On the explicit parametric equation of a general helix with first and second curvature in Nil 3-space

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To cite this article:

Şeyda Kılıçoğlu. On the Explicit Parametric Equation of a General Helix with First and Second Curvature in Nil 3-Space. *Pure and Applied Mathematics Journal*. Special Issue: Applications of Geometry. Vol. 4, No. 1-2, 2015, pp. 19-23. doi: 10.11648/j.pamj.s.2015040102.15

Abstract: Nil geometry is one of the eight geometries of Thurston's conjecture. In this paper we study in Nil 3-space and the Nil metric with respect to the standard coordinates (x, y, z) is $g_{Nil_3} = (dx)^2 + (dy)^2 + (dz - xdy)^2$ in $\mathbb{R}^3$. In this paper, we find out the explicit parametric equation of a general helix. Further, we write the explicit equations Frenet vector fields, the first and the second curvatures of general helix in Nil 3-Space. The parametric equation the Normal and Binormal ruled surface of general helix in Nil 3-space in terms of their curvature and torsion has been already examined in [12], in Nil 3-Space.

Keywords: Nil Space, Helix, Curvatures

1. Introduction

In mathematics, Thurston's conjecture proposed a complete characterization of geometric structures on three-dimensional manifolds. The conjecture was proposed by William Thurston (1982), and implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture. Thurston's geometrization conjecture states that; Certain three-dimensional topological spaces each have a unique geometric structure that can be associated with them. It is an analogue of the uniformization theorem for two-dimensional surfaces, which states that every simply-connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic). In three dimensions, it is not always possible to assign a single geometry to a whole topological space. Instead, the geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure. Thurston's conjecture is that, after you split a three-manifold into its connected sum and the Jaco-Shalen-Johannson torus decomposition, the remaining components each admit exactly one of the following geometries

Euclidean geometry,

Hyperbolic geometry,

Spherical geometry,

The geometry of $S^2 \times \mathbb{R}$,

The geometry of $H^2 \times \mathbb{R}$,

The geometry of the universal cover $SL_2\mathbb{R}$ of the Lie

group $SL_2\mathbb{R}$,

Nil geometry,

Sol geometry.

For more detail see [13].

A nilmanifold is a differentiable manifold which has a transitive nilpotent group of diffeomorphisms acting on it. In the Riemannian category, there is also a good notion of a nilmanifold. A Riemannian manifold is called a homogeneous nilmanifold if there exist a nilpotent group of isometries acting transitively on it. The requirement that the transitive nilpotent group acts by isometries leads to the following rigid characterization: every homogeneous nilmanifold is isometric to a nilpotent Lie group with left-invariant metric (see [4]).

The two-parameter family of metrics first appeared in the works of Bianchi, Cartan and Vranceanu, these spaces are often referred to as Bianchi-Cartan-Vranceanu spaces, or BCV-spaces for short. Some well-known examples of BCV-spaces are the Riemannian product spaces $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$ and the 3-dimensional Heisenberg group [5]. Let κ and τ be real numbers, with $\tau \geq 0$. The Bianchi-Cartan-Vranceanu spaces, (BCV-spaces) $M^3(\kappa, \tau)$ is defined as the set

$$\{(x, y, z) \in \mathbb{R}^3 : 1 + (\kappa/4)(x^2 + y^2) > 0\}$$

equipped with metric

$$ds^2 = ((dx^2 + dy^2) / ((1 + (\kappa/4)(x^2 + y^2))^2))$$

$$+ (dz + \tau((ydx - xdy) / (1 + (\kappa/4)(x^2 + y^2))))^2.$$

* if $\kappa = \tau = 0$, then $M^3(\kappa, \tau) \cong \mathbb{E}^3$

* if $\kappa = 0$ and $\tau \neq 0$, then $M^3(\kappa, \tau) \cong \text{Nil}_3$.

More details can be found in [4] and [1].

In [5], it is restricted to the 3-dimensional Heisenberg group coming from $\mathbb{R}^2$ with the canonical symplectic form $\omega((x, y), (x_1, y_1)) = xy_1 - x_1y$, i.e., they consider $\mathbb{R}^3$ with the group operation

$$(x, y, z) * (x_1, y_1, z_1) = (x + x_1, y + y_1, z + z_1 + ((xy_1)/2) - ((x_1y)/2)).$$

For every non-zero number τ the following Riemannian metric on $(\mathbb{R}^3, *)$ is left invariant:

$$ds^2 = dx^2 + dy^2 + 4\tau^2(dz + ((ydx - xdy)/2))^2.$$

After the change of coordinates $(x, y, 2\tau z) \rightarrow (x, y, z)$, this metric is expressed as

$$ds^2 = dx^2 + dy^2 + (dz + \tau(ydx - xdy))^2.$$

By some authors the notation Nil 3-space is only used if $\tau = (1/2)$. We will use the notation Nil_3 in short. It is well known that Nil space is isometric to Heisenberg space. The geometry of Nil is the three dimensional Lie group of all real 3 triangular matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $(\mathbb{R}^3, g_{\text{Nil}_3})$ denote Nil space, where the metric with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ can be written [14] as

$$g_{\text{Nil}_3} = (dx)^2 + (dy)^2 + (dz - xdy)^2.$$

Hence we get the symmetric tensor field g_{Nil_3} on Nil_3 by components.

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & -x \\ 0 & -x & 1 \end{bmatrix}.$$

Note that the Nil metric can also be written as:

$$ds^2 = \sum_{i=1}^3 \omega_i \otimes \omega_i,$$

where $\omega^1 = dx$, $\omega^2 = dy$, $\omega^3 = dz - xdy$, and the orthonormal basis dual to the 1-forms is

$$E_1 = (\partial/(\partial x)), E_2 = (\partial/(\partial y)) + x(\partial/(\partial z)), E_3 = (\partial/(\partial z)).$$

With respect to this orthonormal basis, the Levi-Civita connection and the Lie brackets can be easily computed as:

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, & \nabla_{E_1} E_2 &= (1/2)E_3, & \nabla_{E_1} E_3 &= ((-1)/2)E_2 \\ \nabla_{E_2} E_1 &= ((-1)/2)E_3, & \nabla_{E_2} E_2 &= 0, & \nabla_{E_2} E_3 &= (1/2)E_1 \\ \nabla_{E_3} E_1 &= ((-1)/2)E_2, & \nabla_{E_3} E_2 &= (1/2)E_1, & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

$$[E_1, E_2] = E_3,$$

$$[E_2, E_3] = 0,$$

$$[E_1, E_3] = 0.$$

Hence

$$\begin{bmatrix} 0 & \frac{1}{2}E_3 & -\frac{1}{2}E_2 \\ -\frac{1}{2}E_3 & 0 & \frac{1}{2}E_1 \\ -\frac{1}{2}E_2 & \frac{1}{2}E_1 & 0 \end{bmatrix}$$

is the matrix with (i, j) -element in the table equals $\nabla_{E_i} E_j$ for the basis $\{E_1, E_2, E_3\}$. See for more details [14].

2. The Parametric Equation of General Helix in Nil 3-Space

2.1. Riemannian Structure of Nil Space

Helix is one of the fascinating curve in science and nature. In this section, we study on the general helices in the Nil_3 . We characterize the general helices in terms of their curvature and torsion. A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [7] and [2] for details) is: A necessary and sufficient condition that a curve be a helix is that the ratio of curvature to torsion be constant. Helices are examined in [9] and [6]. Let α be a helix that lies on the cylinder. A helix which lies on the cylinder is called cylindrical helix or general helix. Assume that $\{T, N, B, \kappa, \tau\}$ be the Frenet apparatus along the curve α . It has been known that the curve α is a cylindrical helix if and only if $((\kappa/\tau))$ is constant, then $((\kappa/\tau))' = 0$ where κ and τ are the curvatures of α . If the curve is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. We call a curve a circular helix if both $\tau \neq 0$ and κ are constant. Then, the Frenet frame satisfies the following Frenet-Serret equations

$$\nabla_T T = \kappa N,$$

$$\nabla_T N = -\kappa T + \tau B,$$

$$\nabla_T B = -\tau N.$$

With respect to the orthonormal basis $\{E_1, E_2, E_3\}$, we can write

$$T = T_1 E_1 + T_2 E_2 + T_3 E_3,$$

$$N = N_1 E_1 + N_2 E_2 + N_3 E_3,$$

$$B = T \times N = B_1 E_1 + B_2 E_2 + B_3 E_3.$$

Parametric equations of general helices in the sol space Sol_3 are examined in [3]. Normal ruled surfaces of general helices in the Sol space Sol_3 are examined in [8].

Normal and Binormal ruled surfaces of general helices in Nil 3-space with the Riemannian Structure of Nil 3-space are examined in [12].

Parametric equation of general helix and all the Frenet

apparatus are examined as in the following theorems.

2.2. The parametric equation of General Helices in Nil Space Nil₃

Theorem:

Let $\alpha: I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the equation of a unit speed non-geodesic general helix α , with respect to the orthonormal basis, $\{E_1, E_2, E_3\}$,

$$\begin{aligned} \alpha(s) = & ((\sin\beta)/(C_1))\sin D + C_3 E_1 + (((-\sin\beta)/(C_1))\cos D + C_4) E_2 \\ & + (((\sin^2\beta)/(4C_1^2))\sin 2D - ((C_4\sin\beta)/(C_1))\sin D \\ & + (((\sin^2\beta)/(2C_1)) + \cos\beta)s - C_3 C_4 + C_5) E_3, \end{aligned}$$

where we take $D = C_1 s + C_2$, $C_1, C_2 \in \mathbb{R}$.

Proof:

Assume that $\alpha: I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix. So, without loss of generality, we take its axis as parallel to the vector E_3 . Then

$$g_{\text{Nil}_3}(T, E_3) = T_3 = \cos\beta,$$

where β is constant angle. On the other hand the tangent vector T is an unit vector, so the following condition is satisfied $T_1^2 + T_2^2 = 1 - \cos^2\beta$. Since $\cos^2\beta + \sin^2\beta = 1$, we have the general solution of $T_1^2 + T_2^2 = \sin^2\beta$ can be written in the following form

$$T_1 = \sin\beta \cos D,$$

$$T_2 = \sin\beta \sin D,$$

$$T_3 = \cos\beta$$

Also, without loss of generality, where we take $D = C_1 s + C_2$ where $C_1, C_2 \in \mathbb{R}$. So, substituting the components T_1 , T_2 and T_3 in the equation, we have the following equation

$$T = \sin\beta \cos D E_1 + \sin\beta \sin D E_2 + \cos\beta E_3.$$

Definition of the tangent vector field T , give us;

$$((dx)/(ds)) = \sin\beta \cos D$$

$$((dy)/(ds)) = \sin\beta \sin D$$

$$((dz)/(ds)) = x \sin\beta \sin D + \cos\beta.$$

Integrating both sides, we have

$$\Rightarrow x(s) = ((\sin\beta)/(C_1))\sin D + C_3$$

$$\Rightarrow y(s) = ((-\sin\beta)/(C_1))\cos D + C_4$$

$$\Rightarrow z(s) = -((\sin^2\beta)/(4C_1^2))\sin 2D$$

$$-((C_3\sin\beta)/(C_1))\cos D$$

$$+(((\sin^2\beta)/(2C_1)) + \cos\beta)s + C_5$$

where C_3, C_4, C_5 are constant of integration. Substituting all them in $\alpha(s)$, this proves our assertion. Thus, the proof of theorem is completed.

3. The Curvatures of the General Helix in Nil 3-Space

3.1. First Curvature of the General Helix in Nil Space Nil₃

Theorem:

The first curvature (curvature) of the general helix in Nil Space Nil₃ is

$$\kappa = (\cos\beta - C_1)\sin\beta, \quad (\cos\beta - C_1)\sin\beta > 0$$

Proof:

Assume that $\alpha: I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix with

$$T = \sin\beta \cos D E_1 + \sin\beta \sin D E_2 + \cos\beta E_3.$$

The Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T T = T_1' E_1 - (1/2) T_1 T_2 E_3 + ((-1)/2) T_1 T_3 E_2$$

$$+ T_2' E_2 + (1/2) T_2 T_1 E_3 + (1/2) T_2 T_3 E_1$$

$$+ T_3' E_3 - (1/2) T_3 T_1 E_2 + (1/2) T_3 T_2 E_1$$

$$= (T_1' + T_2 T_3) E_1 + (T_2' - T_1 T_3) E_2 + (T_3') E_3$$

By substituting T_1, T_2, T_3 and derivatives, we get

$$\nabla_T T = (\cos\beta - C_1)(\sin\beta \sin D E_1 - \sin\beta \cos D E_2)$$

$$= (\cos\beta - C_1)\sin\beta (\sin D E_1 - \cos D E_2).$$

Since $\kappa = g_{\text{Nil}_3}(\nabla_T T, N)$ and $N = (1/\kappa)\nabla_T T$ we have

$$\kappa = g_{\text{Nil}_3}(\nabla_T T, (1/\kappa)\nabla_T T) \quad \kappa^2 = g_{\text{Nil}_3}(\nabla_T T, \nabla_T T)$$

$$\kappa^2 = (C_1 \sin\beta - \sin\beta \cos\beta)^2$$

$$= (\cos\beta - C_1)^2 \sin^2\beta$$

and also for $(\cos\beta - C_1)\sin\beta > 0$, first curvature is

$$\kappa = (C_1 - \cos\beta)\sin\beta.$$

3.1.1. The Normal Vector Fields of the General Helix

The following theorem gives us the explicit parametric equation of normal vector fields in Nil₃.

Theorem:

Let $\alpha: I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the normal vector field of the general helix is

$$N = (\sin D, -\cos D, 0)$$

where we take $D = C_1 s + C_2$, $C_1, C_2 \in \mathbb{R}$.

Proof:

Let $\alpha: I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix. By the use of Frenet formula $\nabla_T T = \kappa N$,

$$\kappa N = (\cos\beta - C_1)(\sin\beta \sin D E_1 - \sin\beta \cos D E_2)$$

$$= (\cos\beta - C_1)\sin\beta (\sin D E_1 - \cos D E_2).$$

Hence the normal vector field of the general helix is

$$N = (1/\kappa)((-C_1 \sin \beta + ((\sin 2\beta)/2)) \sin DE_1 \\ + (C_1 \sin \beta - ((\sin 2\beta)/2)) \cos DE_2)$$

where we take $D=C_1s+C_2$, where $C_1, C_2 \in \mathbb{R}$. Also we know that, $\kappa = (C_1 - \cos \beta) \sin \beta$, so

$$N = (1/\kappa) \nabla_T T$$

$$N = 1/(\sin \beta (\cos \beta - C_1)) (\cos \beta - C_1) (\sin \beta \sin DE_1 - \sin \beta \cos DE_2) \\ = \sin DE_1 - \cos DE_2$$

$$N = (\sin D, -\cos D, 0),$$

or substituting

$$E_1 = (\partial/(\partial x)), E_2 = (\partial/(\partial y)) + x(\partial/(\partial z)), E_3 = (\partial/(\partial z))$$

In

$$N = \sin DE_1 - \cos DE_2$$

$$N = \sin D(\partial/(\partial x)) - \cos D((\partial/(\partial y)) + x(\partial/(\partial z)))$$

$$N = \sin D(\partial/(\partial x)) - \cos D(\partial/(\partial y)) - ((\sin \beta)/(C_1)) \sin D - C_3 \cos D(\partial/(\partial z)).$$

3.2. Second Curvature (Torsion) of the General Helix in Nil Space Nil₃

Theorem:

The second curvature (torsion) of the general helix in Nil Space Nil₃ is

$$\tau = (C_1^2 - C_1 \cos \beta + (1/4))^{1/2}$$

Proof:

With the Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T N = (N'_1 + (1/2)N_2 T_3 + (1/2)N_3 T_2)E_1 \\ + (N'_2 + ((-1)/2)N_1 T_3 + ((-1)/2)N_3 T_1)E_2 \\ + (N'_3 + (1/2)N_2 T_1 + ((-1)/2)N_1 T_2)E_3.$$

Also for $N = \sin D E_1 - \cos D E_2$, we know that

$$N_1 = \sin D; N'_1 = C_1 \cos D \\ N_2 = -\cos D; N'_2 = C_1 \sin D \\ N_3 = 0, N'_3 = 0.$$

Now it is easy to say that for

$$\nabla_T N = ((C_1 - (1/2)\cos \beta) \cos DE_1 \\ + (C_1 - (1/2)\cos \beta) \sin DE_2 \\ + ((-1)/2) \sin \beta) E_3$$

$$\nabla_T N = (1/2)((2C_1 - \cos \beta) \cos D, (2C_1 - \cos \beta) \sin D, -\sin \beta)$$

It is well known that Binormal vector field of a curve is $B = (1/\tau)(\nabla_T N + \kappa T)$. Also torsion is

$$\tau = g_{\text{Nil}_3}(\nabla_T N, B)$$

$$\tau = g_{\text{Nil}_3}(\nabla_T N, (1/\tau)(\nabla_T N + \kappa T))$$

$$\tau^2 = g_{\text{Nil}_3}(\nabla_T N, \nabla_T N + \kappa T)$$

$$\tau^2 = g_{\text{Nil}_3}(\nabla_T N, \nabla_T N) + g_{\text{Nil}_3}(\nabla_T N, \kappa T)$$

$$\tau^2 = (1/4)((2C_1 - \cos \beta) \cos D)^2 + ((2C_1 - \cos \beta) \sin D)^2 + \sin^2 \beta$$

$$\tau^2 = (1/4)((2C_1 - \cos \beta)^2 + \sin^2 \beta) \text{ or } \tau^2 = C_1^2 - C_1 \cos \beta + (1/4).$$

3.2.1. The Binormal Vector Field of the General Helix in Nil Space Nil₃

Theorem:

Let $\alpha : I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix. Then, the binormal vector field of the general helix is

$$B = (1/(C_1^2 - C_1 \cos \beta + (1/4))^{1/2}) \\ ((C_1 - (1/2)\cos \beta + \sin^2 \beta \cos \beta - \sin^2 \beta C_1) \cos DE_1 \\ + (C_1 - (1/2)\cos \beta + \sin^2 \beta \cos \beta - \sin^2 \beta C_1) \sin DE_2 \\ + (\cos^2 \beta - \cos \beta C_1 - (1/2)) \sin \beta E_3)$$

where we take $D=C_1s+C_2$, where $C_1, C_2 \in \mathbb{R}$.

Proof:

With the Levi-Civita connection and Lie brackets can be easily computed as:

$$\nabla_T N = (1/2)((2C_1 - \cos \beta) \cos D, (2C_1 - \cos \beta) \sin D, -\sin \beta).$$

And, using the Frenet-Serret equation $\nabla_T N = -\kappa T + \tau B$, we have $B = (1/\tau)(\nabla_T N + \kappa T)$

$$B = (1/\tau) (((2C_1 - \cos \beta)/(2\kappa)) (\cos \beta - C_1) + \kappa) \sin \beta \cos DE_1 \\ + (((2C_1 - \cos \beta)/(2\kappa)) (\cos \beta - C_1) + \kappa) \sin \beta \sin DE_2 \\ + (\kappa \cos \beta - ((\sin^2 \beta)/(2\kappa)) (\cos \beta - C_1)) E_3.$$

By substituting κ and τ , we complete the proof.

Example:

Assume that $\alpha : I \rightarrow \text{Nil}_3$ be a unit speed non-geodesic general helix and its axis as parallel to the vector E_3 . Also $g_{\text{Nil}_3}(T, E_3) = T_3 = \cos(\pi/6)$, where $\beta = (\pi/6)$ is constant angle. Hence the explicit parametric equation of helix is

$$\alpha(s) = ((1/2) \sin s + 1) E_1 + (((-1)/2) \cos s + 1) E_2 \\ + ((1/16) \sin 2s - (1/2) \sin s + (((1+2\sqrt{3})/4)s)) E_3.$$

Hence; the first curvature is

$$\kappa = (1/2)((\sqrt{3}/2) - 1) = ((\sqrt{3}-2)/4)$$

and the second curvature is

$$\tau = (5-2\sqrt{3})^{1/2}/2.$$

Figure 1. The figure of Helix in Example.

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