Joachimsthal Theorem and Terquem Theorem (known as one of Joachimsthal Theorems but we call Terquem Theorem for easiness) are proved when the helix strip that lies on sphere.

2. The Curve-Surface Pair (Strip)

Definition:
Let and be a surface in and a curve in . We define a surface element of is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve is called a curve-surface pair and is shown as (, ).

Definition 2:
Let be the curve and curve-surface pair’s vector fields. The curve-surface pair’s tangent vector field, normal vector field and binormal vector field is given by

\[
\mathbf{t} = \mathbf{\xi} \times \mathbf{\eta} = \mathbf{N} (\mathbf{N} = \mathbf{n}) \quad \text{and} \quad \mathbf{\eta} = \mathbf{\xi} \times \mathbf{\xi} ([8,10,11]).
\]

2.1. Curvatures of the Curve-Surface Pair and Curvatures of the Curve

Let . Also Joachimsthal Theorem and Terquem Theorem are proved when the helix strip that lies on sphere.
We know that a curve \( \alpha \) has two curvatures \( \kappa \) and \( \tau \). A curve has a strip and a strip has three curvatures \( k, g, k \) and \( r \).

Let \( k, g, k \) and \( r \) be the \( b, c, a \) and \( a \). From last equations we have
\[
\xi' = -\kappa \eta + a \zeta.
\]
and
\[
\eta' = b \xi - c \zeta.
\]
\[
\zeta' = c \eta - b \xi.
\]
(1)

From last two equations we obtain,
\[
\kappa^2 = b^2 + c^2.
\]
This equation is a relation between the curvature \( \kappa \) of a curve \( \alpha \) and normal curvature and geodesic curvature of a curve-surface pair.

By using similar operations, we obtain a new equation as follows
\[
\tau = a + \frac{bc - ca'}{b^2 + c^2},
\]
(8,10,11).

This equation is a relation between \( \tau \) (torsion or second curvature of \( \alpha \) and curvatures of a curve-surface pair that belongs to the curve \( \alpha \)). And also we can write
\[
a = \phi' + \tau.
\]
The special case:
If \( \phi \) is constant, then \( \phi' = 0 \). So the equation is \( a = \tau \). That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

Definition:
Let \( \alpha \) be a curve in \( M \subset E^3 \). If the geodesic curvature (torsion) of the curve \( \alpha \) is equal to zero, then the curve-surface pair \( (\alpha, M) \) is called a curvature curve-surface pair (strip) (8,10,11).

3. General Helix

Definition:

Let \( \alpha \) be a curve in \( E^3 \) and \( V \) be the first Frenet vector field of \( \alpha \). \( U \in \mathcal{X}(E^3) \) be a constant unit vector field.

If
\[
\{V, U\} = \cos \phi \quad \text{(constant)}
\]
\( \alpha, \phi \) and \( \mathcal{S}p[U] \) are called a general helix, the slope and the slope axis ((1,2,6)).

Definition: A regular curve \( \alpha \) is a general helix if its first and second curvatures \( \kappa \) and \( \tau \) are not constant but \( \frac{\kappa}{\tau} \) is constant (1,6).

Definition: A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio \( \frac{\kappa}{\tau} \) is constant ((5,9,12)).

Definition: A helix is a curve in 3-dimensional space. The following parametrisation in Cartesian coordinates defines a helix, see [7].
\[
x(t) = \cos t,
\]
\[
y(t) = \sin t,
\]
\[
z(t) = t.
\]
As the parameter \( t \) increases \((x(t), y(t), z(t))\) traces a right-handed helix of pitch \( 2\pi \) and Radius 1 about the \( z \) axis, in a right-handed coordinate system. In cylindrical coordinates \((r, \theta, h)\) the same helix is parametrised by
\[
r(t) = 1, \quad \theta(t) = t, \quad h(t) = t.
\]

Definition: If the curve \( \alpha \) is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be the constant. The ratio \( \frac{\tau}{\kappa} \) is called first harmonic curvature of the curve and is denoted by \( H \) or \( H \).

Theorem 3.1: A regular curve \( \alpha \subset E^3 \) is a general helix if and only if \( H(s) = \frac{k}{k_2} = \text{const} \) for \( \forall s \in I \), see [7].

Proof: \( \Rightarrow \) Let \( \alpha \) be a general helix. The slope axis of the curve \( \alpha \) is showed \( \mathcal{S}p[U] \). Note that
\[
\langle \alpha'(s), U \rangle = \cos \phi = \text{const}.
\]
If the Frenet Threshold is \( V_1, V_2, V_3 \) at the point \( \alpha(s) \), then we have
\[
\langle V'(s), U \rangle = \cos \phi.
\]
If we take derivative of the both sides of the last equation, then we have
\[ \langle k_i V_i(s), U \rangle = 0 \Rightarrow \langle V_i(s), U' \rangle = 0. \]

Hence
\[ U \in Sp\{V_i(s), V_j(s)\}. \]

Therefore
\[ U = \cos \phi V_i(s) + \sin \phi V_j(s). \]

\( U \) is the linear combination of \( V_i(s) \) and \( V_j(s) \). By differentiating the equation \( \langle V_i(s), U \rangle = 0 \), we obtain
\[ \langle -k_i V_i(s) + k_j V_j(s), U \rangle = 0, \]
\[ -k_i(s) \langle V_i(s), U \rangle + k_j(s) \langle V_j(s), U \rangle = 0, \]
\[ k_i(s) \cos \phi + k_j(s) \sin \phi = 0. \]

By using the last equation, we see that
\[ H = \text{const}. \]

(\( \Leftarrow \)) Let \( H(s) \) be constant for \( \forall s \in I \), and \( \lambda = \tan \phi \), then we obtain
\[ U = \cos \phi V_i(s) + \sin \phi V_j(s). \]

1. If \( U \) is a constant vector, then we have
\[ D_{\alpha} U = (k_i(s) \cos \phi - \sin \phi k_j(s)) V_j(s). \]

By substituting \( H(s) = \tan \phi \) is in the last equation, we see that
\[ k_i(s) \cos \phi - k_j(s) \sin \phi = 0, \]
and so
\[ U = \text{const}. \]

2. If \( \alpha \) is an inclined curve with the slope axis \( Sp\{U\} \), then
\[ \langle \alpha'(s), U \rangle = \langle V_i(s), \cos \phi V_i(s) + \sin \phi V_j(s) \rangle \]
\[ = \cos \phi \langle V_i(s), V_i(s) \rangle + \sin \phi \langle V_i(s), V_j(s) \rangle, \]
and we obtain
\[ \langle \alpha'(s), U \rangle = \cos \phi = \text{const} \]
\[ ([7]). \]

Definition:

Let \( S^2 \) and \( \alpha \) be a sphere in \( E^3 \) and a helix that lies on the sphere \( S^2 \). The curve \( \alpha \) is called a spherical helix which lie on the sphere\([12]\).

**Definition:**

Let \( \alpha \) be a helix in \( M \subset E^3 \). We define a surface element of \( M \) is the part of a tangent plane at the neighbour of the point of the helix that lie on \( M \). Instead of the geometric plane of these surface elements along the helix \( \alpha \) which lie sphere \( M \) is called a helix strip.

**Definition:**

Let \( S^2 \) be a sphere and and \( \alpha \) a helix which lie on \( S^2 \) in \( E^3 \). We define a surface element \( S^2 \) is the part of a tangent plane at the neighbour of the point of the helix that lie on \( S^2 \). The locus of these surface elements along the helix \( \alpha \) which lie on the sphere \( S^2 \) is called spherical helix strip.

**Theorem 3.2**[10]: (Terquem Theorem) Let \( M_1 \) and \( M_2 \) be the different surfaces in \( E^3 \) and \( \alpha \) be a curve but not a planar curve and \( \beta \) be a curve on \( M_2 \),

i. The points of the curves \( \alpha \) and \( \beta \) corresponds to each other 1:1 on a plane \( \varepsilon \) which rolls on the \( M_1 \) and \( M_2 \), such that the distance is constant between corresponding points.

ii. \( (\alpha, M_1) \) is a curvature strip.

iii. \( (\beta, M_2) \) is a curvature strip.

**Proof:** Claim: Two of the three lemmas give the third ([5,10]). It is obviously from the doctoral dissertation by Keles.

By applying the similar way in proof of the Theorem 3.2 in [5,10] to the strip of the spherical helix strip, we give the following theorem.

**Theorem 3.3:** Let \( S^2 \) be a sphere and and \( M \) be a surface in \( E^3 \). Let the tangent planes of the surface \( M \) that along the curve \( \beta \) be the tangent planes of the sphere \( S^2 \) along the helix curve \( \alpha \) at the same time. In this case, if the curve-surface pair \( (\beta, M) \) is a curvature strip, the curve \( \beta \) is a helix and helix strip.

**Proof:**

![Spherical helix and Surface](image)

We use all of the proof of the Terquem Theorem of doctoral thesis of Keles, but only different is we should prove the helix strip. If the curve \( \alpha \) is a helix on \( S^2 \), then it
provides $\frac{K_1}{\tau_1}$ is constant. We have to show that $\beta$ is a helix strip on $M$, that is, $\frac{K_1}{\tau_2}$ is constant.

By the Figure, we have

$$\beta(s_i) = \alpha(s_i) + \lambda(s_i) \nu(s_i)$$

where

$$\alpha(s_i) = m + r \xi_i(s_i)$$

By differentiating both side of (3), we see that

$$\xi_i = \frac{d\alpha}{ds_i} = r \frac{d\xi_i}{ds_i}$$

By (1),

$$\xi_i = r(b_i \xi_i - a_i \eta_i)$$

Since the vector $m$ and $\lambda$ are constant, we obtain the following equation

$$\frac{d\beta}{ds_i} = \frac{d\xi_i}{ds_i} + \lambda(s_i) \left( \cos \phi \frac{d\xi_i}{ds_i} + \sin \phi \frac{d\eta_i}{ds_i} \right)$$

or

$$\frac{d\beta}{ds_i} = \frac{d\xi_i}{ds_i} + \lambda(s_i) \left( \cos \phi \frac{d\xi_i}{ds_i} + \sin \phi \frac{d\eta_i}{ds_i} \right)$$

By (1),

$$\frac{d\beta}{ds_i} = \frac{d\xi_i}{ds_i} + \lambda(s_i) \left( 1 - \frac{d\phi}{ds_i} \sin \phi \frac{d\xi_i}{ds_i} + \cos \phi \frac{d\xi_i}{ds_i} \right)$$

Since the spherical helix and the surface $M$ have the same tangent plane along the curves $\alpha$ and $\beta$, we can write

$$\left\{ \frac{d\beta}{ds_i}, \xi_i \right\} = 0.$$  

By substituting (6) at the last equation, we obtain

$$\frac{d\beta}{ds_i} = (1 \pm \lambda c_i) \xi_i$$  

If we calculate the second and third derivatives of the curve $\beta$, then we get

$$\frac{d^2\beta}{ds_i^2} = \pm \lambda c_i \xi_i + (1 \mp \lambda c_i) \eta_i - (1 \mp \lambda c_i) \xi_i$$

$$\frac{d^3\beta}{ds_i^3} = \pm \lambda \xi_i - (1 \mp \lambda c_i) \xi_i \eta_i - (1 \mp \lambda c_i) \xi_i$$

Since the same result is obtained by using other form of (7), we use the form $\frac{d\beta}{ds_i} = (1 - \lambda c_i) \xi_i$ of (7) at the rest of our proof. By differentiating both sides of (7), we obtain

$$\frac{d\beta}{ds_i} = (1 - \lambda c_i) \xi_i$$

$$\frac{d^2\beta}{ds_i^2} = -\lambda \xi_i + (1 - \lambda c_i) \eta_i - (1 - \lambda c_i) \xi_i$$

$$\frac{d^3\beta}{ds_i^3} = -\lambda \xi_i - (1 - \lambda c_i) \xi_i \eta_i - (1 - \lambda c_i) \xi_i$$

By applying Gram-Schmidt to the $\{\beta^*, \beta^*, \beta^*\}$, then we have

We obtain $a_i = 0$ and $b_i = 1$.

$r$ is the radius of the sphere. We denote $r = 1$. Since $m$ is a position vector that goes to the center of the sphere, $m$ is constant.

Since $\alpha_i = 0$, $(\alpha, S^\perp)$ is a curvature strip. By the strips $(\alpha, S^\perp)$ and $(\beta, M)$ are curvature strips and by the Terquem Theorem, we see that $\lambda$ is non-zero constant. Let $\nu(s_i)$ be a vector in $Sp\{\xi_i, \eta_i\}$, and let $\phi$ be the angle between $\xi_i$ and $\nu(s_i)$. Then we write

$$\nu(s_i) = \cos \phi \xi_i + \sin \phi \eta_i$$

By substituting (3) and (4) in (2), and differentiating both sides, we obtain (5).
\[ F_1 = (1 - \lambda \xi) \xi \]
\[ F_2 = (1 - \lambda \xi) \xi \eta - (1 - \lambda \xi) \eta \]
\[ F_3 = \lambda \xi \eta + \frac{(1 - \lambda \xi) \xi \eta \xi}{\xi^2 + 1} \]

By [10], we have
\[ \kappa_1^2 = h_1^2 + c_1^2, \quad h_1 = 1 \] (8)

and
\[ \tau_1^2 = a_i + \frac{b_i c_i - b_i c_i'}{h_i^2 + c_i^2}, \quad a_i = 0 \] (9)

By (8) and (9), we see that
\[ \tau_1 = \frac{-c_i}{\kappa_1^2}. \] (10)

By using (10) in \( F_3 \), we obtain
\[ F_3 = -(1 - \lambda \xi) \xi \eta - (1 - \lambda \xi) \eta \xi \]

If we calculate \( \kappa_2 \) and \( \tau_2 \), then we have
\[ \kappa_2 = \frac{\kappa_1}{|1 - \lambda \xi|} \]

and
\[ \tau_2 = \frac{\tau_1}{|1 - \lambda \xi|} \]

Dividing by \( \kappa_2 \) to \( \tau_2 \), we obtain
\[ \frac{\kappa_1}{\tau_2} = \frac{\kappa_1}{\tau_1} \] (11)

We obtain the proof of the theorem from the last equation.

References