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Multiparameter System of Operators with Two Parameters in Finite Dimensional Spaces

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Abstract: The authors have proved the existence of the multiple basis on the eigen and associated elements of the two parameter system of operators in finite dimensional spaces. The proof uses the notion of the abstract analog of resultant of two operator pencils, acting, generally speaking, in different Hilbert spaces. In this paper necessary and sufficient conditions of the existence of multiple completeness of the eigen and associated vectors of two parameter system of operators in finite dimensional Hilbert space is given.

Keywords: Multiparameter, Spectrum, Operator, Space, Eigenvector

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding a solution of a system of ordinary differential equations, which are much easier to study. For example, a multivariable problems cause problems in quantum mechanics, diffraction theory, the theory of elastic shells, nuclear reactor calculations, stochastic diffusion processes, Brownian motion, boundary value problems for equations of elliptic-parabolic type, the Cauchy problem for ultraparabolic equations and etc.

Despite the urgency and prescription studies, spectral theory of multiparameter systems was not enough studied. The available results in this area until recently only dealt with systems of self- adjoint multiparameter system of operators, linearly dependent on the spectral parameters.

F.V. Atkinson [1] studied the results available for multiparameter symmetric differential systems, built multiparameter spectral theory of selfadjoint systems in finite-dimensional Euclidean spaces. Further, by taking the limit Atkinson generalized the results obtained for the multiparameter systems with self-adjoint operators in finite dimensional space on the case of the multiparameter system with selfadjoint compact operators in infinite-dimensional Hilbert spaces.

In the further Browne, Sleeman, Roch and other mathematicians built the spectral theory of selfadjoint multiparameter system in infinite-dimensional Hilbert spaces[2],[3]

In particular, in this work the existence of multiple basis on eigen and associated vectors of two parameter non-selfadjoint system of operators in finite-dimentional Hilbert spaces is proved**.** Definitions of the associated vectors, multiple completeness of eigen and associated vectors of two-parameter non-selfadjoint systems, are introduced in [4],[5].

At the proof of these results we essentially use the notion of the analog of an resultant of two polynomial bundles [6],[7].

$$\begin{aligned} A(\lambda,\mu) &= A_0 + \lambda A_1 + \cdots + \lambda^n A_n \\ B(\lambda,\mu) &= B_0 + \lambda B_1 + \cdots + \lambda^m B_m \end{aligned} \tag{1}$$

when operator $A(\lambda,\mu)$ (correspondingly, $B(\lambda,\mu)$) acts in Hilbert space H_1(correspondingly, H_2)

$H_1 \otimes H_2$ is the tensor product of the spaces H_1 and H_2.

Let

$$\begin{aligned} A(\lambda,\mu) &= A_0+\lambda A_1+\cdots+\lambda^{m_1}A_{m_1}+\mu A_{m_1+1}+\cdots+\mu^{n_1}A_{m_1+n_1}+\sum\nolimits_{r<m_1,s<n_1}\lambda^r\mu^s A_{r,s} \\ B(\lambda,\mu) &= B_0+\lambda B_1+\cdots+\lambda^{m_{21}}B_{m_2}+\mu B_{m_2+1}+\cdots+\mu^{n_2}B_{m_2+n_2}+\sum\nolimits_{r<m_2,s<n_2}\lambda^r\mu^s B_{r,s} \end{aligned} \quad (2)$$

be the nonlinear multiparameter system in two parameters. Operators A_i (correspondingly, B_i) act in the the Euclidean space H_1 (correspondingly, H_2), $H_1\otimes H_2$ is a tensor product of spaces H_1 and H_2.

For nonlinear algebraic system with two variables sufficient conditions of existence of solutions are given. The proof of these statements are received as a corollary of more common reviewing considered in this paper.

2. Preliminary Definitions and Remarks

Definition 1[1]

(λ,μ) is an eigenvalue of the system (2) depending on two spectral parameters if there are such non-zero vectors $x\in H_1$, $y\in H_2$ that the equations

$$\begin{aligned} A(\lambda,\mu)x &= (A_0+\lambda A_1+\cdots+\lambda^{m_1}A_{m_1}+\mu A_{m_1+1}+\cdots+\mu^{n_1}A_{m_1+n_1}+\sum\nolimits_{r<m_1,s<n_1}\lambda^r\mu^s A_{r,s})x=0 \\ B(\lambda,\mu)y &= (B_0+\lambda B_1+\ldots+\lambda^{m_2}B_{m_2}+\mu B_{m_2+1}+\ldots+\mu_{n_2}B_{m_2+n_2}+\sum\lambda^r\mu^s B_{r,s})y=0 \end{aligned}$$

are satisfied. Decomposable element $x\otimes y$ is called an eigen vector of multiparameter system (2)

Definition2 [1]

Operator A_i^+ (correspondingly B_i^+) is induced into the space $H=H_1\otimes H_2$ by the operator A_i (correspondingly B_i), acting in the space H_1 (correspondingly, H_2), if the following conditions satisfy : on decomposable tensor $x\otimes y$,

$$A_i^+(x\otimes y)=(A_ix)\otimes y \text{ and } B_i^+(x\otimes y)=x\otimes(B_iy)$$

on other elements of the space $H=H_1\otimes H_2$ -on linearity and continuity

Definition3 ([4], [5])

A tensor z_{m_1,m_2} is called the $(m_1,m_2)-$ th associated vector to an eigenvector $z_{0,0}=x\otimes y$, if the following conditions (3) satisfy.

$$\begin{aligned} &\sum_{0\le r_i\le k_i}\frac{1}{r_1!r_2!}\frac{\partial^{r_1+r_2}A^+(\lambda,\varepsilon)}{\partial^{r_1}\lambda\partial^{r_2}}z_{k_1-r_1,k_2-r_2}=0 \\ &\sum_{0\le r_i\le k_i}\frac{1}{r_1!r_2!}\frac{\partial^{r_1+r_2}B^+(\lambda,\varepsilon)}{\partial^{r_1}\lambda\partial^{r_2}}z_{k_1-r_1,k_2-r_2}=0 \\ &k_s\le m_s;i=1,2;\ s=1,2. \end{aligned} \quad (3)$$

(k_1,k_2) are arrangements from set of the whole nonnegative numbers on 2 with possible recurring and zero.

Definition 4

Systems of elements $\{x_{k,1}\}_1^r$ $(k=1,2,...,n)$ of finite-dimensional space form multiple bases in this space if any n elements. $f_1,f_2,...,f_n$ of space can be spread out in series $f_i=\sum_{k=1}^{s}c_ix_{k,i}$ $(i=1,...,n)$ with coefficients, not depending on indices of the vectors $f_1,f_2,...,f_n$. If system $\{x_{k,1}\}_{k=1}^r$ coincides with the system of eigen and associated vectors of an operator, and systems $\{x_{k,i}\}_{k=1}^r$ are constructed, proceeding from sequences on $\{x_{k,1}\}_{k=1}^r$ to some rules speak about n-fold bases on system of eigen and associated vectors of an operator

Definition5 ([6], [7])

Let be two operator pencils depending on the same parameter and acting in, generally speaking, in various Hilbert spaces

$$\begin{aligned} A(\lambda) &= A_0+\lambda A_1+\lambda^2A_2+\ldots+\lambda^nA_n, \\ B(\lambda) &= B_0+\lambda B_1+\lambda^2B_2+\ldots+\lambda^mB_m \end{aligned}$$

Operator $\mathrm{Re}\,s(A(\lambda),B(\lambda))$ is presented by the matrix

$$\begin{pmatrix} A_0\otimes E_2 & A_1\otimes E_2 & \ldots & A_n\otimes E_2 & \ldots & 0 \\ . & . & \ldots & . & \ldots & . \\ 0 & 0 & \ldots A_0\otimes E_2 & A_1\otimes E_2 & \ldots & A_n\otimes E_2 \\ E_1\otimes B_0 & E_1\otimes B_1 & \ldots & E_1\otimes B_m & .. & 0 \\ . & . & \ldots & . & \ldots & . \\ . & . & \ldots E_1\otimes B_0 & E_1\otimes B_1 & \ldots. & E_1\otimes B_m \end{pmatrix} \quad (4)$$

which acts in the $(H_1\otimes H_2)^{n+m}$ - direct sum of $n+m$ copies of the space $H_1\otimes H_2$ In a matrix (4). number of rows with operators A_i is equal to leading degree of the parameter λ in pencils $B(\lambda)$ and the number of rows with B_i is equal to the leading degree of parameter λ in $A(\lambda)$ Notion of abstract analog of resultant of two operator pencils is considered in the [7] for the case of the same leading degrees of the parameter in both pencils and in the [6], generally speaking, for different degrees of the parameters in the operator pencils.

Theorem 1

Let operators $A(\lambda)$ (correspondingly, $B(\lambda)$) are bounded in corresponding Hilbert spaces, one of operators A_n or B_m has bounded inverse Then operator pencils $A(\lambda)$ and $B(\lambda)$ have a common point of spectra if and only if

$$Ker\,\mathrm{Re}\,s(A(\lambda), B(\lambda)) \neq \{\vartheta\} \qquad (5)$$

Remark1. If the Hilbert spaces H_1 and H_2 are the finite dimensional spaces then a common points of spectra of operator pencils $A(\lambda)$ and $B(\lambda)$ are their common eigenvalues. (see [6], [7].)

3. Nonlinear Nonselfadjoint Multiparameter System in Two Parameters

Consider the system (2). Operators A_i, B_i act in the finite dimensional Hilbert spaces H_1 and H_2 , correspondingly . For study of the spectral properties of the (2) we use the notion of abstract analog of resultant of $A(\lambda)$ and $B(\lambda)$.

Fix the one of the parameters in (2). Let it is the parameter λ and $\lambda = \lambda_0$. Then we have two operator pencils in one parameter μ.

$$A(\lambda_0,\mu)x = (A_0 + \lambda_0 A_1 + \cdots + \lambda_0^{m_1} A_{m_1} + \mu A_{m_1+1} + \cdots + \mu^{n_1} A_{m_1+n_1} + \sum_{r<m_1, s<n_1} \lambda_0^{\,r} \mu^s A_{r,s})x = 0$$

$$B(\lambda_0,\mu)y = (B_0 + \lambda_0 B_1 + \cdots + \lambda_0^{m_{21}} B_{m_2} + \mu B_{m_2+1} + \cdots + \mu^{n_2} B_{m_2+n_2} + \sum_{r<m_2, s<n_2} \lambda_0^{\,r} \mu^s B_{r,s})y = 0$$

Arrange the pencils on increasing of the degree of the parameter μ and denote the operator coefficients of the parameter μ in the degree s in the operator pencil $A(\lambda,\mu)$ though $A_s + \sum_{r<m_1} \lambda^r A_{r,s} = \tilde{A}_s(\lambda)$ and in the pencil $B(\lambda,\mu)$ operator coefficient of the parameter μ in degree s we denote though $B_s + \sum_{r<m_2} \lambda^r B_{r,s} = \tilde{B}_s(\lambda)$ **.**

These operators, induced into the space $H = H_1 \otimes H_2$, we denote $\tilde{A}_s^+(\lambda)$ and $\tilde{B}_s^+(\lambda)$, correspondingly

Introduce the notations

$$\tilde{A}(\lambda_0) = A_0 + \lambda_0 A_1 + \cdots + \lambda_0^{m_1} A_{m_1},$$

$$\tilde{B}(\lambda_0) = B_0 + \lambda_0 B_1 + \cdots + \lambda_0^{m_1} B_{m_1}$$

Construct the resultant of operator pencils $A(\lambda_0,\mu)$ and $B(\lambda_0,\mu)$. (the parameter λ is fixed).

$$\mathrm{Re}\,s(A(\lambda_0,\mu), B(\lambda_0,\mu)) = \begin{pmatrix} \tilde{A}^+(\lambda_0) & \tilde{A}^+_{m_1+1}(\lambda_0) & \ldots & \tilde{A}^+_{m_1+n_1} & 0 & \ldots & 0 \\ 0 & \tilde{A}^+(\lambda_0) & \ldots & \tilde{A}^+_{m_1+n_1-1}(\lambda_0) & \tilde{A}^+_{m_1+n_1} & \ldots & 0 \\ . & . & . & . & . & \ldots & . \\ 0 & 0 & \ldots\tilde{A}^+(\lambda_0) & \tilde{A}^+_{m_1+1}(\lambda_0) & \tilde{A}^+_{m_1+2}(\lambda_0) & \ldots & \tilde{A}^+_{m_1+n_1} \\ \tilde{B}^+(\lambda_0) & \tilde{B}^+_{m_2+1}(\lambda_0) & \ldots & \tilde{B}_{m_2+n_2} & 0 & \ldots & 0 \\ 0 & \tilde{B}^+(\lambda_0) & \ldots & \tilde{B}^+_{m_2+n_2-1}(\lambda_0) & \tilde{B}^+_{m_2+n} & \ldots & 0 \\ . & . & \ldots & . & . & \ldots & . \\ 0 & 0 & \ldots\tilde{B}(\lambda_0)^+ & \tilde{B}^+_{m_2+1}(\lambda_0) & \tilde{B}^2_{m_2+2}(\lambda_0) & \ldots & \tilde{B}^+_{m_2+n_2} \end{pmatrix} \qquad (6)$$

The number of rows with the operators $\tilde{A}_s(\lambda)$ is equal to the leading degree of the parameter μ in the operator pencil $B(\lambda_0,\mu)$ **,** that is n_1 ; number of rows with the operators $\tilde{B}_s^+(\lambda)$ is equal to the leading degree of the parameter μ in the pencil $A(\lambda,\mu)$, that is n_1.

Let $\max_{0\le i\le n_1}(k_i n_2 + i m_2) = k_j n_2 + j m_2$ realizes at $i = j,\ 0 \le j \le n_1$ and also and $\max_{0\le i\le n_1}(k_i n_2 + i m_2) = k_j n_2 + j m_2$ where $i = s,\ 0 \le s \le n_2$ **.** Let the highest degree of λ_0 in the operator coefficient at $\mu.^s$ $s = m_1+1,\ldots,m_1+n_1-1$ in the operator pencil $A(\lambda_0,\mu)$ be k_s**.** By analogy the highest degree of λ_0 in the operator coefficients at $\mu.^s$, $s = m_2+1,\ldots,m_2+n_2-1$ in the operator pencil $B(\lambda_0,\mu)$ be r_s **.** So the parameter λ_0 is fixed arbitrarily, further in the system we miss the index 0 of the parameter λ_0 .

4. Spectral Properties of Two Parameter System

Theorem2. Let all operators A_i (correspondingly, B_i) act in finite dimensional Hilbert space H_1 (correspondingly) and one of the following conditions:

a) $k_j n_2 + jm_2 > r_s n_1 + sm_1$, $Ker(A^{n_2}_{k_j,j} \otimes B^s_{m_2}) = \{\vartheta\}$, $A^{\bullet}_{k_j,j} = A_{k_j,j}; B_{m_2} = B^{\bullet}_{b_2}$

b) $r_s n_1 + sm_1 > k_j n_2 + jm_2$, operator $A^j_{m_1} \otimes B^{n_1}_{r_s,s}$ has inverse and selfadjoint

c) $r_s n_1 + sm_1 = k_j n_2 + jm_2$, operator $A^{n_2}_{k_j,j} \otimes B^s_{m_2} + (-1)^{n_1 n_2} A^j_{m_1} \otimes B^{n_1}_{r_s,s}$ has inverse and selfadjoint

are fulfilled. Then the eigen and associated vectors of the system (2) form $Max(k_j n_2 + jm_2, r_s n_1 + sm_1)$ multiple basis in the tensor product of the spaces $H_1 \otimes H_2$.

Proof of the Theorem 2

Using the results of the theorem1 we have that under the conditions of the theorem2 operator pencils in (1) have the common point of spectra (in the finite dimensional Hilbert space $H_1 \otimes H_2$ the common point of spectra is the common eigenvalues of operators). It is not difficult to see, that under the conditions of the theorem 2 the $\ker \operatorname{Re} s(A(\lambda,\mu), B(\lambda,\mu)) \neq 0$. In fact, decomposition of the (6) is the operator pencil in the parameter λ. The space $H_1 \otimes H_2$ has a finite dimension and in each case operator coefficient of the leading degree of the parameter λ in the obtained in the result of the decomposition of the resultant is selfadjoint operator. Consequently, eigen and associated vectors of the obtained pencil form the corresponding to leading degree of the parameter fold basis in the tensor product space $H_1 \otimes H_2$. Earlier in the [5],[6] is proved that the system of eigen and associated vectors of obtained pencil, depending on parameter λ ,coincides with the system of eigen and associated vectors of the system (2).

Remark1.Besides of these eigen and associated vectors are the first components of the elements of resultant of operator pencils $A(\lambda,\mu)$ and $B(\lambda,\mu)$., when the parameter $\lambda = \lambda_0$ is the first component of the eigenvalue of the system (2).

Remark 2. If the $H_1 \otimes H_2$ is infinite dimensional Hilbert space then the system of eigen and associated vectors of the system (2) coincides with the system of eigen and associated vectors of the operator pencil obtaining as result of decomposition of the resultant of operators $A(\lambda,\mu)$ and $B(\lambda,\mu)$. ,when the parameter λ is fixed. Thus, the systems of eigen and associated and vectors of (2) and the resulting expansion of the resultant simultaneously fold complete or fold up basis in the space $H_1 \otimes H_2$ at the same time.

5. The Nonlinear Algebraic System in Two Variables

The following result for the algebraic system of equations

$$a(x,y) = a_0 + a_1 x + ... + a_{m_1} x^{m_1} + a_{m_1+1} y + ... + a_{m_1+n_1} y^{n_1} + \sum\nolimits_{r<m_1, s<n_1} a_{r,s} x^r y^s = 0$$

$$b(x,y) = b_0 + b_1 x + ... + b_{m_{21}} x^{m_2} + b_{m_2+1} y + ... + b_{m_1+n_1} y^{n_1} + \sum\nolimits_{r<m_1, s<n_1} a_{r,s} x^r y^s = 0 \quad (7)$$

Proving in the work [8]is the corollary of the theorem2.

Really. instead of the spaces H_1 and H_2 we adopt the space R_2, parameters λ, μ play the role of variables x, y.

Let $Max(k_j n_2 + jm_2, r_s n_1 + sm_1) = Max(m_1 n_2, m_2 n_1)$ and one of the following conditions

a) $\max(m_1 n_2, m_2 n_1) = m_1 n_2$, $a_{m_1} \neq 0$, $b_{m_2+n_2} \neq 0$

b) $\max(m_1 n_2, m_2 n_1) = m_2 n_1$,

c) $m_1 n_2 = m_2 n_1$, $a^{n_2}_{m_1+n_1} b^{n_1}_{m_2} + (-1)^{n_1 n_2} a^{n_2}_{m_1} b^{n_1}_{m_2+n_2} \neq 0$,

are satisfied then the algebraic system (7) has not less than $\max(m_1 n_2, m_2 n_1)$ the solutions.

6. Conclusion

In this paper the conditions of existence of multiple basis of eigen and associated vectors of the system (2) in the finite dimensional tensor product space $H_1 \otimes H_2$ are proved.

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