Pure and Applied Mathematics Journal
2015; 4(4-1): 38-44
Published online August 21, 2015 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.s.2015040401.18
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Research Methods of Multiparameter System in Hilbert Spaces

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To cite this article:
Rakhshanda Dzhabarzadeh. Research Methods of Multiparameter System in Hilbert Spaces. *Pure and Applied Mathematics Journal*. Special Issue: Spectral Theory of Multiparameter Operator Pencils and Its Applications. Vol. 4, No. 4-1, 2015, pp. 38-44.
doi: 10.11648/j.pamj.s.2015040401.18

Abstract: The work is devoted to the presentation of the methods, available in the literature, of the study of multiparameter spectral problems in Hilbert space. In particular, the method of Atkinson and his followers for a purely self-adjoint multiparameter systems and methods proposed by the author for the study, in general, non- selfadjoint multiparameter system in Hilbert space. These approaches solve questions of completeness, multiple completeness, the basis and a multiple basis property of eigen and associated vectors of multiparameter systems with a complex dependence on the parameters.

Keywords: Method, Atkinson, Multiparameter Systems, Basis, Complete

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The physical sciences open more and more challenges for mathematical researches. In particular, the resolution of the problems associated with the physical processes and, consequently, the study of partial differential equations and mathematical physics equations required a new approach. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding of a solution of a system with ordinary differential equations, which are much easier to study. For example, a multivariable problems cause problems in quantum mechanics, diffraction theory, the theory of elastic shells, nuclear reactor calculations, stochastic diffusion processes, Brownian motion, boundary value problems for equations of elliptic-parabolic type, the Cauchy problem for ultra-parabolic equations, etc.

Despite the urgency and prescription studies, spectral theory of multiparameter systems was not enough investigated? The available results in this area until recently only dealt with systems of selfadjoint multiparameter operator, linearly depending on the spectral parameters.

F.V. Atkinson is the founder of the research of the spectral multiparameter system of operators. Atkinson [1] studied the results available for multiparameter symmetric differential systems, built multiparameter spectral theory in Euclidean spaces. Further, by taking the limit Atkinson summarized these results on the case of multiparameter systems with self-adjoint compact operators in infinite-dimensional Hilbert spaces.

Later, method of investigation, introduced by Atkinson to study multiparameter systems in finite-dimensional spaces is used by Browne, Sleeman and others for constructing the spectral theory of selfadjoint multiparameter system in Hilbert spaces [2], [3], etc.

2. Some Aspects of the Method of Separation of Variables: Abstract Analog of a Separation of Variables for the Operational Equations

In the works of Morse, Feshbach [4], Roche [5], Sleeman [6] are paid the great attention to detail shown in the essence of the emergence and development of a multiparameter system as a result of applying of the method of separation of variables in the partial differential equations and equations of mathematical physics.

The method of separation of variables is applied to various boundary value problems, when the initial conditions are considered as a special case of the border $t = 0$.

[4], in particular, contain a series of coordinate systems in

which the equations admit separation of variables. Equations which may be separated have the special form .and they can be written as:

$$(A_{1,i}\otimes E_2\otimes\ldots\otimes E_n+E_1\otimes A_{2.p}\otimes\ldots\otimes E_n+\ldots\_+E_1\otimes E_2\otimes\ldots\otimes A_{n,i})x=0,\qquad \hat{x}\in H$$

where $H=H_1\otimes H_2\otimes\ldots\otimes H_n$ is the tensor product of the spaces $H_1,\ldots,H_n$. E_i are unity operators of H_s.

It is interesting that all solutions of partial differential equations can be obtained by linear combinations of family members separated solutions.

Let $A_{s,k}$ be linear operators acting in Hilbert space H_s, and let $H=H_1\otimes\ldots\otimes H_n$ be the tensor product of spaces $H_1,\ldots,H_n$. Suppose that an equation is given in the form

$$\sum_{i=1}^{k}A_{1,i}\otimes A_{2,i}\otimes\ldots\otimes A_{n,i}\hat{x}=0,\quad \hat{x}\in H \tag{1}$$

Theorem 1. [7] For every decomposable tensor $\hat{x}=x_1\otimes\ldots\otimes x_n$, that is solution of (1), there exist complex numbers $\{\alpha_{i,j}\}_{j=1}^{k-1}$, $\quad i=1,2,\ldots,n-1$ such, that

$$\begin{cases} A_{1,i}x_1+(-1)^{n-1}a_{1,i}a_{2,i}\ldots a_{n-1,i}A_{1,k}x_1=0\\ \left(a_{n-1,i}A_{2,i}+A_{2,k}\right)x_2=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \left(a_{2,i}A_{n-1,i}+A_{n-1,k}\right)x_{n-1}=0\qquad ,\ i=\overline{1,k-1}\\ \left(a_{1,1}A_{n,1}+\ldots+a_{1,k-1}A_{n,k-1}+A_{n,k}\right)x_n=0\end{cases}$$

For the proof of the *Theorem1* we use the famous property of the elements of tensor product space. It is known that the representation of the element in tensor product space is not unique. For each element of tensor product space there is the number coinciding with the minimal number of decomposable tensors, necessary for the representation of this element. This number is named the rank of element. If the number of decomposable tensors in the representation of element are more than the rank of element then one-nominal components of decomposable tensors in the representation of the given element are linear dependent

3. Selfadjoint Multiparameter Systems

Research method in spectral theory of self adjoint multiparameter systems.

The first works on spectral theory of multiparameter system of operators were the investigations of Faierman[8] concerning to the system of differential equations. Faierman [8] considered the system

$$\frac{d^2y_r(x_r)}{dx_r^2}+q_r(x_r)y_r(x_r)+\sum_{s=1}^{k}\lambda_s a_{r,s}y_r(x_r)=0,\ r=1,2,\ldots,k\,, \tag{2}$$

when $q_r(x_r),a_{r,s}(x_r)$, $r,s=1,2,\ldots,k$ are continuous ,real valued and differentiable on the interval $[a_r,b_r]$ of real axis.

System (2) with the common boundary conditions

$$\begin{matrix} y_r(a_r)Cos\alpha_r+y_r^{/}Sin\alpha_r=0;\alpha_r\in[0,\pi)\\ y_r(b_r)Cos\beta_r+y_r^{/}Sin\beta_r=0;\beta_r\in(0,\pi]\end{matrix},\ r=1,2,\ldots,k \tag{3}$$

is the $k-$ parameter Sturm-Luiville system.

If condition $a(x)=\det\{a_{r,s}(x_r)\}>0$ for $\forall x=\{x_1,x_2,\ldots,x_k\}$ in the systems (2) and (3) are satisfied then their eigen values belong to R^k and do not have a finite limit point.

Let $\{p_1,p_2,\ldots,p_k\}$ be the set of non-negative integers, then there is exactly one eigenvalue $\lambda\in R^k$ of the problem (2) and (3), for which the function $y_r(x_r)$ has exactly p_r zeroes in the interval of $[a_r,b_r]$.

Assuming $a_{r,s}(x)\in C[0,1]$, $a_{r,s}(x)$ and $q(x)$ are differentiable functions Faerman [8] also proves that the eigenfunctions of the problem (2) and (3) form a complete orthogonal system with respect to the weight function $\det\{a_{r,s}(x_s)\}\in L^2(I_k)$, where $\forall x=\{x_1,x_2,\ldots,x_k\}\in I_k$ the Cartesian product of intervals.

Later Browne established this result without conditions of differentiable of functions

A brief proof of the fundamental theorem of Browne gives research methods for study of selfadjoint multiparameter systems

For presenting of the method of investigation of multiparameter systems we present some results in this direction.

Famous results in the spectral theory of selfadjoint multiparameter systems:

Let

$$T_r^+f_r+\sum_{s=1}^{n}\lambda_s V_{r,s}^+f_r=0;f_r\in H_r;r=1,2,\ldots,n \tag{4}$$

be n -parameter system

Operators $T_r,V_{r,s}:H_r\to H_r,s=1,2,\ldots,n$ are bounded and selfadjoint in the space H_r. For any each set of elements $f_r\in H_r,f_r\neq 0,r=1,2,\ldots,n$ determinant $\det(V_{r,s}f_r,f_r)>0$, where (.,.) is the inner product in H_r.

Operators $\Delta_s;H\to H,s=0,1,\ldots,n$ are defined as follows: let $f=f_1\otimes f_2\otimes\ldots\otimes f_n$ be a decomposable tensor in H and $\alpha_0,\alpha_1,\ldots,\alpha_n$ be an arbitrary complex numbers. Then $\Delta_0 f,\Delta_1 f,\ldots,\Delta_n f$ are determined by equation

$$\sum_{s=0}^{n}\alpha_s\Delta_s f=\otimes\begin{vmatrix}\alpha_0 & \alpha_1 & \ldots & \alpha_n\\ T_1f_1 & V_{1,1}f_1 & \ldots & V_{1,n}f_1\\ . & . & \ldots & .\\ T_nf_n & V_{n,1} & \ldots & V_{n,n}f_n\end{vmatrix} \tag{5}$$

where the determinant can be extended to the whole space with the help of the tensor product.

Δ_s is determined on the decomposable tensor $f = f_1 \otimes f_2 \otimes ... \otimes f_n$ of the space H when $a^+_{0,1},...,-a^+_{0,n},-t^+_{2,n+1},0,...,0,-t_{2,n+k_1-1},...,0,...,-t^+_{2,k_1+k_2+...+k_{n-1}+1},0,...)$ in the (5) with help of(5) and on all other elements of the space H is defined on linearity and continuity.

The inner product $[f,g]$. is given by the expression $(\Delta_0 f, g)$ .The norms, induced by these inner products are equivalent, and thus topological concepts as continuity of the operators and the convergence of sequences of elements are equal with respect to these standards. Further, we will denote Γ_i the operator $\Delta_0^{-1}\Delta_i (i = 1,2,...,n)$.

Theorem 2. [1,2]. Suppose, $D(\Delta_0^{-1}) \subset R(\Delta_i); i = 1,2,...,n$ then $\Delta_i = \Delta_i^* (i = 0,1,...,n)$.

Theorem3. ([1],[2]).Suppose $D(\Delta_0^{-1}) \subset R(\Delta_i); i = 1,2,...,n$, the inner product $[f,g]$ is given by the expression $(\Delta_0 f, g)$,then Γ_i = $\Delta_0^{-1}\Delta_i (i = 1,2,...,n)$ are selfadjont operators.

Operators $E_r(\cdot)$ are projection operators, as operators Γ_r are mutual commute. Thus, we adopt the $E_r(\cdot)$ as the spectral measure on the Borel subsets of the space R^k , which carrier is set σ_0 , and for each pair of elements has a function $\{E(M)f, g\}$ with complex valued Borel measure, turns to zero out λ The type of measures $[E(\cdot)f, f]$ are nonnegative, essentially finite Borel measures, vanishing outside λ

The spectrum σ of the system $\{T_r, V_{r,s}\}$ is defined in ([1].[2]) as a vehicle operator-valued measure. Then σ there is a compact subset R^k of measures $\{E(M)f, g\}$ and indeed fade out λ If $E(\lambda) = E(\lambda_1, \lambda_2, ..., \lambda_n)$ there is a Borel function defined on σ , and we can define $F(\Gamma) = F(\Gamma_1, ..., \Gamma_n)$ the operator as follows:

$$i) DF(\Gamma) = \{F \in H / \int_\sigma |F(\lambda)|^2 [E(d\lambda)f, f] < \infty\}$$

$$(ii) f \in D(F(\Gamma))$$

and to have $[F(\Gamma)f, g] = \int_\sigma F(\lambda)[E(d\lambda)f, g]$ for an arbitrary $g \in H$

If $F(\lambda)$ is a bounded function, then $DF(\Gamma) = H$. If $F(\lambda)$ is unbounded and has a dense domain, the details of these results can be found in [17] E. Prugovečku.

Definition 1. Operator A_i^+ is named by operator, induced to space H by A_i and is constructed by following:: on decomposable tensor $f = f_1 \otimes f_2 \otimes ... \otimes f_n$ of $A_r^+ f = f_1 \otimes f_2 \otimes ... \otimes A_r f_r \otimes f_{r+1} \otimes ... \otimes f_n$ and on other elements of H operator A_i^+ is defined on linearity and continuity.

Theorem 4 ([1], [2]) For each $f \in H$ there are elements $g_1, g_2, ..., g_n \in H$ such that $T_r^+ f + \sum_{s=1}^n V_{r,s}^+ g_s = 0; r = 1,2,...,n$, the operators Γ_i ; $i = 1,2,...,n$ are mutual commute.

4. Non-Selfadjoint Multiparameter System of Operators

We research the multiparameter system

$$A_i(\lambda)x_i = (A_{0,i} + \lambda_1 A_{1,i} + ... + \lambda_n A_{m,i})x_i = 0 \quad i = 1,2,...n, \quad (6)$$

In (6) operators $A_{i,k}$ act in separable Hilbert space H_k and bounded.

Definition 2. ([1]], [2]) $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in C^n$ is eigen value of multiparameter system (6) (see [1,2]), if there are such n nonzero elements $x_i \in H_i (i = 1,2,...,n)$ that equalities (6) are fulfilled. Vector $x_1 \otimes x_2 \otimes ... \otimes x_n$ named eigenvector of the system (6), corresponding to eigen value $\lambda \in C^n$..

Definition 3. ([8],[9]).Let A_1^{-1} be an eigenvector of the system (6), corresponding to its eigen value $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$; the $x_{m_1, m_2, ..., m_n}$ is $m_1, m_2, ..., m_n$ - th associated vector to an eigenvector $x_{0,...,0}$ of the system (6) if there is a set of vectors $(x_{i_1, i_2, ..., i_n}) \subset H_1 \otimes ... \otimes H_n$, satisfying to conditions

$$A_{0,i}^+(\lambda)x_{s_1, s_2, ..., s_n} + A_{1,i}^+ x_{s_1 - 1, s_2, ..., s_n} + ... + A_{k,i}^+ x_{s_1, s_2, ..., s_n} = 0 .$$

Indices $s_1, s_2, ..., s_n$ are the various arrangements of set of integers on n with $0 \le s_r \le m_r$, $r = 1,2,...,n$. $0 \le s_r \le m_r, r = 1,...,n; \quad i = 1,...,n, \quad x_{i_1, i_2, ..., i_n} = 0 ; \sum_{k=1}^n i_k < 0$.

Under canonical system e.a. elements we understand system

$$\left\{z^{(k)}_{i_1, i_2, ..., i_n}\right\}_{0 \le i_r \le m_r}, r = 1,2,...,n \quad (7)$$

having the following properties: elements $z^{(k)}_{0,...,0}$ form base of an eigen subspace $M(\lambda^0)$; there is $z^{(1)}_{0,...,0}$ eigenvector which multiplicity reaches a possible maxima $p_1 + 1$; there is $z^{(k)}_{0,...,0}$ eigenvector which is not expressing linearly through $z^{(1)}_{0,...,0}, ..., z^{(k-1)}_{0,...,0}$ which sum of multiplicities reaches a possible maxima $p_k + 1$ for everyone fixed value of number k, $k = 1,2,...,s$.

Elements from (7) form a chain of e.a. elements at everyone fixed value k, $k = 1,2,...,s$ .The sum $p_1 + p_2 + ... + p_s + s$ - we name a multiplicity of an eigen value $\lambda^0 = \left(\lambda_1^0, ..., \lambda_n^0\right)$.

We introduce the necessary definitions and notions for the statement of the main results of the spectral theory of nonselfadjoint multiparameter system in Hilbert space.

Some positions play the essential role in the investigation of multiparameter system of operators

Definition 4. ([10],[11]). Let be two polynomial bundles

$$A(\lambda)=A_0+\lambda A_1+...+\lambda^m A_m,\ B(\lambda)=B_0+\lambda B_1+...+\lambda^n B_n \quad (8)$$

depending on the same parameters λ and acting, generally speaking, in various spaces H_1, H_2, accordingly. The operator $\operatorname{Re} z(A(\lambda),B(\lambda))$ is given by matrix

$$\operatorname{Re} z(A(\lambda),B(\lambda))=\otimes\begin{pmatrix} A_0^+ & A_1^+ & A_2^+ & \cdots & A_m^+ & 0 & \cdots & 0 \\ 0 & A_0^+ & A_1^+ & \cdots & A_{m-1}^+ & A_m^+ & \cdots & 0 \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots A_0^+ & A_1^+ & A_2^+ & \cdots & A_m^+ \\ B_0^+ & B_1^+ & B_2^+ & \cdots & B_n^+ & 0 & \cdots & 0 \\ 0 & B_0^+ & B_1^+ & \cdots & B_{m-1}^+ & B_n^+ & \cdots & 0 \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots B_0^+ & B_1^+ & B_2^+ & \cdots & B_n^+ \end{pmatrix} \quad (9)$$

In [10],[11] operator $\operatorname{Re} s(A(\lambda),B(\lambda))$ is named by abstract analog of a Resultant for polynomial bundles (8).

In definition of a Resultant (9) of bundles (8) the rows with operators A_i are repeated n times, and rows with operators B_i are repeated exactly m times. m, n there are the highest degrees of parameter λ in bundles of $A(\lambda)$ and $B(\lambda)$, accordingly. Thus, the Resultant (9) is an operator, acting in space $(H_1\otimes H_2)^{m+n}$ that is a direct sum $m+n$ of copies of tensor product spaces $H_1\otimes H_2$. Value of Resultant $\operatorname{Re} s(A(\lambda),B(\lambda))$ is equal to its formal expansion when each term of this expansion is tensor product of operators. Let all operators $A_i\ (i=0,1,...,n)$ (correspondingly, $B_i\ (i=0,1,...,m)$ are bounded in the Hilbert space (correspondingly, H_2) and operator A_n or B_m is invertible.

By [10],[11] it follows that the existence non-zero kernel of the operator $\operatorname{Re} s(A(\lambda),B(\lambda))$ is the necessary and sufficient conditions for the existing the common point of spectra of operators $A(\lambda)$ and $B(\lambda)$ .If the spectrum of each operator $A(\lambda)$ and $B(\lambda)$ contains only eigen values then a common point of spectra of these operators $A(\lambda)$ and $B(\lambda)$ is their eigenvalue.

Let now we have n the bundles depending on the same parameter λ

$$\begin{cases} B_i(\lambda)=B_{0,i}+\lambda B_{1,i}+...+\lambda^{K_i}B_{K_i,i} \\ i=1,2,...,n \end{cases} \quad (10)$$

$B_i(\lambda)$ are operational bundles with the discrete spectrum, acting in a Hilbert space H_i, accordingly. Without loss of generality we shall suppose, that $k_1\ge k_2\ge...\ge k_n$. We shall introduce operators $R_i\ (i=1,...,n-1)$ which act in space $H^{k_1+k_2}$ (the direct sum of k_1+k_2 copies of tensor-product $H=H_1\otimes...\otimes H_n$ of spaces $H_1,H_2,...,H_n$ and these operators $R_i\ (i=1,...,n-1)$ are defined by means of operational matrixes

$$R_{i-1}=\begin{pmatrix} B_{0,1}^+ & B_{1,1}^+ & \cdots\cdots & B_{k_1,1}^+ & \cdots\cdots & 0 \\ 0 & B_{0,1}^+ & B_{1,1}^+\cdots & B_{k_1-1.1}^+ & B_{k_1,1}^+\cdots & 0 \\ . & . & \cdots\cdots & . & \cdots\cdots & . \\ 0 & 0 & \cdots B_{0,1}^+ & B_{1,1}^+ & \cdots\cdots & B_{k_1,1}^+ \\ B_{0,i}^+ & B_{1,i}^+ & \cdots\cdots & B_{k_i,i}^+ & 0\cdots\cdots & 0 \\ 0 & B_{0,i}^+ & B_{1,i}^+\cdots & \cdot & B_{k_i,i}^+\cdots & 0 \\ . & . & \cdots\cdots & . & \cdots\cdots & . \\ 0 & 0 & \cdots B_{0,i}^+ & B_{1,i}^+ & \cdots\cdots & B_{k_i,i}^+ \end{pmatrix}$$

$$i=2,...,n. \quad (11)$$

We shall designate $\sigma_p\left(B_i(\lambda)\right)$ the set of eigen values of an operator $B_i(\lambda)$.

Theorem 5 [12]. Let the operator B_{k_1} has inverse and operators $\{B_i(\lambda)\ (i=1,2,...,n)$ have discrete spectrums. Then $\bigcap_{i=1}^{n}\sigma_p(B_i(\lambda))\ne\{\theta\}$ in the only case if $\bigcap_{i=1}^{n} KerR_i\ne\{\theta\}$.

A decisive result in the spectral theory of nonselfadjoint multiparameter systems began with the following Theorem 5. If the number of parameters in (6) is equal 2 with the help of the abstract resultant of two operator bundles in one parameter (the other parameter is fixed) we obtain one equation in one parameter in the tensor product of the Hilbert spaces. The obtained equation is studied. It is proved that the eigen subspace of this pencil coincides with the subspace spanned by eigen and associated vectors of kind $x_{0,k}$ or $x_{k,0}$ (index 0 stands on the place of the fixed parameter). Associated vectors of the pencil are also the associated vectors of the system (6). The author proves that the system (6) and the obtained the pencil have the common system of eigen and associated vectors.

Theorem 6 [9]. Let $n=2$ in (6). Operators $KerA_1=\{\theta\}$ and $KerB_1=\{\theta\}$, eigenvectors and associated vectors of an operator $(A_0+\lambda_1 A_1+\lambda_2 A_2)$ at any fix meanings of parameter λ_2 form basis in Hilbert space H_1 ; operators A_0, A_2, B_0, B_1, B_2, $KerA_1=\{\theta\}$ are bounded in corresponding spaces. Then the eigen and associated vectors on the direction λ_1 is a linear combination of the elements of an aspect $U_i\otimes V_0+U_{i-1}\otimes V_1+...+U_0\otimes V_i$.

where $U_0,U_1,...,U_i$ (accordingly, $V_0,V_1,...,V_i$) there is a restricted chain of e.a. vectors of an operator $(A_0+\lambda_1 A_1+\lambda_2^0 A_2)$ (accordingly, $\left(B_0+\lambda_1 B_1+\lambda_2^0 B_2\right)$) corresponding to some common eigen value of both operators $A_0+\lambda_1 A_1+\lambda_2^0 A_2$ and $B_1^{-1}(B_0+\lambda_1 B_1+\lambda_2^0 B_2^0)$

Let's designate through $M(\lambda^0)$ a subspace spanned by eigenvectors and associated vectors of system (6), corresponding to an eigenvalue λ^0.

Linearly-independent elements from the set $\left(z_{i_1,...,i_n}\right)\subset H$ form a chain of eigenvector and associated (e.a) vectors

Theorem 7. [8],[9].. Let following conditions satisfies: operators $A_{i,k}$ are bounded for all meanings i and k in space H_k, the operator Δ_0^{-1} exists and bounded. Then system of e.a. vectors (6) coincides with e.a. vectors of each of operators $\Gamma_i (i=1,2,...,n)$.

The proof .when $n>2$ in (6). We fix parameters $\lambda_1,\lambda_2,...,\lambda_{n-1}$. Then we have n the operational bundles linearly depending on parameter λ_n. From Theorem 5 it follows, that bundles have a common eigenvalue if and only if $\bigcap_{i=1}^{n-1} KerR_i\neq\theta$ under conditions, that one of operators $A_{n,i}^{-1}$ (at fixed n) exists and bounded . Existence $A_{n,i}^{-1}$ (not limiting a generality, we suppose, that $i=1$) follows from existence and bounded of an operator $\Delta_0^{-1}=\Delta_{0,n}^{-1}$. As the result of use the Theorem 5 we obtain $n-1$ equation with the $n-1$ parameters. The main operator for the obtained $\Delta_{0,n-1}^{-1}$ exists and bounded. Continue this process, at last we have the one equation in one parameter in the tensor product space H. Further, it is proved that all eigenvalues and the systems of eigen and associated vectors of original , intermediate and last multiparameter systems, presenting in the process of proof theorem, coincide

Theorem 7. [18] Let the eigen and associated vectors of operator $A_1(\lambda)$ in (6),when any $n-1$ of parameters are fixed, form a basis in space H_1, $KerA_{r,1}=\{\theta\}$. If $x_{1,n}^r, x_{2,n}^r \ \dots, x_{k_n,n}^r$ is the chain of e.a. vectors of an operator $A_r(\lambda)$ on parameter λ_n,corresponding to its eigen value, then associated vectors $x_{s_1,s_2,...,s_k}$ on direction λ_n of (6) is the linear span of the sum of various combinations of decomposable tensors $x_{r_1,1}^1\otimes x_{r_2,2}^2\otimes...\otimes x_{r_n,n}^n$, $r_1+r_2+...+r_n=i$ for which $0\le r_i\le s_i$, $i=1,2,...,n.$

5. Some Approaches to the Investigation of the Multiparameter Systems of Operators Complicated Depending on Parameters in the Hilbert Spaces

Below we give two approaches by which we study multiparameter systems with complex dependence on the parameters.

For the example we consider

$$\begin{cases} A_i(\lambda)x_i=(A_{0,i}+\sum_{k=1}^{k_{1,i}}\lambda_1^k A_{1,k,i}+\sum_{k=1}^{k_{2,i}}\lambda_2^k A_{2,k,i}+...+\sum_{k=1}^{k_{n,i}}\lambda_n^k A_{n,k,i})x_i=0 \\ i=1,2,...,n \end{cases} \tag{12}$$

when $A_{s,j,i}$, $A_{0,i}$ bounded operators in a separable Hilbert space.

Definition 5. Elements

$$x_{i_1,...,i_n}\in\tilde{H},\ 0<i_s<m_s,\ s=1,2,...,k$$

are named the associated vectors to the eigen vector $x_{0,...,0}$ of the system (12) if the following conditions

$$\sum_{s=1}^{k}\sum_{0\le r\le i_s}\frac{\partial^r\tilde{A}_{i,j,s}^{+}(\lambda^0)}{r!\partial\lambda_s^r}\tilde{x}_{i_1,i_2,...,i_s-r,...,i_k}=0,\quad i=1,2,...,n,\ k=\sum_{i=1}^{n}k_i$$

are satisfied. If $i_s<0$ for some meanings s $(0\le s\le n)$, then element $x_{i_1,...,i_n}=0$. Linear independent elements of $\{x_{i_1,...,i_n}\}$ form the chain of eigen and associated vectors of multiparameter system (12), corresponding to the eigenvalue λ_0.

One of methods of investigation of the system(12) is the method of converting into a linear multiparameter system of type (6)

Introduce the notations

$$A_{r,s,i}=\tilde{A}_{k_1+k_2+...+k_{r-1}+s,i}$$

$$\lambda_i^s=\tilde{\lambda}_{k_1+k_2+...+k_{i-1}+s};\ i=1,2,...,n;\ k_0=0,\ s=1,2,...,k_i.$$

and add system (12) by following equation (13)

$$\begin{aligned}
&(t_2+\lambda_1 t_0+\lambda_2 t_1)x_{n+1}=0\\
&(\lambda_1 t_2+\lambda_2 t_0+\lambda_3 t_1)x_{n+2}=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\lambda_{k_1-2}t_2+\lambda_{k_1-1}t_0+\lambda_{k_1}t_1)x_{n+k-1}=0\\
&(t_2+\lambda_{k_1+1}t_0+\lambda_{k_1+2}t_1)x_{n+k_1}=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\lambda_{k_1+k_2-2}t_2+\lambda_{k_1+k_2-1}t_0+\lambda_{k_1+k_2}t_1)x_{n+k_1+k_2-2}=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(t_2+\lambda_{k_1+k_2+\ldots+k_{n-1}+1}t_0+\lambda_{k_1+k_2+\ldots+k_{n-1}+k_n}t_1)x_{k_1+\ldots+k_{n-1}+1}=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\lambda_{k_1+k_2+\ldots+k_n-2}t_2+\lambda_{k_1+k_2+\ldots+k_n-1}t_0+\lambda_{k_1+\ldots+k_n}t_1)x_{k_1+k_2+\ldots+k_n}=0
\end{aligned} \tag{13}$$

where operators t_0,t_1,t_2 are set by means of matrixes

$$t_0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ t_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}\ t_2=\begin{pmatrix}0&0\\0&1\end{pmatrix} \tag{14}$$

If the $\tilde{\lambda}=(\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots,\tilde{\lambda}_{k_1+k_2+\ldots+k_n})$ is the eigenvalue of the system [(12),(13)] and $x_1\otimes x_2\otimes\ldots\otimes x_n\otimes(\alpha_{n+1},\beta_{n+1})\otimes\ldots\otimes(\alpha_{k_1+k_2+\ldots+k_n-n},\beta_{k_1+k_2+\ldots+k_n-n})$ is the corresponding eigenvector, rhen on the eigenvalues and the corresponding eigenvectors of the system (12) and (13) the equations (13) are realized connections between parameters λ_i according to requirements of system (12).

((12), (13)), considered together, form the multiparameter system consisting of $k_1+k_2+\ldots+k_n$ the equations and containing $k_1+k_2+\ldots+k_n$ parameters.

To obtained system all procedures on investigations under certain conditions by the method of Atkinson and by the method offered by author are possible. The main goal of this approach is to reduce the investigation of complicated system (12) to the investigation of linear multiparameter system. On the eigenvalues of linear system we have $\tilde{\lambda}_4=\tilde{\lambda}_1^4$, …, $\tilde{\lambda}_{k_1}=\tilde{\lambda}_1^{k_1}$, … , $\tilde{\lambda}_{k_1+k_2+\ldots+k_r+s}=\tilde{\lambda}^s_{k_1+k_2+\ldots+k_r+1}$, $r=1,2,\ldots,n-1;s=1,2,\ldots,k_n$. If the $(\Delta_0 x,x)\geq\delta(x,x)$ $Ker\Gamma_{k_1+k_2+\ldots+k_{r-1}+1}=\theta;r=1,2,\ldots,n-1$, all operators , forming the system selfadjoind, then under the additional conditions Browne 's theorem states the existence and a reality of a spectrum of a multiparameter system [(12),(13)]. If the operators in (12) are bounded but the system [(12),(13)] is not selfadjoint, $Ker\Gamma_{k_1+k_2+\ldots+k_{r-1}+1}=\theta;r=1,2,\ldots,n-1$ we can apply the results of the work [9], or [8].

Such approach allows solving more complex multiparameter systems containing products of parameters. In this case the additional equations contain non- selfadjoint operator giving with the help of matrixes. And it is possible the applying only the results of [8],[9].

Second approach of investigation of n-multiparameter system is to use the criterion of existence of common point of spectra of several polynomial pencils, acting, generally speaking, in different Hilbert spaces. Fixed all parameters in the system besides one parameter we obtain several operator pencils, depending on one parameter. With help of criterion (Theorem5) we come the multiparameter system in which number of parameters is equal to $n-1$. Continue this process we at last obtain the one equation in the tensor product of spaces with the one parameter. Further we proved the coincidence of the system of eigen and associated vectors considering multiparameter system and the last equation.

This method allows solving the problems when the number of equations in the multiparameter system is greater than the number of parameters.

The special case of the multiparameter system is the nonlinear algebraic systems. In the case the complicated nonlinear algebraic systems with many variables with help of these two approaches we find the number of solutions and prove the reality of solutions [13],[15],,ets.

6. Conclusion

The methods of investigations of non-self- adjoint multiparameter system are stated.

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