Connection forms of an orthonormal frame field in the Minkowski space

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Abstract: In this work, connection formulas and forms of an orthonormal frame field in the Minkowski space $\mathbb{IR}^3$ were introduced and then the variation of connection forms was studied. In addition, the relation between the matrix of connection forms and the transition matrix of an orthonormal basis of tangent space were established, and an example was illustrated.

Keywords: Minkowski Space, One-Form, Connection Forms

1. Introduction

It is well-know that Euclidean geometry is a very useful tool in classical mechanics. On the other hand, Riemannian geometry has tremendous amount of applications in general relativity. Therefore, differential geometry has always given rise to new branches of physics [3,7]. Over the years, differential forms have generated a considerable amount of interest not only because they are interesting, but also important as they influenced the research direction both in Euclidean and Lorentzian geometries. Some of the works in this direction are given in [2,4,6] where the authors studied connections forms. In particular, they investigated covariant derivatives of frame elements as connected to this frame and they obtained connection forms and their matrices.

In the spirit of this study, we investigated the connection formulas and forms of an orthonormal frame field in the Minkowski space $\mathbb{IR}^3$. This paper is organized as follows: subsequently, we provided the background material concerning the basic concepts and definitions. Then, we study the variation of connection forms. In particular, we establish the relationship between the matrix of connection forms and the transition matrix of an orthonormal basis of the tangent space, and we present an example. In the final section, we summarize our results.

2. Preliminaries

We consider the Minkowski 3-space $\mathbb{IR}^3$ with the scalar product

$$\langle \tilde{X}, \tilde{Y} \rangle = x_1y_1 + x_2y_2 - x_3y_3 \quad \text{where } \tilde{X} = (x_1, x_2, x_3) \text{ and } \tilde{Y} = (y_1, y_2, y_3) \text{ are vectors in } \mathbb{IR}^3.$$

$\tilde{X}$ and $\tilde{Y}$ are called perpendicular if $\langle \tilde{X}, \tilde{Y} \rangle = 0$. The norm of $\tilde{X}$ is defined by $\|\tilde{X}\| = \sqrt{\langle \tilde{X}, \tilde{X} \rangle}$. $\tilde{X}$ is called space-like if $\langle \tilde{X}, \tilde{X} \rangle > 0$ or $\tilde{X} = \tilde{0}$, time-like if $\langle \tilde{X}, \tilde{X} \rangle < 0$ and light-like (null) if $\langle \tilde{X}, \tilde{X} \rangle = 0$. The cross product of $\tilde{X}$ and $\tilde{Y}$ is defined by [1]

$$\tilde{X} \times \tilde{Y} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

Let $\tilde{a}: I \rightarrow \mathbb{IR}^3, I \subset \mathbb{IR}$, be a regular curve in $\mathbb{IR}^3$, and consider the tangent vector $\tilde{a}(s), s \in I \subset \mathbb{IR}$. Then in [5],

1) $\tilde{a}$ is a space-like curve if $\langle \tilde{a}(s), \tilde{a}(s) \rangle > 0$,

2) $\tilde{a}$ is a time-like curve if $\langle \tilde{a}(s), \tilde{a}(s) \rangle < 0$,

3) $\tilde{a}$ is a null curve if $\langle \tilde{a}(s), \tilde{a}(s) \rangle = 0$.

Let $\tilde{a}(s)$ be a space-like curve of unit speed in $\mathbb{IR}^3$ with the natural curvature $\kappa(s)$ and torsion $\tau(s)$. Let us consider the Frenet frame $\{\tilde{t}, \tilde{n}, \tilde{b}\}$ of $\tilde{a}(s)$ where $\tilde{t}$, $\tilde{n}$ and $\tilde{b}$ are the space-like unit tangent vector, time-like unit principal normal vector and space-like unit binormal vector, respectively. Then scalar and cross product of $\tilde{t}$, $\tilde{n}$ and $\tilde{b}$ are given by

$$\langle \tilde{t}, \tilde{t} \rangle = -\langle \tilde{n}, \tilde{n} \rangle = \langle \tilde{b}, \tilde{b} \rangle = 1, \quad \langle \tilde{t}, \tilde{n} \rangle = \langle \tilde{t}, \tilde{b} \rangle = \langle \tilde{n}, \tilde{b} \rangle = 0,$$
Finally, Frenet formulas are given by \cite{8}
\[ \dot{t} = \kappa(s) \mathbf{n}, \quad \dot{\mathbf{n}} = \kappa(s) \mathbf{t} + \tau(s) \mathbf{b}, \quad \dot{b} = \tau(s) \mathbf{n}. \]

3. Main Results

Let \( \{E_1, E_2, E_3\} \) be an orthonormal frame field in the Minkowski space in which \( E_1 \) is a time-like vector. Consider a tangent vector \( v_p \in T_p(\mathbb{R}^3) \) at any \( p \in \mathbb{R}^3 \). Let \( D \) be the Levi-Civita connection on \( \mathbb{R}^3 \). Inspired by Frenet formulas, we can consider the covariant derivative of vector fields \( E_i \), \( 1 \leq i \leq 3 \), with respect to the tangent vector \( v_p \) as connected to this frame field. Since \( D_{v_p} E_i \in T_p(\mathbb{R}^3) \), and \( \{E_i(p), E_j(p), E_3(p)\} \) is an orthonormal basis of tangent space \( T_p(\mathbb{R}^3) \) with \( v_p \in \mathbb{R}^3 \), the covariant derivative of \( E_i \), \( 1 \leq i \leq 3 \), with respect to \( v_p \) can be written by

\[ (3.1) \quad D_{v_p} E_i = \sum_{j=1}^{3} w_{ij}(v_p) E_j(p) \]

where

\[ (3.2) \quad w_{ij}(v_p) = \langle D_{v_p} E_i, E_j(p) \rangle \]

From a geometric point of view, this equation extracts the number \( w_{ij}(v_p) \) that is component of the variation of vector \( E_i \) with respect to \( E_j(p) \) where the tangent vector \( v_p \) is the velocity vector along a curve.

Let \( v_p, u_p \in T_p(\mathbb{R}^3) \) and let \( a, b \in \mathbb{R} \). Since

\[ w_{ij}(a v_p + bu_p) = aw_{ij}(v_p) + bw_{ij}(u_p) \]

the transformation \( (w_{ij}) : T_p(\mathbb{R}^3) \to \mathbb{R} \), defined by

\[ (w_{ij})_p(v_p) = w_{ij}(v_p) \]

is linear. Thus \( w_{ij} \) corresponds to a linear transformation from the tangent space \( T_p(\mathbb{R}^3) \) to \( \mathbb{R} \) for all \( p \) in the Minkowski space \( \mathbb{R}^3 \). In this case, note that \( (w_{ij})_p \) is an element of the cotangent space \( T^*_p(\mathbb{R}^3) \), that is, a one-form in \( \mathbb{R}^3 \).

Theorem 3.1: One-forms \( w_{ij} \) of the orthonormal frame field \( \{E_i, E_j, E_3\} \) in which \( E_3 \) is a time-like vector are given by

\[ w_{ij} = -\epsilon_{ij3}(v_p), \quad 1 \leq i, j \leq 3. \]

Proof: Since \( \{E_i, E_j, E_3\} : \mathbb{R}^3 \to \mathbb{R} \) is a constant function, we have \( v_p \{E_i, E_j\} = 0 \) for all \( v_p \in T_p(\mathbb{R}^3) \). On the other hand, the equation

\[ v_p \{E_i, E_j\} = \{D_{v_p} E_i, E_j(p)\} + \{E_i(p), D_{v_p} E_j\} \]

implies that

\[ \{D_{v_p} E_i, E_j(p)\} = -\{E_i(p), D_{v_p} E_j\} \]

Thus

\[ w_{ij}(v_p) = \epsilon_{ij3}(E_i(p)) \]

Since the equation is true for all \( v_p \in T_p(\mathbb{R}^3) \), we obtain

\[ w_{ij} = -\epsilon_{ij3}(E_i) \]

Definition 3.1: One-forms \( w_{ij} \) are called connection forms of the orthonormal frame field \( \{E_i, E_j, E_3\} \).

Definition 3.2: The equation given in (3.1) is called connection formulas of the orthonormal frame field \( \{E_i, E_j, E_3\} \).

Using Theorem 3.1, we can obtain the matrix \( W = [w_{ij}]_{3 \times 3} \) of one-forms as

\[ W = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ w_{13} & w_{23} & 0 \end{bmatrix} \]

Note that \( W \) is a skew-adjoint matrix in the sense that \( W^\top = -\epsilon W \), where \( \epsilon \) is the signature matrix given by

\[ \epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

Let \( \frac{\partial}{\partial x_i} \) be the natural frame field in the Minkowski space \( \mathbb{R}^3 \) in which \( \frac{\partial}{\partial x_3} \) is a time-like vector.

Then the vector \( E_i \), \( 1 \leq i \leq 3 \), can be written as

\[ (3.3) \quad E_i = \sum_{j=1}^{3} a_{ij} \frac{\partial}{\partial x_j}, \]

where \( a_{ij} : \mathbb{R}^3 \to \mathbb{R} \) is a differentiable function.

Let \( A = [a_{ij}]_{3 \times 3} \) be the transition matrix between the orthonormal bases \( \{E_i(p), E_j(p), E_3(p)\} \) and \( \{\frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial x_2}(p), \frac{\partial}{\partial x_3}(p)\} \) of the tangent space \( T_p(\mathbb{R}^3) \). The equation given in (3.3) can be written as
Now we are ready to state the relation between the matrices $A$ and $W$. Let $v_p$ be an element of $T_p(\mathbb{R}^3)$. Then, we have

$$D_y E_i = D_y \left( \sum_{k=1}^3 a_{ik} \frac{\partial}{\partial x_k} \right) = \sum_{k=1}^3 D_y (a_{ik}) \frac{\partial}{\partial x_k} = \sum_{k=1}^3 v_p [a_{ik}] \frac{\partial}{\partial x_k} (p)$$

and therefore

$$w_y (v_p) = \left( D_y E_i, E_j (p) \right) \epsilon_j = \sum_{k=1}^3 v_p [a_{ik}] \frac{\partial}{\partial x_k} (p) \sum_{j=1}^3 a_{jk} \frac{\partial}{\partial x_j} (p) \epsilon_j = (d_{ik}) \epsilon_j + (d_{kj}) \epsilon_i - (d_{ij}) \epsilon_k$$

Since this equation is correct for all $v_p \in T_p(\mathbb{R}^3)$, we obtain

$$\epsilon_j w_y = d_{ij} \delta_{ij} + d_{kj} a_{ij} - d_{ij} a_{jk}.$$
References


