On an Almost $C(\alpha)$ –Manifold Satisfying Certain Conditions on the Conircular Curvature Tensor

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Abstract: We classify almost $C(\alpha)$ –manifolds, which satisfy the curvature conditions $\tilde{Z}(\xi, X)R = 0$, $\tilde{Z}(\xi, X)\tilde{Z} = 0$, $\tilde{Z}(\xi, X)S = 0$ and $\tilde{Z}(\xi, X)P = 0$, where $\tilde{Z}$ is the concircular curvature tensor, P is the Weyl projective curvature tensor, S is the Ricci tensor and R is Riemannian curvature tensor of manifold.

Keywords: Almost $C(\alpha)$ –Manifold, Conircular Curvature Tensor, Projective Curvature Tensor

1. Introduction

An odd-dimensional Riemannian manifold (M, g) is said to be an almost co-Hermitian or almost contact metric manifold if there exist on M a (1,1) tensor field ϕ , a vector field ξ (called the structure vector field) and 1-form η such that

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi \quad (1.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (1.2)$$

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0 \quad (1.3)$$

for any vector fields X, Y on M .

The Sasaki form (or fundamental 2-form) Φ of an almost co-Hermitian manifold (M, g, ϕ, ξ, η) is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for all X, Y on M and this form satisfies $\eta \wedge \Phi^n \neq 0$. This means that every almost co-Hermitian manifold is orientable and (η, Φ) defines an almost cosymplectic structure on M . If this associated structure is cosymplectic $d\Phi = d\eta = 0$, then M is called an almost co-Kahler manifold. On the other hand, when $\Phi = d\eta$, the associated almost cosymplectic structure is a contact structure and is an almost Sasakian manifold. It is well known every contact manifold has an almost Sasakian structure.

The Nijenhuis tensor of type(1,1) –tensor field ϕ is type (1,2) $[\phi, \phi]$ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y]$$

$$-\phi[X, \phi Y] \quad (1.4)$$

where $[X, Y]$ is the Lie bracket of $X, Y \in \chi(M)$.

On the other hand, an almost co-complex structure is called integrable if $[\phi, \phi] = 0$ and normal $[\phi, \phi] + 2d\eta \otimes \xi = 0$. An integrable almost cocomplex structure is a cocomplex structure. A co-Kahler manifold (or normal cosymplectic manifold) is an integrable (or equivalently, a normal) almost co-Kahler manifold, while a Sasakian manifold is a normal almost contact metric manifold [3].

2. Preliminaries

In [2], $N(k)$ – contact metric manifolds satisfying $\tilde{Z}(\xi, X)\tilde{Z} = \tilde{Z}(\xi, X)R = R(\xi, X)\tilde{Z} = 0$ were classified.

In [1], $\tilde{Z}(\xi, X)R = R(\xi, X)R = \tilde{Z}(\xi, X)S = \tilde{Z}(\xi, X)C = 0$ on P –Sasakian manifolds and obtained the some results.

M. M. Tripathi and J. S. Kim gave a classification of (μ, κ) –manifolds satisfying the conditions $\tilde{Z}(\xi, X)S = 0$ [7].

Definition 2.1. An almost $C(\alpha)$ –manifold M is an almost co-Hermitian manifold such that the Riemann curvature tensor satisfies the following property, $\exists \alpha \in R$ such that

$$\begin{aligned} R(X, Y, W, Z) &= R(X, Y, \phi Z, \phi W) \\ &+ \alpha \{-g(X, Z)g(Y, W) + g(X, W)g(Y, Z) \\ &+ g(X, \phi Z)g(Y, \phi W) \\ &- g(X, \phi W)g(Y, \phi Z)\}. \end{aligned} \quad (2.1)$$

Moreover, if such a manifold has constant ϕ –sectional curvature equal to c , then its curvature tensor is given by

$$\begin{aligned}
R(X, Y)Z = & \left(\frac{c+3\alpha}{4} \right) \{g(Y, Z)X - g(X, Z)Y\} \\
& + \left(\frac{c-\alpha}{4} \right) \{g(X, \phi Y)\phi Z - g(Y, \phi Z)\phi X \\
& + 2g(X, \phi Y)\phi Z - \eta(X)\eta(Z)Y \\
& + \eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi \\
& - g(Y, Z)\eta(X)\xi\}, \quad (2.2)
\end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

A normal almost $C(\alpha)$ -manifold is said to be a $C(\alpha)$ -manifold. For example, Co-Kählerian, Sasakian and Kenmotsu manifolds are $C(0)$, $C(1)$ and $C(-1)$ -manifolds, respectively [3].

Theorem 2.1.

- (i) An almost co-Hermitian manifold M is α -Sasakian if and only if

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(X)Y\}, \quad (2.3)$$

for all $X, Y \in \chi(M)$.

- (ii) If M is α -Sasakian, then ξ is Killing vector field and

$$\nabla_X \xi = -\alpha\phi X \quad (2.4)$$

- (iii) An α -Sasakian manifold is a $C(\alpha^2)$ -manifold [3].

Theorem 2.2.

- (i) An almost co-Hermitian manifold M is an α -Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \quad (2.5)$$

for all $X, Y \in \chi(M)$.

- (iii) An α -Kenmotsu manifold is a $C(-\alpha^2)$ -manifold.

3. An Almost $C(\alpha)$ -Manifold Satisfying Certain Conditions on the Conircular Curvature Tensor

In this section, we will give the main results for this paper.

Let M be a $(2n+1)$ -dimensional almost $C(\alpha)$ -manifold and denote Riemannian curvature tensor of R , then we have from (2.2), for $X = \xi$

$$R(\xi, Y)Z = \alpha\{g(Y, Z)\xi - \eta(Z)Y\}. \quad (3.1)$$

In the same way, choosing $Z = \xi$ in (2.2), we have

$$R(X, Y)\xi = \alpha\{\eta(Y)X - \eta(X)Y\}. \quad (3.2)$$

In (3.2), choosing $Y = \xi$, we obtain

$$R(X, \xi)\xi = \alpha\{X - \eta(X)\xi\}. \quad (3.3)$$

Also from (3.2), we obtain

$$\eta(R(X, Y)Z) = \alpha\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}. \quad (3.4)$$

From (2.2), we can state

$$\begin{aligned}
R(X, e_i)e_i + R(X, \phi e_i)\phi e_i + R(X, \xi)\xi \\
= \left(\frac{c+3\alpha}{4} \right) \{nX - g(X, e_i)e_i + nX \\
- g(X, \phi e_i)\phi e_i + X - g(X, \xi)\xi\} \\
+ \left(\frac{c-\alpha}{4} \right) \{3g(X, \phi e_i)\phi e_i - 2n\eta(X)\xi \\
+ 3g(X, \phi^2 e_i)\phi^2 e_i \eta(X)\xi \\
- X\} \quad (3.5)
\end{aligned}$$

for $\{e_1, e_2, \dots, \phi e_1, \phi e_2, \dots, \phi e_n, \xi\}$ orthonormal basis of M . From (3.5), for $\in \chi(M)$, we obtain

$$\begin{aligned}
S(X, Y) \\
= \left(\frac{\alpha(3n-1) + c(n+1)}{2} \right) \\
+ \left(\frac{(\alpha-c)(n+1)}{2} \right) \eta(X)\eta(Y), \quad (3.6)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
QX \\
= \left(\frac{\alpha(3n-1) + c(n+1)}{2} \right) X \\
+ \left(\frac{(\alpha-c)(n+1)}{2} \right) \eta(X)\xi. \quad (3.7)
\end{aligned}$$

From (3.7) we can give the following corollary.

Corollary 3.1. An almost $C(\alpha)$ -manifold is always an η -Einstein manifold.

Also, from (3.6), we can easily see

$$\tau = n[\alpha(3n+1) + c(n+1)], \quad (3.8)$$

$$S(X, \xi) = 2n\alpha\eta(X) \quad (3.9)$$

and

$$Q\xi = 2n\alpha\xi. \quad (3.10)$$

Definition 3.1. Let (M, g) be an $(2n+1)$ -dimensional Riemannian manifold. Then the Weyl conircular curvature tensor $\tilde{Z}$ is defined by

$$\begin{aligned}
\tilde{Z}(X, Y)W = R(X, Y)W \\
- \frac{\tau}{2n(2n+1)} \{g(Y, W)X \\
- g(X, W)Y\} \quad (3.11)
\end{aligned}$$

for all $X, Y, W \in \chi(M)$, where τ is the scalar curvature of M [5].

In (3.11), choosing $X = \xi$, we obtain

$$\tilde{Z}(\xi, Y)W = \left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{g(Y, W)\xi - \eta(W)Y\} \quad (3.12)$$

Theorem 3.1. Let M be a $(2n+1)$ -dimensional almost $C(\alpha)$ -manifold. Then $\tilde{Z}(\xi, X)R = 0$ if and only if M either has c -sectional curvature or the scalar curvature $\tau = 2n\alpha(2n+1)$.

Proof: Suppose that $\tilde{Z}(\xi, X)R = 0$. Then from (3.11), we have

$$\begin{aligned} \tilde{Z}(\xi, X)R(U, W)Z &= \tilde{Z}(\xi, X)R(U, W)Z - R(\tilde{Z}(\xi, X)U, W)Z \\ &\quad - R(U, \tilde{Z}(\xi, X)W)R - R(U, W)\tilde{Z}(\xi, X)Z \\ &= 0. \end{aligned} \quad (3.13)$$

Using (3.12) in (3.13), we obtain

$$\begin{aligned} &= \left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{g(X, R(U, W)Z)\xi - \eta(R(U, W)Z)X \\ &\quad - g(X, U)R(\xi, W)Z + \eta(U)R(X, W)Z \\ &\quad - g(X, W)R(U, \xi)Z + \eta(W)R(U, X)Z \\ &\quad - g(X, Z)R(U, W)\xi + \eta(Z)R(U, W)X\} \\ &= 0. \end{aligned} \quad (3.14)$$

Using (3.2), (3.3) and putting $U = \xi$ in (3.14), we get

$$\left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{R(X, U)W - \alpha\{g(U, W)X - g(X, W)U\}\} = 0.$$

Therefore, manifold has either α -sectional curvature or $\tau = 2n\alpha(2n+1)$. This implies that $\alpha = c$.

Theorem 3.2. Let M be a $(2n+1)$ -dimensional almost $C(\alpha)$ -manifold. $\tilde{Z}(\xi, X)\tilde{Z} = 0$ if and only if M either has α -sectional curvature or the scalar curvature $\tau = 2n\alpha(2n+1)$.

Proof: Suppose that $\tilde{Z}(\xi, X)\tilde{Z} = 0$, we have

$$\begin{aligned} \tilde{Z}(\xi, X)\tilde{Z}(Y, U)W &= \tilde{Z}(\xi, X)\tilde{Z}(Y, U)W - \tilde{Z}(\tilde{Z}(\xi, X)Y, U)W \\ &\quad - \tilde{Z}(Y, \tilde{Z}(\xi, X)U)W - \tilde{Z}(Y, U)\tilde{Z}(\xi, X)W \\ &= 0. \end{aligned} \quad (3.15)$$

Using the equations (3.12) and (3.2), (3.3) in (3.15), we have

$$\begin{aligned} &= \tilde{Z}(\xi, X)R(Y, U)W - \tilde{Z}(R(\xi, X)Y, U)W \\ &\quad - \tilde{Z}(Y, R(\xi, X)U)W - \tilde{Z}(Y, U)R(\xi, X)W \\ &\quad + \left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{g(Y, W)\tilde{Z}(\xi, X)U \\ &\quad - g(U, W)\tilde{Z}(\xi, X)Y + g(X, Y)\tilde{Z}(\xi, U)W \\ &\quad - \eta(Y)\tilde{Z}(X, U)W + g(X, U)\tilde{Z}(Y, \xi)W \\ &\quad - \eta(U)\tilde{Z}(Y, X)W + g(X, W)\tilde{Z}(Y, U)\xi \\ &\quad - \eta(W)\tilde{Z}(Y, U)X\} \\ &= 0. \end{aligned} \quad (3.16)$$

Putting $Y = \xi$ in (3.16), we get

$$\left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{R(X, U)W - \alpha\{g(U, W)X - g(X, W)U\}\} = 0.$$

This tells us that M has either α -sectional curvature or the scalar curvature $\tau = 2n\alpha(2n+1)$.

The converse is obvious.

Theorem 3.3. Let M be a $(2n+1)$ -dimensional almost $C(\alpha)$ -manifold. Then $\tilde{Z}(\xi, X)S = 0$ if and only if M reduce an Einstein manifold.

Proof: We suppose that $\tilde{Z}(\xi, X)S = 0$, which implies that

$$S(\tilde{Z}(\xi, X)U, W) = +S(U, \tilde{Z}(\xi, X)W) = 0. \quad (3.17)$$

Using (3.12) in (3.17), we get

$$\begin{aligned} &S\left(R(\xi, X)U - \frac{\tau}{2n(2n+1)}\{g(X, U)\xi - \eta(U)X\}, W\right) \\ &+ S\left(U, R(\xi, X)W - \frac{\tau}{2n(2n+1)}\{g(X, W)\xi - \eta(W)X\}\right) \\ &= 0 \end{aligned} \quad (3.18)$$

Using (3.1), (3.9) in (3.18), we obtain

$$\begin{aligned} &= 2n\alpha^2 g(X, U)\eta(W) - \alpha\eta(U)S(X, W) \\ &\quad - \frac{\tau}{2n(2n+1)}(2n\alpha g(X, U)\eta(W) - \eta(U)S(X, W)) \\ &\quad + 2n\alpha^2 g(X, W)\eta(U) - \alpha\eta(W)S(X, U) \\ &\quad - \frac{\tau}{2n(2n+1)}(2n\alpha g(X, W)\eta(U) - \eta(W)S(X, U)) \\ &= 0. \end{aligned} \quad (3.19)$$

Putting $U = \xi$ in (3.19), we get

$$\begin{aligned} &S(X, W)\left\{\frac{\tau}{2n(2n+1)} - \alpha\right\} + g(X, W)\left\{2n\alpha^2 - \alpha\frac{\tau}{2n+1}\right\} \\ &= 0, \end{aligned}$$

under the condition $\alpha \neq \frac{\tau}{2n(2n+1)}$,

$$S(X, W) = 2n\alpha g(X, W).$$

Therefore, the manifold is Einstein manifold.

The converse is obvious.

If M is an Einstein manifold, the scalar curvature τ of M is

$$\tau = 2n\alpha(2n+1). \quad (3.20)$$

By corresponding (3.8) and (3.20) we obtain $\alpha = c$ which implies that M is of constant sectional curvature c .

Definition 3.2. Let (M, g) be a $(2n+1)$ -dimensional Riemannian manifold. Then Weyl projective curvature tensor P is defined by

$$\begin{aligned} P(X, Y)Z &= R(X, Y)Z \\ &\quad - \frac{1}{2n}\{S(Y, Z)X - S(X, Z)Y\}, \end{aligned} \quad (3.21)$$

where R is Riemannian curvature tensor and S is Ricci tensor [5].

Theorem 3.2. Let M be a $(2n+1)$ -dimensional almost $C(\alpha)$ -manifold. Then, $\tilde{Z}(\xi, X)P = 0$ if and only if M reduce an Einstein manifold.

Proof: Suppose that $\tilde{Z}(\xi, X)P = 0$. Then we have,

$$\begin{aligned} \tilde{Z}(\xi, X)P(Y, U)W &= \tilde{Z}(\xi, X)P(Y, U)W - P(\tilde{Z}(\xi, X)Y, U)W \\ &\quad - P(Y, \tilde{Z}(\xi, X)U)W - P(Y, U)\tilde{Z}(\xi, X)W \\ &= 0, \end{aligned} \quad (3.22)$$

for $X, Y, U, W \in \chi(M)$. Using (3.12) in (3.22), we get

$$\left(\alpha - \frac{\tau}{2n(2n+1)} \right) \{\alpha g(X, U)\eta(W)\xi - \alpha g(U, W)X +$$

$$\frac{1}{2n}\{S(U, W)X + S(X, W)\eta(U)\xi - S(U, X)\eta(W)\xi\} \\ - \alpha g(X, W)\eta(U)\xi + P(U, X)W\}$$

$$= 0. \quad (3.23)$$

Taking inner product both sides of (3.23) by $\xi \in \chi(M)$, we obtain

$$\left(\alpha - \frac{\tau}{2n(2n+1)}\right)\{\alpha g(X, U)\eta(W) - \alpha g(U, W)\eta(X) + \\ \frac{1}{2n}\{S(U, W)\eta(X) + S(X, W)\eta(U) - S(U, X)\eta(W)\} \\ - \alpha g(X, W)\eta(U) + \eta(P(U, X)W)\}$$

$$= 0. \quad (3.24)$$

Also making use of (3.21), we obtain

$$\eta(P(U, X)W) = \alpha\{g(X, W)\eta(U) - g(U, W)\eta(X)\} \\ - \frac{1}{2n}\{S(X, W)\eta(U) - S(U, W)\eta(X)\}. \quad (3.25)$$

Using (3.25) in (3.24) and choosing $W = \xi$, we have provided that $\alpha \neq \frac{\tau}{2n(2n+1)}$,

$$S(U, X) = 2n\alpha g(U, X).$$

So, the manifold is an Einstein manifold. The converse is obvious.

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