On the explicit parametric equation of a general helix with first and second curvature in Nil 3-space

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Abstract: Nil geometry is one of the eight geometries of Thurston's conjecture. In this paper we study in Nil 3-space and the Nil metric with respect to the standard coordinates \((x,y,z)\) is \(g_{\text{Nil}^3} = (dx)^2 + (dy)^2 + (dz - xdy)^2\) in \(\mathbb{R}^3\). In this paper, we find out the explicit parametric equation of a general helix. Further, we write the explicit equations Frenet vector fields, the first and the second curvatures of general helix in Nil 3-Space. The parametric equation the Normal and Binormal ruled surface of general helix in Nil 3-space in terms of their curvature and torsion has been already examined in [12], in Nil 3-Space.

Keywords: Nil Space, Helix, Curvatures

1. Introduction

In mathematics, Thurston's conjecture proposed a complete characterization of geometric structures on three-dimensional manifolds. The conjecture was proposed by William Thurston (1982), and implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture. Thurston's geometrization conjecture states that; Certain three-dimensional topological spaces each have a unique geometric structure that can be associated with them. It is an analogue of the uniformization theorem for two-dimensional surfaces, which states that every simply-connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic). In three dimensions, it is not always possible to assign a single geometry to a whole topological space. Instead, the geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure. Thurston's conjecture is that, after you split a three-manifold into its connected sum and the Jaco-Shalen-Johannson torus decomposition, the remaining components each admit exactly one of the following geometries
- Euclidean geometry,
- Hyperbolic geometry,
- Spherical geometry,
- The geometry of \(S^2 \times \mathbb{R}\),
- The geometry of \(H^2 \times \mathbb{R}\),
- The geometry of the universal cover \(SL_2\mathbb{R}\)– of the Lie group \(SL_2\mathbb{R}\),
- Nil geometry,
- Sol geometry.

For more detail see [13].

A nilmanifold is a differentiable manifold which has a transitive nilpotent group of diffeomorphisms acting on it. In the Riemannian category, there is also a good notion of a nilmanifold. A Riemannian manifold is called a homogeneous nilmanifold if there exist a nilpotent group of isometries acting transitively on it. The requirement that the transitive nilpotent group acts by isometries leads to the following rigid characterization: every homogeneous nilmanifold is isometric to a nilpotent Lie group with left-invariant metric (see [4]).

The two-parameter family of metrics first appeared in the works of Bianchi, Cartan and Vranceanu, these spaces are often referred to as Bianchi-Cartan-Vranceanu spaces, or BCV-spaces for short. Some well-known examples of BCV-spaces are the Riemannian product spaces \(S^3 \times \mathbb{R}\), \(H^2 \times \mathbb{R}\) and the 3-dimensional Heisenberg group [5]. Let \(\kappa\) and \(\tau\) be real numbers, with \(\kappa \geq 0\). The Bianchi-Cartan-Vranceanu spaces, (BCV-spaces) \(M^3(\kappa, \tau)\) is defined as the set

\[
\{(x,y,z) \in \mathbb{R}^3 : 1 + \frac{\kappa}{4}(x^2 + y^2) > 0\}
\]

equipped with metric

\[
ds^2 = (dx^2 + dy^2)/((1 + \frac{\kappa}{4}(x^2 + y^2))^2)
\]

\[
+ (dz + \tau((ydx - xdy))/((1 + \frac{\kappa}{4}(x^2 + y^2)))^2).
\]
More details can be found in [4] and [1].

In [5], it is restricted to the 3-dimensional Heisenberg group coming from \( \mathbb{R}^2 \) with the canonical symplectic form \( \omega((x,y),(x',y'))=xy_1-x_1y \), i.e., they consider \( \mathbb{R}^2 \) with the group operation 

\[
(x,y,z)\mapsto (x_1,y_1,z_1)=(x+y_1,y_1,z_1+(xy_1/2)-(x_1y/2)).
\]

For every non-zero number \( \tau \) the following Riemannian metric on \((\mathbb{R}^3,*)\) is left invariant:

\[
ds^2=dx^2+dy^2+4\tau^2(dz+(ydx-xdy)/2))^2.
\]

After the change of coordinates \((x, y, 2\tau z)\rightarrow(x,y,z)\), this metric is expressed as

\[
ds^2=dx^2+dy^2+(dz+\tau(ydx-xdy))^2.
\]

By some authors the notation Nil 3-space is only used if \( \tau=1/2 \). We will use the notation \( \text{Nil}_3 \) in short. It is well known that Nil space is isometric to Heisenberg space. The geometry of Nil is the three dimensional Lie group of all real 3 triangular matrices of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]

Let \((\mathbb{R}^3, g_{\text{Nil}})\) denote Nil space, where the metric with respect to the standard coordinates \((x,y,z)\) in \( \mathbb{R}^3 \) can be written [14] as

\[
g_{\text{Nil}}=(dx)^2+(dy)^2+(dz-xydxdy)^2.
\]

Hence we get the symmetric tensor field \( g_{\text{Nil}} \) on \( \text{Nil}_3 \) by components.

\[
g_{ij}=
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1+x^2-x & 0 \\
0 & -x & 1
\end{pmatrix}
\]

Using the formula with \( \omega^i=dx, \omega^i=dy, \omega^i=dx\wedge dy \), we can also write:

\[
ds^2=\sum_{i=1}^3\omega^i\otimes\omega^i,
\]

where the orthonormal basis dual to the 1-forms is

\[
E_1=(\partial/(\partial x)), E_2=(\partial/(\partial y))+(x(\partial/(\partial z))), E_3=(\partial/(\partial z)).
\]

With respect to this orthonormal basis, the Levi-Civita connection and the Lie brackets can be easily computed as:

\[
\nabla_{E_1}E_2=0, \quad \nabla_{E_1}E_3=0, \quad \nabla_{E_2}E_1=(1/2)E_2, \quad \nabla_{E_2}E_3=0, \quad \nabla_{E_3}E_1=(1/2)E_3, \quad \nabla_{E_3}E_2=0.
\]

Hence

\[
\begin{pmatrix}
0 & \frac{1}{2}E_1 & -\frac{1}{2}E_2 \\
\frac{-1}{2}E_1 & 0 & \frac{1}{2}E_3 \\
\frac{-1}{2}E_2 & -\frac{1}{2}E_3 & 0
\end{pmatrix}
\]

is the matrix with \((i,j)\)-element in the table equals \( \nabla_{E_i}E_j \) for the basis \( \{E_1, E_2, E_3\} \). See for more details [14].

### 2. The Parametric Equation of General Helix in Nil 3-Space

#### 2.1. Riemannian Structure of Nil Space

Helix is one of the fascinating curve in science and nature. In this section, we study on the general helices in the \( \text{Nil}_3 \). We characterize the general helices in terms of their curvature and torsion. A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [7] and [2] for details) is: A necessary and sufficient condition that a curve be a helix is that the ratio of curvature to torsion be constant. Helices are examined in [9] and [6]. Let \( \alpha \) be a helix that lies on the cylinder. A helix which lies on the cylinder is called cylindrical helix or general helix. Assume that \( \{T,N,B,\kappa,\tau\} \) be the Frenet apparatus along the curve \( \alpha \). It has been known that the curve \( \alpha \) is a cylindrical helix if and only if \((\kappa/\tau)=\text{constant}\) then \((\kappa/\tau)'=0\) where \( \kappa \) and \( \tau \) are the curvatures of \( \alpha \). If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. We call a curve a circular helix if both \( \tau\neq 0 \) and \( k \) are constant. Then, the Frenet frame satisfies the following Frenet-Serret equations

\[
\nabla_{T_1}T=\kappa N, \quad \nabla_{T_1}N=-\kappa T+\tau B, \quad \nabla_{T_1}B=-\tau N.
\]

With respect to the orthonormal basis \( \{E_1, E_2, E_3\} \), we can write

\[
T=T_1E_1+T_2E_2+T_3E_3, \quad N=N_1E_1+N_2E_2+N_3E_3, \quad B=B_1E_1+B_2E_2+B_3E_3.
\]

Parametric equations of general helices in the sol space \( \text{Sol}_3 \) are examined in [3]. Normal ruled surfaces of general helices in the sol space \( \text{Sol}_3 \) are examined in [8].

Normal and Binormal ruled surfaces of general helices in Nil 3-space with the Riemannian Structure of Nil 3-space are examined in [12].

Parametric equation of general helix and all the Frenet
apparatus are examined as in the following theorems.

2.2. The parametric equation of General Helices in Nil Space Nil$_3$

**Theorem:**
Let $\alpha: I \to \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the equation of a unit speed non-geodesic general helix $\alpha$, with respect to the orthonormal basis, $\{E_1, E_2, E_3\}$,

$$a(s) = ((\sin \beta)/(C_1))\sin D + C_3 E_1 + ((-\sin \beta)/(C_1))\cos D + C_4 E_2$$
$$+ ((\sin^2 \beta)/(4C_1^2))\sin 2D - ((\sin^2 \beta)/(2C_1))\cos D$$
$$+ ((\sin^2 \beta)/(2C_1)) + \cos \beta) s - C_3 C_4 + C_2 E_3,$$

where we take $D = C_1$.

**Proof:**
Let $\alpha(I \to \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the equation of a unit speed non-geodesic general helix $\alpha$, with respect to the orthonormal basis, $\{E_1, E_2, E_3\}$,

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$$+ ((\sin^2 \beta)/(4C_1^2))\sin 2D - ((\sin^2 \beta)/(2C_1))\cos D$$
$$+ ((\sin^2 \beta)/(2C_1)) + \cos \beta) s - C_3 C_4 + C_2 E_3,$$

where we take $D = C_1$.

3. The Curvatures of the General Helix in Nil 3-Space

3.1. First Curvature of the General Helix in Nil Space Nil$_3$

**Theorem:**
The first curvature (curvature ) of the general helix in Nil Space Nil$_3$ is

$$\kappa = (\cos \beta - C_1) \sin \beta,$$

$$(\cos \beta - C_1) \sin \beta > 0$$

**Proof:**
Assume that $\alpha: I \to \text{Nil}_3$ be a unit speed non-geodesic general helix with

$$T = \sin \beta \cos DE_1 + \sin \beta \sin DE_2 + \cos \beta E_3.$$ 

The Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T = T_1 E_1 + (1/2)T_2 E_2 + ((-1/2)T_3 E_2$$

$$+ T_2 E_2 + (1/2)T_3 E_3 + (1/2)T_1 E_1$$

$$+ T_3 E_3 - (1/2)T_1 E_1 - (1/2)T_3 E_1$$

$$= (T_1 + T_2) E_1 + (T_2 - T_3) E_2 + (T_3) E_3$$

By substituting $T_1, T_2, T_3$ and derivates, we get

$$\nabla_T = (\cos \beta - C_1) (\sin \beta \sin DE_1 - \sin \beta \cos DE_2)$$

$$= (\cos \beta - C_1) \sin \beta (\sin DE_1 - \cos DE_2).$$

Since $\kappa = g_{\text{Nil}_3}(\nabla_T, N)$ and $N = (1/\kappa) \nabla_T T$ we have

$$\kappa = g_{\text{Nil}_3}(\nabla_T, (1/\kappa) \nabla_T T) = g_{\text{Nil}_3}(\nabla_T T, \nabla_T T)$$

$$\kappa = (C_1 \sin \beta - \sin \beta \cos \beta)^2$$

$$= (\cos \beta - C_1)^2 \sin^2 \beta$$

and also for $(\cos \beta - C_1) \sin \beta > 0$, first curvature is

$$\kappa = (C_1 - \cos \beta) \sin \beta.$$

3.1.1. The Normal Vector Fields of the General Helix

The following theorem gives us the explicit parametric equation of normal vector fields in Nil$_3$.

**Theorem:**
Let $\alpha: I \to \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the normal vector field of the general helix is

$$N = (\sin D, -\cos D, 0).$$

where we take $D = C_1 \sin \beta$.

**Proof:**
Let $\alpha: I \to \text{Nil}_3$ be a unit speed non-geodesic general helix. By the use of Frenet formula $\nabla_T T = \kappa N$,

$$\kappa N = (\cos \beta - C_1)(\sin \beta \sin DE_1 - \sin \beta \cos DE_2)$$

$$= (\cos \beta - C_1) \sin \beta (\sin DE_1 - \cos DE_2).$$

Hence the normal vector field of the general helix is
\[ N = (1/κ)((-C₁\sinβ - ((\sin2β)/2))\sin DE₁ \]
\[ + (C₁\sinβ - ((\sin2β)/2))\cos DE₂) \]

where we take \( D = C₁s + C₂ \), where \( C₁, C₂ \in \mathbb{E}_3 \). Also, we know that: \( κ = (C₁ - \cos β)\sin β \), so
\[ N = (1/κ) \nabla T \]
\[ N = \frac{1}{\sin(\cos β - C₁)}((\cos β - C₁)(\sin β)\sin DE₁ - \sin β \cos DE₂) \]
\[ = \sin DE₁ - \cos DE₂ \]
\[ N = (\sin D, -\cos D, 0), \]
or substituting
\[ E₁ = (\partial/(\partial x)), E₂ = (\partial/(\partial y)) + x(\partial/(\partial z)), E₃ = (\partial/(\partial z)) \]

In
\[ N = \sin DE₁ - \cos DE₂ \]
\[ N = \sin D(\partial/(\partial x)) - \cos D(\partial/(\partial y)) + x(\partial/(\partial z)) \]
\[ N = \sin D(\partial/(\partial x)) - \cos D(\partial/(\partial y)) + ((\sin β)/(C₁))\sin D - C₁(\cos D)(\partial/(\partial z)). \]

### 3.2. Second Curvature (Torsion) of the General Helix in Nil Space \( \mathbb{E}_3 \)

**Theorem:**
The second curvature (torsion) of the general helix in Nil Space \( \mathbb{E}_3 \) is
\[ τ = (C₁^2 - C₁\cos β + (1/4))^{1/2} \]

**Proof:**
With the Levi-Civita connection and Lie brackets, the second curvature is easily computed as:
\[ \nabla T N = N₁ = (1/2)N₂ T₃ + (1/2)N₃ T₂ E₁ \]
\[ + (N₁ + ((-1)/2)N₂ T₃ + ((-1)/2)N₃ T₂) E₂ \]
\[ + (N₂ + ((1/2)N₁ T₃ + ((-1)/2)N₃ T₂) E₃. \]

Also for \( N = \sin D E₁ - \cos D E₂ \), we know that
\[ N₁ = \sin D; N₁ = C₁ \cos D \]
\[ N₂ = -\cos D; N₂ = C₁ \sin D \]
\[ N₃ = 0, N₃ = 0. \]

Now it is easy to say that for
\[ \nabla T N = ((C₁ - (1/2)\cos β)\cos DE₁ \]
\[ + (C₁ - (1/2)\cos β)\sin DE₂ \]
\[ + ((-1)/2)\sin β) E₃ \]
\[ \nabla T N = (1/2)((2C₁ - \cos β)\cos D(2C₁ - \cos β)\sin D - \sin β) \]

It is well known that Binormal vector field of a curve is \( B = (1/τ)(\nabla T N + κ T) \). Also torsion is
\[ τ = g_{\mathbb{E}_3}(\nabla T N, B) \]
\[ τ = g_{\mathbb{E}_3}(\nabla T N, (1/τ)(\nabla T N + κ T)) \]
\[ τ = g_{\mathbb{E}_3}(\nabla T N, (1/τ)(\nabla T N + κ T)) \]
\[ τ = (1/4)((2C₁ - \cos β)\cos D)² + ((2C₁ - \cos β)\sin D)² + \sin β \]
\[ τ = (1/4)((2C₁ - \cos β)² + \sin β) \] or \( τ = C₁² - C₁\cos β + (1/4) \).
Figure 1. The figure of Helix in Example.

References


