

Spectral Problems of Two-Parameter System of Operators

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Abstract: The author has proved the existence of the multiple basis of the eigen and associated vectors of the two parameter system of operators in Hilbert spaces. The proof essentially uses the theorem of the existence of multiple basis of operator bundles and the notion of the abstract analog of resultant of two operator pencils, acting, generally speaking, in different Hilbert spaces. Considerable non-selfadjoint two parameter systems depend on both parameters in a complicated manner.

Keywords: Multiparameter, Spectrum, Operator, Space, Eigenvector

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding a solution of a system of ordinary differential equations, which are much easier to study.

F.V. Atkinson [1] studied the fragmentary results for multiparameter symmetric differential systems, built multiparameter spectral theory of selfadjoint systems in finite-dimensional Euclidean spaces. Further, by taking the limit, Atkinson generalized the results obtained for the multiparameter systems with self-adjoint operators in finite dimensional space on the case of the multiparameter system with compact selfadjoint operators in Hilbert spaces.

Later Browne, Sleeman, Roch and other mathematicians built the spectral theory of selfadjoint multiparameter system in infinite-dimensional Hilbert spaces [2],[3].

In this work the existence of multiple basis on eigen and associated vectors of two parameter non-selfadjoint system of operators in Hilbert spaces is proved. Definitions of the associated vectors, multiple completeness of eigen and associated vectors of two-parameter non-selfadjoint systems, are introduced in [4],[6].

2. Preliminary Definitions and Remarks

Let

$$\begin{aligned} A(\lambda, \mu) &= A_0 + \lambda A_1 + \dots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \dots + \mu^{n_1} A_{m_1+n_1} + \sum_{r < m_1, s < n_1} \lambda^r \mu^s A_{r,s} \\ B(\lambda, \mu) &= B_0 + \lambda B_1 + \dots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \dots + \mu^{n_2} B_{m_2+n_2} + \sum_{r < m_2, s < n_2} \lambda^r \mu^s B_{r,s} \end{aligned} \quad (1)$$

be the non-linear multiparameter system in two parameters.

Operators A_i (correspondingly, B_i) act in the Hilbert space H_1 (correspondingly, H_2), $H_1 \otimes H_2$ is a tensor product of spaces H_1 and H_2 .

For nonlinear algebraic system with two variables sufficient conditions of existence of solutions are given. The proof of these statements are received as a corollary of more

common reviewing considered in this paper.

Definition 1.

(λ, μ) is an eigenvalue of the system (1) depending on two spectral parameters if there are such non-zero pair vectors $x \in H_1$, $y \in H_2$ that the equations (2)

$$\begin{aligned}
 A(\lambda, \mu)x &= (A_0 + \lambda A_1 + \dots + \lambda^{m_1} A_{m_1} + \mu A_{m_1+1} + \dots + \mu^{n_1} A_{m_1+n_1} + \sum_{r < m_1, s < n_1} \lambda^r \mu^s A_{r,s})x = 0 \\
 B(\lambda, \mu)y &= (B_0 + \lambda B_1 + \dots + \lambda^{m_2} B_{m_2} + \mu B_{m_2+1} + \dots + \mu^{n_2} B_{m_2+n_2} + \sum \lambda^r \mu^s B_{r,s})y = 0
 \end{aligned}
 \tag{2}$$

are satisfied .Decomposable element $x \otimes y$ is called an eigen vector of multiparameter system (1).

Definition 2 [1]

Operator A_i^+ (correspondingly, B_i^+) is induced into the space $H = H_1 \otimes H_2$ by the operator A_i (correspondingly B_i), acting in the space H_1 (correspondingly, H_2), if the following conditions satisfy: on decomposable tensor $x \otimes y$,

$A_i^+(x \otimes y) = (A_i x) \otimes y$ and $B_i^+(x \otimes y) = x \otimes (B_i y)$, on other elements of the space $H = H_1 \otimes H_2$ -on linearity and continuity

Definition 3 ([4], [5])

A tensor z_{m_1, m_2} is named (m_1, m_2) - th the associated vector to an eigenvector $z_{0,0} = x \otimes y$, if the following conditions (3) are satisfied

$$\sum_{0 \leq r_1 \leq k_1} \frac{1}{r_1! r_2!} \frac{\partial^{r_1+r_2} A^+(\lambda, \epsilon)}{\partial \lambda^{r_1} \partial \epsilon^{r_2}} z_{k_1-r_1, k_2-r_2} = 0, \quad \sum_{0 \leq r_1 \leq k_1} \frac{1}{r_1! r_2!} \frac{\partial^{r_1+r_2} B^+(\lambda, \epsilon)}{\partial \lambda^{r_1} \partial \epsilon^{r_2}} z_{k_1-r_1, k_2-r_2} = 0, \quad k_s \leq m_s; i = 1, 2; s = 1, 2. \tag{3}$$

(k_1, k_2) - arrangements from set of the whole nonnegative numbers on 2 with possible recurring and zero.

Under canonical system of e.a.vectors (this definition is generalization of the definition of canonical system, introduced in [10] in the case polynomial pencils in one parameter) for (λ, μ) we understand the system

$$\left\{ z_{i_1, i_2}^{(k)} \right\}_{0 \leq i_r \leq m_r}, \quad r = 1, 2, \tag{4}$$

possessing the following properties: elements $z_{0,0}^{(k)}$ form basis of a eigen subspace $M(\lambda^0)$; there is $z_{0,0}^{(1)}$ eigenvector which multiplicity reaches a possible maxima $p_1 + 1$; there is $z_{0,0}^{(k)}$ eigenvector which is not expressing linearly through $z_{0,0}^{(1)}, \dots, z_{0,0}^{(k-1)}$ which sum of multiplicities reaches a possible maxima $p_k + 1$ Let's designate through $M((\lambda^0, \mu^0))$ a subspace spanned by eigen and associated vectors of the system (1), corresponding to an eigenvalue λ^0 .

Linearly-independent elements form a chain of a set $(z_{i_1, i_2}) \subset H$ eigen and associated (e.a) vectors of (1). The multiplicity of eigenvalue designates the greatest number of associated vectors to $z_{0,0,0}$ a plus 1.

The sum $p_1 + p_2 + \dots + p_s + s$ is a multiplicity of an eigen value (λ^0, μ^0) .

Elements in (4) form a chain of eigen and associated vectors for every fixed value $k, k = 1, 2, \dots, s$.

Definition 4.

Systems of elements $\{x_{k,i}\}_1^r (k = 1, 2, \dots, n)$ of Hilbert space form multiple bases in this space if any n elements f_1, f_2, \dots, f_n of space can be spread out in series

$$f_i = \sum_{k=1}^s c_i x_{k,i} \quad (i = 1, \dots, n)$$

with coefficients, not dependent on an index of vectors f_1, f_2, \dots, f_n . If system $\{x_{k,i}\}_{k=1}^r$ coincides with the system of eigen and associated vectors of an

operator, and systems $\{x_{k,i}\}_{k=1}^r$ are constructed, proceeding

from sequences on $\{x_{k,i}\}_{k=1}^r$ according to some rules then speak about n -multiple basis on system of eigen and associated vectors of an operator

Definition5([6],[7])

Let be two operator pencils depending on the same parameter and acting in, generally speaking, various Hilbert spaces

$$\begin{aligned}
 A(\lambda) &= A_0 + \lambda A_1 + \lambda^2 A_2 + \dots + \lambda^n A_n, \\
 B(\lambda) &= B_0 + \lambda B_1 + \lambda^2 B_2 + \dots + \lambda^m B_m
 \end{aligned}$$

Operator $Res(A(\lambda), B(\lambda))$ is presented by the matrix

$$\begin{pmatrix}
 A_0 \otimes E_2 & A_1 \otimes E_2 & \dots & A_n \otimes E_2 & \dots & 0 \\
 \cdot & \cdot & \dots & \cdot & \dots & \cdot \\
 0 & 0 & \dots A_0 \otimes E_2 & A_1 \otimes E_2 & \dots & A_n \otimes E_2 \\
 E_1 \otimes B_0 & E_1 \otimes B_1 & \dots & E_1 \otimes B_m & \dots & 0 \\
 \cdot & \cdot & \dots & \cdot & \dots & \cdot \\
 \cdot & \cdot & \dots E_1 \otimes B_0 & E_1 \otimes B_1 & \dots & E_1 \otimes B_m
 \end{pmatrix} \tag{5}$$

which acts in the $(H_1 \otimes H_2)^{n+m}$ - direct sum of $n + m$ copies of the space $H_1 \otimes H_2$. In a matrix (5) number of rows with operators A_i is equal to leading degree of the parameter λ in pencils $B(\lambda)$ and the number of rows with B_i is equal to the leading degree of parameter λ in $A(\lambda)$. Notion of abstract analog of resultant of two operator pencils is considered in the [8] for the case of the same leading degree of the parameter in both pencils and in the [7] for, generally speaking, different leading degrees of the parameters in the operator pencils.

Theorem 1.

Let all the operators be bounded in corresponding Hilbert spaces, one of operators A_n or B_m has bounded inverse Then operator pencils $A(\lambda)$ and $B(\lambda)$ have a common point of spectra if and only if

$$\text{Ker Re } s(A(\lambda), B(\lambda)) \neq \{\varnothing\} \tag{6}$$

Remark 1. If the Hilbert spaces H_1 and H_2 are the finite dimensional spaces then a common points of spectra of operator pencils $A(\lambda)$ and $B(\lambda)$ are their common eigenvalues.(see [6], [7].)

3. Multiple Basis of Eigen and Associated Vectors of Multiparameter System with Two Parameters

Consider the system (2). Operators A_i, B_i act in the Hilbert spaces H_1 and H_2 , correspondingly. For study of the spectral properties of the system (1) we use the notion of abstract analog of resultant of $A(\lambda)$ and $B(\lambda)$. Fix the one of the parameters in (1). Let it is the parameter λ and $\lambda = \lambda_0$. Then we have two operator pencils in one parameter μ .

$$\begin{aligned} A(\lambda_0, \mu)x &= (A_0 + \lambda_0 A_1 + \dots + \lambda_0^{m_1} A_{m_1} + \mu A_{m_1+1} + \dots + \mu^{n_1} A_{m_1+n_1} + \sum_{r < m_1, s < n_1} \lambda_0^r \mu^s A_{r,s})x = 0 \\ B(\lambda_0, \mu)y &= (B_0 + \lambda_0 B_1 + \dots + \lambda_0^{m_2} B_{m_2} + \mu B_{m_2+1} + \dots + \mu^{n_2} B_{m_2+n_2} + \sum_{r < m_2, s < n_2} \lambda_0^r \mu^s B_{r,s})y = 0 \end{aligned} \tag{7}$$

Arrange the pencils on increasing of the degree of the parameter μ and denote the operator coefficients of the parameter μ in the degree s in the operator pencil $A(\lambda, \mu)$ though $A_s + \sum_{r < m_1} \lambda^r A_{r,s} = \tilde{A}_s(\lambda)$ and in the pencil $B(\lambda, \mu)$ operator coefficient of the parameter μ in degree s we

denote though $B_s + \sum_{r < m_2} \lambda^r B_{r,s} = \tilde{B}_s(\lambda)$. Operators, induced into the space $H = H_1 \otimes H_2$, we denote $\tilde{A}_s^+(\lambda)$ and $\tilde{B}_s^+(\lambda)$, correspondingly

Introduce the notations

$$\tilde{A}(\lambda_0) = A_0 + \lambda_0 A_1 + \dots + \lambda_0^{m_1} A_{m_1}, \quad \tilde{B}(\lambda_0) = B_0 + \lambda_0 B_1 + \dots + \lambda_0^{m_2} B_{m_2}$$

Construct the resultant of operator pencils $A(\lambda_0, \mu)$ and $B(\lambda_0, \mu)$ (the parameter λ is fixed).

$$\text{Re } s(A(\lambda_0, \mu), B(\lambda_0, \mu)) = \begin{pmatrix} \tilde{A}^+(\lambda_0) & \tilde{A}_{m_1+1}^+(\lambda_0) & \dots & \tilde{A}_{m_1+n_1}^+(\lambda_0) & 0 & \dots & 0 \\ 0 & \tilde{A}^+(\lambda_0) & \dots & \tilde{A}_{m_1+n_1-1}^+(\lambda_0) & \tilde{A}_{m_1+n_1}^+(\lambda_0) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \tilde{A}^+(\lambda_0) & \tilde{A}_{m_1+1}^+(\lambda_0) & \tilde{A}_{m_1+2}^+(\lambda_0) & \dots & \tilde{A}_{m_1+n_1}^+(\lambda_0) \\ \tilde{B}^+(\lambda_0) & \tilde{B}_{m_2+1}^+(\lambda_0) & \dots & \tilde{B}_{m_2+n_2}^+(\lambda_0) & 0 & \dots & 0 \\ 0 & \tilde{B}^+(\lambda_0) & \dots & \tilde{B}_{m_2+n_2-1}^+(\lambda_0) & \tilde{B}_{m_2+n_2}^+(\lambda_0) & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \tilde{B}(\lambda_0)^+ & \tilde{B}_{m_2+1}^+(\lambda_0) & \tilde{B}_{m_2+2}^+(\lambda_0) & \dots & \tilde{B}_{m_2+n_2}^+(\lambda_0) \end{pmatrix} \tag{8}$$

The number of rows with the operators $\tilde{A}_s(\lambda)$ is equal to the leading degree of the parameter μ in the operator pencil $B(\lambda_0, \mu)$, that is n_1 ; number of rows with the operators $\tilde{B}_s^+(\lambda)$ is equal to the leading degree of the parameter μ in the pencil $A(\lambda, \mu)$, that is n_1 .

Let $\max_{0 \leq i \leq n_1} (k_i n_2 + i m_2) = k_j n_2 + j m_2$ realizes at $i = j, 0 \leq j \leq n_1$ where $i = s, 0 \leq s \leq n_2$. Let the greatest degree of λ_0 in the operator coefficient of μ^s ($s = m_1 + 1, \dots, m_1 + n_1 - 1$) in the operator pencil $A(\lambda_0, \mu)$ be k_s . By analogy the greatest degree of λ_0 in the operator coefficients at $\mu^s, s = m_2 + 1, \dots, m_2 + n_2 - 1$ in the operator pencil $B(\lambda_0, \mu)$ be r_s . So the parameter λ_0 is fixed arbitrarily,

further, in the system we miss the index 0 of the parameter λ_0 .

Let K denotes the term of the expansion of the resultant (7) free from parameter λ and let K be a selfadjoint operator with the $\text{ker } K = \{\varnothing\}$.

It is known that the self-adjoint completely continuous operator has a discrete spectrum, i.e. the spectrum of the operator contains only eigenvalues of finite multiplicity. If the series of modules of eigenvalues of completely continuous operator in some positive degree converges then this operator belongs to class σ_p .

Arrange the bundle produced by the decomposition of the resultant (7), to increase the power of the parameter λ and denote the operator coefficients of λ^s though T_i^+ .

Let's remind, that under an eigen value of polynomial bundle $L(\lambda) = A_0 + \lambda A_1 B + \dots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^n B^n$ is

understood such non-zero vector x_0 , that equality $L(\lambda_0)x_0 = x_0$ is satisfied. m -th associated vector x_k to an eigenvector x_0 is the vector, satisfying to conditions

$$x_k = L(\lambda_0)x_k + \frac{d}{d\lambda}L(\lambda_0)x_{k-1} + \dots + \frac{1}{k!} \frac{d^k}{d\lambda^k}L(\lambda_0)x_0; k = 1, \dots, m$$

Let $\{x_{k,0}\}$ be system of eigen and associated vectors of a bundle $L(\lambda)$.

On system of eigen and associated vectors of polynomial bundle $L(\lambda)$ derivative systems of vectors are constructed by rules

$$x_{i,j} = \left\{ \frac{d^j}{dt^j} e^{\lambda_0 t} (x_{i,0} + x_{i-1,0} \frac{t}{1!} + \dots + x_{0,0} \frac{t^i}{i!}) \right\}_{t=0}, i = 1, \dots$$

4. Spectral Properties of Two Parameter System

For the study of the spectral problems of two parameter system (1) we use the following result

In a separable Hilbert space we consider an operator pencil

$$L(\lambda) = A_0 + \lambda A_1 B + \lambda^2 A_2 B^2 + \dots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^n B^n$$

when the operators A_i, B are completely continuous in the space $H, \lambda \in C$

Let $B (Ker B = \{0\})$ be the normal operator, characteristic values of which lie on a finite number of rays, emanating from the origin. Denote an increasing sequence of modules characteristic numbers of the operator B , taking into account their multiplicity.

Theorem 2.[5] Suppose that one of the following two conditions:

a) $0 < p < 1, \lim_{k \rightarrow \infty} k \mu_k^{-p} = 0$, operators

$A_i B^{-p} (i = 0, 1, 2, \dots, n-1)$ are bounded

b) $0 < p < 1, \lim_{k \rightarrow \infty} k \mu_k^{-p} < \infty, A_i B^{-p} (i = 0, 1, 2, \dots, n-1)$

completely continuous operators are satisfied.

Then different eigenvalues $\{\lambda_n\}_1^\infty$ of $L(\lambda)$ can be arranged in a sequence such that for some increasing sequence $\{n_j\}_1^\infty (n_0 = 0)$ of positive integers, and for all n elements f_0, f_1, \dots, f_{n-1} of the space H , satisfying the conditions $f_i \in R(B^i)$, the expansions

$$f_j = \sum_{i=0}^{\infty} \left(\sum_{k=n_{i-1}}^{n_i-1} \sum_{s=0}^{m_{k_s}-1} \sum_{r=0}^{p_{k_s}-1} a_{sr}^{(k)} Z_{jsr}^{(k)} \right)$$

where p_{s_k} the number of different eigenvectors of multiplicity m_{kr} corresponding to eigenvalue λ_k with eigenvector $L(\lambda)$

$$Z_{jsr}^{(k)} = \left[\frac{d^j}{dt^j} e^{\lambda_k t} \left(Z_{0sr}^{(k)} + Z_{0,s-1,r}^{(k)} \frac{t}{1!} + \dots + Z_{00r}^{(k)} \frac{t^s}{s!} \right) \right]_{t=0}$$

$\{Z_{0sk}^{(k)}\}$ -canonical system, corresponding eigen values λ_k are satisfied.

Arrange the bundle produced by the decomposition of the resultant (8), to increase the power of the parameter λ and denote the operator coefficients of λ^s through T_s^+ and the modules of characteristic numbers of operators S, S_1, S_2 through $\mu_s, \mu_{s_1}, \mu_{s_2}$, correspondingly. S, S_1, S_2 are defined below.

Theorem3. Let be

$$k_j n_2 + j m_2 > r_s n_1 + s m_1, S = K^{-\frac{1}{2}} A_{k_j, j}^{n_2} \otimes B_{m_2}^s K^{-\frac{1}{2}} \in \sigma_p$$
 is

selfadjoint completely continuous operator and one of following conditions are fulfilled

a) $0 < p < 1, \lim_{k \rightarrow \infty} k (\mu_s)_k^{-p} = 0$ operators $(T_i^*) S^{-\frac{ip}{k_j n_2 + j m_2}}$,

$i = 1, \dots, k_j n_2 + j m_2, Ker S = \{0\}$ are bounded,

b) $0 < p < 1, \lim_{k \rightarrow \infty} k (\mu_s)_k^{-p} < \infty$ operators $(T_i^*) S^{-\frac{ip}{k_j n_2 + j m_2}}$

are the completely continuous

Then characteristic values $\{\mu_n\}_1^\infty$ of $Re s(A(\mu(\lambda)), B(\mu(\lambda)))$ can be arranged in a sequence such that for some increasing sequence $\{n_j\}_1^\infty (n_0 = 0)$ of positive integers, and for all n elements f_0, f_1, \dots, f_{n-1} of the space H , satisfying the conditions $f_i \in R(S^i)$, we have the expansions

$$f_j = \sum_{i=0}^{\infty} \left(\sum_{k=n_{i-1}}^{n_i-1} \sum_{s=0}^{m_{k_s}-1} \sum_{r=0}^{p_{k_s}-1} a_{sr}^{(k)} Z_{jsr}^{(k)} \right)$$

where p_{s_k} the number of different eigenvectors of multiplicity

m_{kr} corresponding to eigenvalue $\lambda_k = \frac{1}{\mu_k}$.

Proof of the Theorem3.

If any of the conditions a), b) is satisfied then conditions of Theorem 2 are fulfilled, therefore, multiple basis of eigen and associated vectors of the operator bundle $Re s(A(\lambda, \mu), B(\lambda, \mu))$ exists. Studies conducted in the papers [4],[5] and [6] show that the eigenvectors of the bundle (8)(decompositions of resultant) are either eigenvectors or associated vectors of the system (1) in the definition of which there is no differentiation on λ . The associated vectors of the bundle (8) (decomposition of the resultant (7)) are also eigenvectors of the system (1). The results of the theorem1 mean if the conditions a), b) of the theorem 2 hold operator pencils in (1) have the common point of spectra (in the finite dimensional Hilbert space $H_1 \otimes H_2$ the common point of spectra is the common eigenvalues). Really, each condition a), b) of the theorem 2 means the $ker Re s(A(\lambda, \mu), B(\lambda, \mu)) \neq 0$. Consider the decomposition of the resultant (8) which is the operator

pencil in the parameter λ . Consequently, eigen and associated vectors of this operator pencil form the multiple basis in the tensor product space $H_1 \otimes H_2$ and the multiplicity of this basis coincides with the greatest degree of the parameter λ in the operator pencil $\text{Res}(A(\lambda, \mu), B(\lambda, \mu))$ (the parameter λ is fixed). Earlier in the [4],[5],[6] it is proved that the system of eigen and associated vectors of obtained pencil, depending on parameter λ and coincides with the system of eigen and associated vectors of the system (1). Therefore, the eigen and associated vectors of the two parameter system (1) form the $\text{Max}(k_j n_2 + j m_2, r_s n_1 + s m_1)$ multiple basis in $H_1 \otimes H_2$. Thus, the systems of eigen and associated vectors of (1) and the resulting expansion of the resultant $\text{Res}(A(\lambda, \mu), B(\lambda, \mu))$ simultaneously $\text{Max}(k_j n_2 + j m_2, r_s n_1 + s m_1)$ multiple basis with brackets.

Theorem 4

Let be $r_s n_1 + s m_1 > k_j n_2 + j m_2$, operator $S_1 = K^{-\frac{1}{2}} A_{m_1}^s \otimes B_{r_s, s}^{n_1} K^{-\frac{1}{2}} \in \sigma_p$ has inverse, selfadjoint completely continuous and one of following conditions

- a) $0 < p < 1, \lim_{k \rightarrow \infty} k(\mu_{s_1})_k^{-p} = 0$
 $\tilde{T}_i S_1^{-\frac{ip}{r_s n_1 + s m_1}} (i = 1, 2, \dots, r_s n_1 + s m_1)$ are bounded operators
- b) $0 < p < 1, \lim_{k \rightarrow \infty} k(\mu_{s_1})_k^{-p} < 0$
 $\tilde{T}_i S_1^{-\frac{ip}{r_s n_1 + s m_1}} (i = 1, 2, \dots, r_s n_1 + s m_1)$ are completely complete

operators is satisfied. Then the assertions of theorem 3 is fulfilled.

Theorem 5. Let be $r_s n_1 + s m_1 = k_j n_2 + j m_2$, operator

$S_2 = K^{-\frac{1}{2}} (A_{k_j, j}^{n_2} \otimes B_{m_2}^s + (-1)^{n_1 n_2} A_{m_1}^j \otimes B_{r_s, s}^{n_1}) K^{-\frac{1}{2}} \in \sigma_p$ has inverse, selfadjoint completely continuous and one of following conditions

- c₁) $0 < p < 1, \lim_{k \rightarrow \infty} k(\mu_{s_1})_k^{-p} = 0$
 $\tilde{T}_i S_2^{-\frac{ip}{r_s n_1 + s m_1}} (i = 1, 2, \dots, r_s n_1 + s m_1)$ are bounded
- c₂) $0 < p < 1, \lim_{k \rightarrow \infty} k(\mu_{s_1})_k^{-p} = 0$
 $\tilde{T}_i S_2^{-\frac{ip}{r_s n_1 + s m_1}} (i = 1, 2, \dots, r_s n_1 + s m_1)$ completely continuous

operators is satisfied. Then the statements of the theorem 3 is fulfilled.

The next theorem is the special case of the theorem2

Theorem 6. Let all operators A_i (correspondingly, B_i) act in finite dimensional Hilbert space H_1 (correspondingly, H_2), and one of following conditions:

- a) $k_j n_2 + j m_2 > r_s n_1 + s m_1, \text{Ker}(A_{k_j, j}^{n_2} \otimes B_{m_2}^s) = \{\emptyset\}, A_{k_j, j}^* = A_{k_j, j}; B_{m_2}^* = B_{m_2}^*$
- b) $r_s n_1 + s m_1 > k_j n_2 + j m_2$, operator $A_{m_1}^s \otimes B_{r_s, s}^{n_1}$ has inverse

and selfadjoint

- c) $r_s n_1 + s m_1 = k_j n_2 + j m_2$, operator $A_{k_j, j}^{n_2} \otimes B_{m_2}^s + (-1)^{n_1 n_2} A_{m_1}^j \otimes B_{r_s, s}^{n_1}$ has inverse and selfadjoint

is satisfied. Then the eigen and associated vectors of the system (2) form $\text{Max}(k_j n_2 + j m_2, r_s n_1 + s m_1)$ multiple basis in the tensor product of the space $H_1 \otimes H_2$.

Remark1. Eigen and associated vectors are the first components of the elements of the kernel of the resultant of operator pencils $A(\lambda, \mu)$ and $B(\lambda, \mu)$.

Remark2. If the $H_1 \otimes H_2$ is infinite dimensional Hilbert space then the system of eigen and associated vectors of the system (1) coincides with the system of eigen and associated vectors of the operator pencil obtaining as result of decomposition of the resultant of operators

Theorems 4,5,6 are proved by analogy with the proof of the Theorem 3.

5. Conclusion

In this paper it is given the conditions of multiple basis of eigen and associated vectors of two parameter system of operators in Hilbert space $H_1 \otimes H_2$.

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