Multiparameter System of Operators with Two Parameters in Finite Dimensional Spaces

Rakhshanda Dzhabarzadeh, Afet Jabrailova

Department of functional analysis, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku

Email address: rakhshanda.dzhabarzade@rambler.ru (R. Dzhabarzadeh), afet.cebrayilova@mail.ru (A. Jabrailova)

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Abstract: The authors have proved the existence of the multiple basis on the eigen and associated elements of the two parameter system of operators in finite dimensional spaces. The proof uses the notion of the abstract analog of resultant of two operator pencils, acting, generally speaking, in different Hilbert spaces. In this paper necessary and sufficient conditions of the existence of multiple completeness of the eigen and associated vectors of two parameter system of operators in finite dimensional Hilbert space is given.

Keywords: Multiparameter, Spectrum, Operator, Space, Eigenvector

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding a solution of a system of ordinary differential equations, which are much easier to study. For example, a multivariable problems cause problems in quantum mechanics, diffraction theory, the theory of elastic shells, nuclear reactor calculations, stochastic diffusion processes, Brownian motion, boundary value problems for equations of elliptic-parabolic type, the Cauchy problem for ultra parabolic equations and etc.

Despite the urgency and prescription studies, spectral theory of multiparameter systems was not enough studied. The available results in this area until recently only dealt with systems of self-adjoint multiparameter system of operators, linearly dependent on the spectral parameters.

F.V. Atkinson [1] studied the results available for multiparameter symmetric differential systems, built multiparameter spectral theory of selfadjoint systems in finite-dimensional Euclidean spaces. Further, by taking the limit Atkinson generalized the results obtained for the multiparameter systems with self-adjoint operators in finite dimensional space on the case of the multiparameter system with selfadjoint compact operators in infinite-dimensional Hilbert spaces.

In the further Browne, Sleeman, Roch and other mathematicians built the spectral theory of selfadjoint multiparameter system in infinite-dimensional Hilbert spaces[2],[3]

In particular, in this work the existence of multiple basis on eigen and associated vectors of two parameter non-selfadjoint system of operators in finite-dimensional Hilbert spaces is proved. Definitions of the associated vectors, multiple completeness of eigen and associated vectors of two-parameter non-selfadjoint systems, are introduced in [4],[5].

At the proof of these results we essentially use the notion of the analog of two polynomial bundles [6],[7].

\[ A(\lambda, \mu) = A_0 + \lambda A_1 + \cdots + \lambda^n A_n \]
\[ B(\lambda, \mu) = B_0 + \lambda B_1 + \cdots + \lambda^n B_n \]  

when operator $A(\lambda, \mu)$ (correspondingly, $B(\lambda, \mu)$) acts in Hilbert space $H_1$ (correspondingly, $H_2$)

$H_1 \otimes H_2$ is the tensor product of the spaces $H_1$ and $H_2$.

Let
be the nonlinear multiparameter system in two parameters.

Operators $A_i$ (correspondingly, $B_i$) act in the the Euclidean space $H_i$ (correspondingly, $H_2$), $H_1 \otimes H_2$ is a tensor product of spaces $H_1$ and $H_2$.

For nonlinear algebraic system with two variables sufficient conditions of existence of solutions are given. The proof of these statements are received as a corollary of more common reviewing considered in this paper.

\[
A(\lambda, \mu) = A_0 + \lambda A_1 + \cdots + \lambda^n A_{m_n} + \mu A_{m_1} + \cdots + \mu^n A_{m_1+m_2} + \sum_{r<s<m_n, r<s<n} \lambda^r \mu^s A_{r,s}
\]

\[
B(\lambda, \mu) = B_0 + \lambda B_1 + \cdots + \lambda^n B_{m_n} + \mu B_{m_1} + \cdots + \mu^n B_{m_1+m_2} + \sum_{r<s<m_n, r<s<n} \lambda^r \mu^s B_{r,s}
\]

(2)

2. Preliminary Definitions and Remarks

Definition 1 [1]

$(\lambda, \mu)$ is an eigenvalue of the system (2) depending on two spectral parameters if there are such non-zero vectors $x \in H_1$, $y \in H_2$ that the equations

\[
A(x) x = (A_0 + \lambda A_1 + \cdots + \lambda^n A_{m_n} + \mu A_{m_1} + \cdots + \mu^n A_{m_1+m_2} + \sum_{r<s<m_n, r<s<n} \lambda^r \mu^s A_{r,s}) x = 0
\]

\[
B(x) y = (B_0 + \lambda B_1 + \cdots + \lambda^n B_{m_n} + \mu B_{m_1} + \cdots + \mu^n B_{m_1+m_2} + \sum_{r<s<m_n, r<s<n} \lambda^r \mu^s B_{r,s}) y = 0
\]

are satisfied. Decomposable element $x \otimes y$ is called an eigen vector of multiparameter system (2).

Definition 2 [1]

Operator $A_i^*$ (correspondingly $B_i^*$) is induced into the space $H = H_1 \otimes H_2$ by the operator $A_i$ (correspondingly $B_i$), acting in the space $H_i$ (correspondingly, $H_2$), if the following conditions satisfy: on decomposable tensor $x \otimes y$,

\[
A_i^*(x) y = (A_i x) \otimes y \quad \text{and} \quad B_i^*(x) y = x \otimes (B_i y)
\]

Definition 3 \cite{4, 5}

A tensor $z_{m,n}$ is called the $(m_i, n_i)$ – th associated vector to an eigenvector $z_{0,0} = x \otimes y$, if the following conditions (3) satisfy.

\[
\sum_{0 \leq j \leq k_i} \frac{1}{r_1! r_2!} \frac{\partial^{r_1+n} A_i^* (\lambda, \mu)}{\partial \lambda^{r_1}} \frac{\partial^{n} \lambda^s}{\partial \lambda^{r_2}} z_{k_1-j, k_2-j} = 0
\]

\[
\sum_{0 \leq j \leq k_i} \frac{1}{r_1! r_2!} \frac{\partial^{r_1+n} B_i^* (\lambda, \mu)}{\partial \lambda^{r_1}} \frac{\partial^{n} \lambda^s}{\partial \lambda^{r_2}} z_{k_1-j, k_2-j} = 0
\]

(3)

$(k_1, k_2)$ are arrangements from set of the whole nonnegative numbers on 2 with possible recurring and zero.

Definition 4

Systems of elements \( \{x_{k,1}\}_{k=1}^n \) of finite-dimensional space form multiple bases in this space if any $n$ elements $f_i, f_2, \ldots, f_n$ of space can be spread out in series $f_i = \sum_{k=1}^n c_{k,i} x_{k,j} (i = 1, \ldots, n)$ with coefficients, not depending on indices of the vectors $f_1, f_2, \ldots, f_n$. If system \( \{x_{k,1}\}_{k=1}^n \) coincides with the system of eigen and associated vectors of an operator, and systems \( \{x_{k,1}\}_{k=1}^n \) are constructed, proceeding from sequences on \( \{x_{k,1}\}_{k=1}^n \) to some rules speak about $n$-fold bases on system of eigen and associated vectors of an operator.

Definition 5 \cite{6, 7}

Let be two operator pencils depending on the same parameter and acting in, generally speaking, in various Hilbert spaces

\[
A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^n A_n,
\]

\[
B(\lambda) = B_0 + \lambda B_1 + \lambda^2 B_2 + \cdots + \lambda^n B_n
\]

Operator $Re s(A(\lambda), B(\lambda))$ is presented by the matrix

\[
\begin{pmatrix}
A_0 \otimes E_2 & A_1 \otimes E_2 & \cdots & A_n \otimes E_2 & 0 \\
0 & 0 & \cdots & \cdots & \cdots \\
E_1 \otimes B_0 & E_1 \otimes B_1 & \cdots & E_1 \otimes B_n & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
E_n \otimes B_0 & E_n \otimes B_1 & \cdots & E_n \otimes B_n & 0
\end{pmatrix}
\]

(4)

which acts in the $(H_1 \otimes H_2)^{n+m}$ - direct sum of $n+m$ copies of the space $H_1 \otimes H_2$. In a matrix (4), number of rows with operators $A_i$ is equal to leading degree of the parameter $\lambda$ in pencils $B(\lambda)$ and the number of rows with $B_i$ is equal to the leading degree of parameter $\lambda$ in $A(\lambda)$ Notion of abstract analog of resultant of two operator pencils is considered in the [7] for the case of the same leading degrees of the parameter in both pencils and in the [6], generally speaking, for different degrees of the parameters in the operator pencils.
3. Nonlinear Nonselfadjoint
Multiparameter System in Two Parameters

Consider the system (2). Operators \( A_i, B_i \) act in the finite dimensional Hilbert spaces \( H_i \) and \( H_j \), correspondingly. For study of the spectral properties of the (2) we use the notion of abstract analog of resultant of \( A(\lambda) \) and \( B(\lambda) \).

Fix the one of the parameters in (2). Let it is the parameter \( \lambda \) and \( \mu = \lambda_0 \). Then we have two operator pencils in one parameter \( \mu \).

Introduce the notations
\[
\tilde{A}(\lambda_0) = A_0 + \lambda_0 A_1 + \cdots + \lambda_0^n A_n, \\
\tilde{B}(\lambda_0) = B_0 + \lambda_0 B_1 + \cdots + \lambda_0^n B_n.
\]

Construct the resultant of operator pencils \( \tilde{A}(\lambda_0, \mu) \) and \( \tilde{B}(\lambda_0, \mu) \). (the parameter \( \lambda \) is fixed).

4. Spectral Properties of Two Parameter System

Theorem 2. Let all operators \( A_i \) (correspondingly, \( B_i \)) act in finite dimensional Hilbert space \( H_i \) (correspondingly)
and one of the following conditions:

a) \( k,n_2 + jm_2 > r,n_1 + sm_1 \), \( \ker(A_{\nu_\lambda} \otimes B_{\nu_\mu}) = \{0\} \), \( A_{\nu_\lambda} = A_{\nu_\lambda}^*; B_{\nu_\mu} = B_{\nu_\mu}^* \)

b) \( r,n_1 + sm_1 > k,n_2 + jm_2 \), operator \( A_{\nu_\lambda} \otimes B_{\nu_\mu}^* \) has inverse and selfadjoint

c) \( r,n_1 + sm_1 = k,n_2 + jm_2 \), operator \( A_{\nu_\lambda}^* \otimes B_{\nu_\mu}^* \) has inverse and selfadjoint.

Thus, the eigenvectors of the resultant of operators \( A_{\nu_\lambda} \) and \( B(\lambda,\mu) \) are fulfilled. Then the eigen and associated vectors of the system (2) form Max\((k,n_2 + jm_2; r,n_1 + sm_1)\) multiple basis in the tensor product of the spaces \( H_1 \odot H_2 \).

Proof of the Theorem 2

Using the results of the theorem 1 we have that under the conditions of the theorem 2 operator pencils in (1) have the common point of spectra (in the finite dimensional Hilbert space \( H_1 \odot H_2 \) the common point of spectra is the common eigenvalues of operators). It is not difficult to see, that under the conditions of the theorem 2 the \( \ker(\lambda, \mu) \neq 0 \). In fact, decomposition of the (6) is the operator pencil in the parameter \( \lambda \). The space \( H_1 \odot H_2 \) has finite dimension and in each case operator coefficient of the leading degree of the parameter \( \lambda \) in the obtained in the result of the decomposition of the resultant is selfadjoint operator. Consequently, eigen and associated vectors of the obtained pencil form the corresponding to leading degree of the parameter fold basis in the tensor product space \( H_1 \odot H_2 \).

Remark 1. Besides of these eigen and associated vectors are the first components of the elements of resultant of operator pencils \( A(\lambda, \mu) \) and \( B(\lambda, \mu) \), when the parameter \( \lambda = \lambda_0 \) is the first component of the eigenvalue of the system (2).

Remark 2. If the \( H_1 \odot H_2 \) is infinite dimensional Hilbert space then the system of eigen and associated vectors of the system (2) coincides with the system of eigen and associated vectors of the operator pencil obtaining as result of decomposition of the resultant of operators \( A(\lambda, \mu) \) and \( B(\lambda, \mu) \), when the parameter \( \lambda \) is fixed. Thus, the systems of eigen and associated vectors of (2) and the resulting expansion of the resultant simultaneously fold complete or fold up basis in the space \( H_1 \odot H_2 \) at the same time.

5. The Nonlinear Algebraic System in Two Variables

The following result for the algebraic system of equations

\[
\begin{align*}
a(x,y) &= a_0 + a_1 x + \ldots + a_{n_1} x^{n_1} + a_{n_1+1} y + \ldots + a_{n_1+n_2} y^{n_2} + \sum_{r,s,c} a_{r,s} x^r y^s = 0 \\
b(x,y) &= b_0 + b_1 x + \ldots + b_{m_1} x^{m_1} + b_{m_1+1} y + \ldots + b_{m_1+m_2} y^{m_2} + \sum_{r,s,c} b_{r,s} x^r y^s = 0
\end{align*}
\]

Proving in the work [8] is the corollary of the theorem 2.

Really, instead of the spaces \( H_1 \) and \( H_2 \) we adopt the space \( R_2 \), parameters \( \lambda, \mu \) play the role of variables \( x, y \).

Let \( \max(k,n_2 + jm_2; r,n_1 + sm_1) = \max(m,n_2, m,n_1) \) and one of the following conditions

a) \( \max(m,n_2, m,n_1) = m,n_2; a_{n_1} \neq 0, b_{m_1+n_2} \neq 0 \)

b) \( \max(m,n_2, m,n_1) = m,n_1 \)

c) \( m,n_2 = m,n_1; a_{n_1}^* b_{m_1}^* + (-1)^{n_2} a_{n_1}^* b_{m_1}^* = 0 \)

are satisfied then the algebraic system (7) has not less than \( \max(m,n_2, m,n_1) \) the solutions.

6. Conclusion

In this paper the conditions of existence of multiple basis of eigen and associated vectors of the system (2) in the finite dimensional tensor product space \( H_1 \odot H_2 \) are proved.

References


