Research Methods of Multiparameter System in Hilbert Spaces

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To cite this article:

Abstract: The work is devoted to the presentation of the methods, available in the literature, of the study of multiparameter spectral problems in Hilbert space. In particular, the method of Atkinson and his followers for a purely self-adjoint multiparameter systems and methods proposed by the author for the study, in general, non-selfadjoint multiparameter system in Hilbert space. These approaches solve questions of completeness, multiple completeness, the basis and a multiple basis property of eigen and associated vectors of multiparameter systems with a complex dependence on the parameters.

Keywords: Method, Atkinson, Multiparameter Systems, Basis, Complete

1. Introduction

Spectral theory of operators is one of the important directions of functional analysis. The physical sciences open more and more challenges for mathematical researches. In particular, the resolution of the problems associated with the physical processes and, consequently, the study of partial differential equations and mathematical physics equations required a new approach. The method of separation of variables in many cases turned out to be the only acceptable, since it reduces finding a solution of a complex equation with many variables to finding of a solution of a system with ordinary differential equations, which are much easier to study. For example, a multivariable problems cause problems in quantum mechanics, diffraction theory, the theory of elastic shells, nuclear reactor calculations, stochastic diffusion processes, Brownian motion, boundary value problems for equations of elliptic-parabolic type, the Cauchy problem for ultra-parabolic equations, etc.

Despite the urgency and prescription studies, spectral theory of multiparameter systems was not enough investigated? The available results in this area until recently only dealt with systems of selfadjoint multiparameter operator, linearly depending on the spectral parameters.

F.V. Atkinson is the founder of the research of the spectral multiparameter system of operators. Atkinson [1] studied the results available for multiparameter symmetric differential systems, built multiparameter spectral theory in Euclidean spaces. Further, by taking the limit Atkinson summarized these results on the case of multiparameter systems with self-adjoint compact operators in infinite-dimensional Hilbert spaces.

Later, method of investigation, introduced by Atkinson to study multiparameter systems in finite-dimensional spaces is used by Browne, Sleeman and others for constructing the spectral theory of selfadjoint multiparameter system in Hilbert spaces [2], [3], etc.

2. Some Aspects of the Method of Separation of Variables: Abstract Analog of a Separation of Variables for the Operational Equations

In the works of Morse, Feshbach [4], Roche [5], Sleeman [6] are paid the great attention to detail shown in the essence of the emergence and development of a multiparameter system as a result of applying of the method of separation of variables in the partial differential equations and equations of mathematical physics.

The method of separation of variables is applied to various boundary value problems, when the initial conditions are considered as a special case of the border $t = 0$.

[4], in particular, contain a series of coordinate systems in
The equations may be separated have the special form and they can be written as:

\[
(A_{i_1} \otimes E_2 \otimes \ldots \otimes E_n + E_1 \otimes A_{r_1} \otimes \ldots \otimes E_n + \ldots + E_1 \otimes E_2 \otimes \ldots \otimes A_{r_k})x = 0, \quad \hat{x} \in H
\]

where \( H = H_1 \otimes H_2 \otimes \ldots \otimes H_n \) is the tensor product of the spaces \( H_1, \ldots, H_n \). \( E_i \) are unity operators of \( H_1 \). For any each set of elements \( a_{r,k} \) and \( b_{r,k} \), let \( \lambda \) be a non-negative integer, then there is exactly one eigenvalue \( \lambda \) in the interval of \( [a_{r,k}, b_{r,k}] \).

Assuming \( a_{r,k}(x) \in C[0,1] \) and \( b_{r,k}(x) \) are differentiable functions Faerman [8] also proves that the eigenfunctions of the problem (2) and (3) form a complete orthogonal system with respect to the weight function \( \det(a_{r,k}(x)) \) on \( I_k \), where \( \forall x = \{x_1, x_2, \ldots, x_k\} \in I_k \) the Cartesian product of intervals.

Later Browne established this result without conditions of differentiable functions.

A brief proof of the fundamental theorem of Browne gives research methods for study of selfadjoint multiparameter systems.

For presenting of the method of investigation of multiparameter systems we present some results in this direction.

Famous results in the spectral theory of selfadjoint multiparameter systems:

Let

\[
T^r f + \sum_{s=1}^n \lambda_s V^r_{s} f_r = 0; f_r \in H_r; r = 1, 2, \ldots, n
\]

be \( n \) -parameter system

where

\[
T = T_{r_1} \otimes \ldots \otimes T_{r_k}, \quad \alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_n}
\]

are arbitrary complex numbers. Then \( \Delta_\alpha T f \) are defined as follows:

For each set of elements \( f_r \in H_r, f_r \neq 0, r = 1, 2, \ldots, n \) determinant \( \det(V^r f_r) > 0 \) where \( (\ldots) \) is the inner product in \( H_r \).

Operators

\[
\Delta_\alpha T : H \rightarrow H, s = 1, 2, \ldots, n
\]

are bounded and selfadjoint in the space \( H_r \). For any each set of elements \( f_r \in H_r, f_r \neq 0, r = 1, 2, \ldots, n \) determinant \( \det(V^r f_r) > 0 \), where \( (\ldots) \) is the inner product in \( H_r \).

Operators \( \Delta_\alpha T : H \rightarrow H, s = 1, 2, \ldots, n \) are defined as follows:

\[
\Delta_\alpha T = T_{r_1} f_{r_1} \otimes \ldots \otimes T_{r_k} f_{r_k}
\]

be an arbitrary complex numbers. Then \( \Delta_\alpha T f \) are determined by equation

\[
\sum_{r=0}^n \alpha_r \Delta_\alpha T f = \otimes \begin{bmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_n \\ T_{r_1} f_{r_1} & V_{r_1} f_{r_1} & \ldots & V_{r_k} f_{r_k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{r_n} f_{r_n} & V_{r_1} f_{r_n} & \ldots & V_{r_k} f_{r_n} \end{bmatrix}
\]
where the determinant can be extended to the whole space with the help of the tensor product.

$\Delta_\gamma$ is determined on the decomposable tensor $f = f_1 \otimes f_2 \otimes \ldots \otimes f_n$ of the space $H$ when $\sigma_{a_1,\ldots,a_n,0,0,\ldots,0,\ldots,a_1,\ldots,a_n,0}$ in the (5) with help of(5) and on all other elements of the space $H$ is defined on linearity and continuity.

The inner product $\langle f, g \rangle$ is given by the expression $(\Delta_\gamma f, g)$. The norms, induced by these inner products are equivalent, and thus topological concepts as continuity of the operators and the convergence of sequences of elements are equal with respect to these standards. Further, we will denote $\Gamma$, the operator $\Delta_\gamma^{-1}\gamma(i = 1,2,\ldots,n)$.

Theorem 2. [1,2]. Suppose, $D(\Delta_\gamma^{-1}\gamma) \subset R(\gamma); i = 1,2,\ldots,n$ then $\Delta_\gamma = \Delta_\gamma^{-1}\gamma(i = 0,1,\ldots,n)$.

Theorem 3. ([1],[2]). Suppose $D(\Delta_\gamma^{-1}\gamma) \subset R(\gamma); i = 1,2,\ldots,n$, the inner product $\langle f, g \rangle$ is given by the expression $(\Delta_\gamma f, g)$, then $\Gamma = \Delta_\gamma^{-1}\gamma(i = 1,2,\ldots,n)$ are selfadjont operators.

Operators $E, (\cdot)$ are projection operators, as operators $\Gamma$, are mutual commute. Thus, we adopt the $E, (\cdot)$ as the spectral measure on the Borel subsets of the space $R^4$, which carrier is set $\sigma_0$, and for each pair of elements has a function $\{E(M)f, g\}$ with complex valued Borel measure, turns to zero out $\lambda$. The type of measures $\{E(f), f\}$ are nonnegative, essentially finite Borel measures, vanishing outside $\lambda$.

The spectrum $\sigma$ of the system $\{T, V_{rs}\}$ is defined in ([1],[2]) as a vehicle operator-valued measure. Then there is a compact subset $R^4$ of measures $\{E(M)f, g\}$ and indeed fade out $\lambda$. If $E(\lambda) = E(\lambda_1, \lambda_2,\ldots,\lambda_n)$ there is a Borel function defined on $\sigma$, and we can define $F(\Gamma) = F(\Gamma_1,\ldots,\Gamma_n)$ the operator as follows:

(i) $DF(\Gamma) = \{F \in H_f \left[ F(\lambda) \right] (\lambda) (\lambda) f) (f, f) < \infty$ (ii) $f \in D(F(\Gamma))$

and to have $\left\{ F(\lambda) \right\} (\lambda) (\lambda) \left( f, g \right)$ for an arbitrary $g \in H$

If $F(\lambda)$ is a bounded function, then $DF(\Gamma) = H$. If $F(\lambda)$ is unbounded and has a dense domain, the details of these results can be found in [17] E. Prugovečku.

Definition 1. Operator $A_i$ is named by operator, induced to space $H$ by $A$ and is constructed by following:: on decomposable tensor $f = f_1 \otimes f_2 \otimes \ldots \otimes f_n$ of $A_i f = f_1 \otimes f_2 \otimes \ldots \otimes A_i f, f = f_1 \otimes f_2 \otimes \ldots \otimes f_n$ and on other elements of $H$ operator $A_i$ is defined on linearity and continuity.

### 4. Non-Selfadjont Multiparameter System of Operators

We research the multiparameter system

$A(\lambda)x_0 = (A_{0,\lambda} + \lambda A_{1,\lambda} + \ldots + \lambda^n A_{n,\lambda})x_0 = 0; i = 1,2,\ldots,n$ (6)

In (6) operators $A_{i, \lambda}$ act in separable Hilbert space $H_i$ and bounded.

Definition 2. ([1],[2]) $\lambda = (\lambda_1, \lambda_2,\ldots,\lambda_n) \in C^n$ is eigen value of multiparameter system (6) (see [1,2]), if there are such $n$ nonzero elements $x_i \in H_i (i = 1,2,\ldots,n)$ that equalities (6) are fulfilled. Vector $x_0 \otimes x_1 \otimes \ldots \otimes x_n$ named eigenvector of the system (6), corresponding to eigen value $\lambda \in C^n$.

Definition 3. ([8],[9]). Let $A_i$ be an eigenvector of the system (6), corresponding to its eigen value $\lambda = (\lambda_1, \lambda_2,\ldots,\lambda_n)$; the $x_{m_1,m_2,\ldots,m_n}$ is $m_1,m_2,\ldots,m_n$ - th associated vector to an eigenvector $x_{0,\lambda}$ of the system (6) if there is a set of vectors $x_0 \otimes x_1 \otimes \ldots \otimes x_n$, satisfying to conditions

$A_i(\lambda)x_{s_1,s_2,\ldots,s_n} + A_i(\lambda)x_{s_1,s_2,\ldots,s_n} + \ldots + A_i(\lambda)x_{s_1,s_2,\ldots,s_n} = 0$.

Indices $s_1,s_2,\ldots,s_n$ are the various arrangements of set of integers $n$ with $0 \leq s_i \leq m_i, r = 1,2,\ldots,n$. $0 \leq m_i, s_i = 1,\ldots,n, x_{s_1,s_2,\ldots,s_n} = 0 \sum_{s_i} i_s < 0$ .

Under canonical system e.a. elements we understand system

$z_{k,\lambda}^{(j)}(\lambda) \sum_{i<s} r = 1,2,\ldots,n$ (7)

having the following properties: elements $z_{k,\lambda}^{(j)}$ form base of an eigen subspace $M(\lambda)$; there is $z_{0,\lambda}^{(1)}$ eigenvector which multiplicity reaches a possible maxima $p_1 + 1$; there is $z_{0,\lambda}^{(k)}$ eigenvector which is not expressing linearly through $z_{0,\lambda}^{(1)},\ldots, z_{0,\lambda}^{(k)}$ which sum of multiplicities reaches a possible maxima $p_1 + 1$ for every fixed value of number $k$, $k = 1,2,\ldots,s$ .

Elements from (7) form a chain of e.a. elements at everyone fixed value $k$, $k = 1,2,\ldots,s$ and we name a multiplicity of an eigen value $\lambda^0 = (\lambda_1^0,\ldots,\lambda_n^0)$.
We introduce the necessary definitions and notions for the statement of the main results of the spectral theory of nonselfadjoint multiparameter system in Hilbert space.

Some positions play the essential role in the investigation of multiparameter system of operators

**Definition 4.** ([10],[11]). Let be two polynomial bundles

\[
\text{Re } z(A(\lambda), B(\lambda)) = \bigotimes \left(\begin{array}{cccc}
A_0^+ & A_1^+ & \cdots & A_n^+ \\
0 & A_0^+ & \cdots & A_{n-1}^+ \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_0^+ \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_0^+ \\
0 & 0 & \cdots & B_0^+ \\
\end{array}\right) = \bigotimes
\]

In [10],[11] operator \(\text{Re } z(A(\lambda), B(\lambda))\) is named by abstract analog of a Resultant for polynomial bundles (8).

In definition of a Resultant (9) of bundles (8) the the rows with operators \(A_i\) are repeated \(n\) times, and rows with operators \(B_i\) are repeated exactly \(m\) times, \(m, n\) there are the highest degrees of parameter \(\lambda\) in bundles of \(A(\lambda)\) and \(B(\lambda)\), accordingly. Thus, the Resultant (9) is an operator, acting in space \((H_1 \otimes H_2)^{\mathbb{N}}\) that is a direct sum \(m+n\) of copies of tensor product spaces \(H_1 \otimes H_2\). Value of Resultant \(\text{Re } z(A(\lambda), B(\lambda))\) is equal to its formal expansion when each term of this expansion is tensor product of operators. Let all operators \(A_i\) (correspondingly, \(B_i\)) are bounded in the Hilbert space (correspondingly, \(H_2\)) and operator \(A_0\) or \(B_0\) is invertible.

By [10],[11] it follows that the existence non-zero kernel of the operator \(\text{Re } z(A(\lambda), B(\lambda))\) is the necessary and sufficient conditions for the existing the common point of spectra of operators \(A(\lambda)\) and \(B(\lambda)\). If the spectrum of each operator \(A(\lambda)\) and \(B(\lambda)\) contains only eigen values then a common point of spectra of these operators \(A(\lambda)\) and \(B(\lambda)\) is their eigenvalue.

Let now we have \(n\) the bundles depending on the same parameter \(\lambda\)

\[
B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + \cdots + \lambda^{k_i} B_{k_i,i} \quad (i=1,2,...,n)
\]

\(B_i(\lambda)\) are operational bundles with the discrete spectrum, acting in a Hilbert space \(H_i\), accordingly. Without loss of generality we shall suppose, that \(k_1 \geq k_2 \geq \cdots \geq k_n\). We shall introduce operators \(R_i\) (\(i=1,...,n-1\)) which act in space \(H^{k_i+1}\) (the direct sum of \(k_i+1\) copies of tensor-product \(H = H_1 \otimes \cdots \otimes H_n\) of spaces \(H_1, H_2, \ldots, H_n\) and these operators \(R_i\) (\(i=1,...,n-1\)) are defined by means of

\[
A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^n A_n, \quad B(\lambda) = B_0 + \lambda B_1 + \cdots + \lambda^n B_n
\]

depending on the same parameters \(\lambda\) acting in the Hilbert spaces. The obtained equation is studied. It is proved that the eigen subspace of this pencil coincides with the subspace spanned by eigen and associated vectors of kind \(\lambda_{0,k}\) or \(\lambda_{k,0}\) (index 0 stands on the place of the fixed parameter). Associated vectors of the pencil are also the associated vectors of the system (6). The author proves that the system (6) and the obtained the pencil have the common system of eigen and associated vectors.

**Theorem 5** [12]. Let the operator \(B_i\) has inverse and operators \(\{B_i(\lambda)\} (i=1,2,...,n)\) have discrete spectrums.

Then \(\bigcap_{i=1}^n \sigma_p (B_i(\lambda)) \neq \{\theta\}\) in the only case if \(\bigcap_{i=1}^n \text{Ker } R_i \neq \{\theta\}\).

A decisive result in the spectral theory of nonselfadjoint multiparameter systems began with the following Theorem 5. If the number of parameters in (6) is equal 2 with the help of the abstract resultant of two operator bundles in one parameter (the other parameter is fixed) we obtain one equation in one parameter in the tensor product of the Hilbert spaces. The obtained equation is studied. It is proved that the eigen subspace of this pencil coincides with the subspace spanned by eigen and associated vectors of kind \(\lambda_{0,k}\) or \(\lambda_{k,0}\) (index 0 stands on the place of the fixed parameter). Associated vectors of the pencil are also the associated vectors of the system (6). The author proves that the system (6) and the obtained the pencil have the common system of eigen and associated vectors.

**Theorem 6** [9]. Let \(n=2\) in (6). Operators \(\text{Ker } A_i = \{\theta\}\) and \(\text{Ker } B_i = \{\theta\}\), eigenvectors and associated vectors of an
operator \((A_0 + \lambda A_1 + \lambda_2 A_2)\) at any fix meanings of parameter \(\lambda\) form basis in Hilbert space \(H_i\); operators \(A_0, A_1, B_0, B_1, B_2, \text{Ker}A = \{\theta\}\) are bounded in corresponding spaces. Then the eigen and associated vectors on the direction \(\lambda\) is a linear combination of the elements of an aspect \(U_i \otimes V_0 + U_{i-1} \otimes V_1 + \ldots + U_0 \otimes V_i\), where \(U_i, U_{i-1}, \ldots, V_i\) (accordingly, \(V_0, V_1, \ldots, V_i\)) there is a restricted chain of e.a. vectors of an operator \((A_0 + \lambda A_1 + \lambda_2 A_2)\) (accordingly, \((B_0 + \lambda B_1 + \lambda_2 B_2)\)) corresponding to some common eigen value of both operators \(A_0 + \lambda A_1 + \lambda_2 A_2\) and \(B_0 + \lambda B_1 + \lambda_2 B_2\).

Let’s designate through \(\mathcal{M}(\lambda)\) a subspace spanned by eigenvectors and associated vectors of system (6), corresponding to an eigenvalue \(\lambda\).

Linearly-independent elements from the set \(\{z_{n,k}\} \subset H\) form a chain of eigenvector and associated (e.a) vectors.

**Theorem 7.** [8] Let following conditions satisfies: operators \(A_{ij}\) are bounded for all meanings \(i\) and \(k\) in space \(H_i\), the operator \(\Delta_{ij}\) exists and bounded. Then system of e.a. vectors \(A_{ij}\) (at fixed \(n\)) exists and bounded. Existence \(A_{ij}\) (not limiting a generality, we suppose, that \(i = 1\)) follows from existence and bounded of an operator \(\Delta_{ij}^{-1} = \Delta_{ij}^{-1}\). As the result of use the Theorem 5 we obtain \(n-1\) equation with the \(n-1\) parameters. The main operator for the obtained \(\Delta_{ij}^{-1}\) exists and bounded. Continue this process, at last we have the one equation in one parameter in the tensor product space \(H\). Further, it is proved that all eigenvalues and the systems of eigen and associated vectors of original, intermediate and last multioperator systems, presenting in the process of proof theorem, coincide.

**Theorem 7.** [18] Let the eigen and associated vectors of operator \(A(\lambda)\) in (6), when any \(n-1\) of parameters are fixed, form a basis in space \(H_i, \text{Ker}A_{ij} = \{\theta\}\). If \(x_{i,n}, x_{i,n-1}, \ldots, x_{i,0}\) is the chain of e.a. vectors of an operator \(A(\lambda)\) on parameter \(\lambda\), corresponding to its eigen value, then associated vectors \(x_{i,n}, x_{i,n-1}, \ldots, x_{i,0}\) on direction \(\lambda\) of (6) is the linear span of the sum of various combinations of decomposable tensors \(x_{i,0} \otimes x_{i,1} \otimes \ldots \otimes x_{i,n}\), \(r_i + r_2 + \ldots + r_n = i\) for which \(0 \leq r_i \leq s_i, i = 1, 2, \ldots, n\).

5. Some Approaches to the Investigation of the Multiparameter Systems of Operators Complicated Depending on Parameters in the Hilbert Spaces

Below we give two approaches by which we study multiparameter systems with complex dependence on the parameters.

For the example we consider

\[
\begin{align*}
A_i(\lambda)x_i &= (A_0 + \sum_{k=1}^{k_i} A_{ik}^{(0)} \lambda^k A_{ik} + \sum_{k=1}^{k_i} A_{jk}^{(1)} \lambda^k A_{jk} + \ldots + \sum_{k=1}^{k_i} A_{nk}^{(n)} \lambda^k A_{nk})x_i = 0 \\
i = 1, 2, \ldots, n
\end{align*}
\]

(12)

when \(A_{ij}\), \(A_{ij}\) bounded operators in a separable Hilbert space.

**Definition 5.** Elements

\[x_{i,j} \in \tilde{H}, 0 < i < m, s = 1, 2, \ldots, k\]

are named the associated vectors to the eigen vector \(x_{0,s}\) of the system (12) if the following conditions

\[
\sum_{i=0}^{i} \sum_{j=0}^{j} \frac{\partial A_{ij}^{(s)}}{\partial \lambda_j}(\lambda_j) \cdot x_{i,j} = 0, \quad i = 1, 2, \ldots, n, \quad k = \sum_{i=1}^{i}
\]

are satisfied. If \(i < 0\) for some meanings \(s(0 \leq s \leq n)\), then element \(x_{i,j} = 0\). Linear independent elements of \(\{x_{i,j}\}\) form the chain of eigen and associated vectors of multiparameter system (12), corresponding to the eigenvalue \(\lambda\).

One of methods of investigation of the system (12) is the method of converting into a linear multiparameter system of type (6).

Introduce the notations

\[A_{i,s} = \tilde{A}_{i+k+\ldots+k+s}, \quad \lambda_i^* = \tilde{\lambda}_{i+k+\ldots+k+s}; i = 1, 2, \ldots, n; \quad k_0 = 0, \quad s = 1, 2, \ldots, k_i\]

and add system (12) by following equation (13)
$$(t_2 + \lambda_i t_0 + \lambda_i t_1)x_{n+1} = 0$$
$$(\lambda_i t_2 + \lambda_i t_0 + \lambda_i t_1)x_{n+2} = 0$$

$$(\lambda_{i-1} t_2 + \lambda_{i-1} t_0 + \lambda_{i-1} t_1)x_{n+1} = 0$$
$$(t_2 + \lambda_{i+1} t_0 + \lambda_{i+1} t_1)x_{n+4} = 0$$

$$(\lambda_{i+1} t_2 + \lambda_{i+1} t_0 + \lambda_{i+1} t_1)x_{n+2} = 0$$

$$(t_2 + \lambda_{i+2} t_0 + \lambda_{i+2} t_1)x_{n+3} = 0$$
$$(\lambda_{i+2} t_2 + \lambda_{i+2} t_0 + \lambda_{i+2} t_1)x_{n+4} = 0$$

$$(\lambda_{i+k_1+...+k_s+1} t_2 + \lambda_{i+k_1+...+k_s+1} t_0 + \lambda_{i+k_1+...+k_s+1} t_1)x_{n+k_1+...+k_s+1} = 0$$

where operators $t_0, t_1, t_2$ are set by means of matrices

$$t_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If the $\lambda_i = (\lambda_i, \ldots, \lambda_i)$ is the eigenvalue of the system \((12), (13)\) and $x_i \otimes x_i \otimes \ldots \otimes x_i \otimes (\alpha_{i1}, \beta_{i1}) \otimes \ldots \otimes (\alpha_{i+k_1+\ldots+k_s}, \beta_{i+k_1+\ldots+k_s})$ is the corresponding eigenvector, then on the eigenvalues and the corresponding eigenvectors of the system \((12)\) and \((13)\) the equations \((13)\) are realized connections between parameters $\lambda_i$ according to requirements of system \((12)\).

\((12), (13)\), considered together, form the multiparameter system consisting of $k_1 + k_2 + \ldots + k_s$ the equations and containing $k_1 + k_2 + \ldots + k_s$ parameters.

To obtained system all procedures on investigations under certain conditions by the method of Atkinson and by the method offered by author are possible. The main goal of this approach is to reduce the investigation of complicated system \((12)\) to the investigation of linear multiparameter system. On the eigenvalues of linear system we have $\lambda_i = \lambda_i^i$, $\ldots$, $\lambda_i = \lambda_i^{k_1}$, $\ldots$, $\lambda_i = \lambda_i^{k_1+\ldots+k_s}, s = 1,2,\ldots,k_s$. If the $\Delta x(x) x \geq 0(x,x)$

$Ker\Gamma_{\lambda,\ldots,\lambda,\ldots,\lambda} = \emptyset; r = 1,2,\ldots,n-1$, all operators forming the system selfadjoint, then under the additional conditions Browne’s theorem states the existence and a reality of a spectrum of a multiparameter system \((12), (13)\). If the operators in \((12)\) are bounded but the system \((12), (13)\) is not selfadjoint, $Ker\Gamma_{\lambda,\ldots,\lambda,\ldots,\lambda} = \emptyset; r = 1,2,\ldots,n-1$ we can apply the results of the work \(9\), or \(8\).

Such approach allows solving more complex multiparameter systems containing products of parameters. In this case the additional equations contain non-selfadjoint operator giving with the help of matrices. And it is possible the applying only the results of \(8\), \(9\).

Second approach of investigation of n-multiparameter system is to use the criterion of existence of common point of spectra of several polynomial pencils, acting, generally speaking, in different Hilbert spaces. Fixed all parameters in the system besides one parameter we obtain several operator pencils, depending on one parameter. With help of criterion \((\text{Theorem} 5)\) we come the multiparameter system in which number of parameters is equal to $n-1$. Continue this process we at last obtain the one equation in the tensor product of spaces with the one parameter. Further we proved the coincidence of the system of eigen and associated vectors considering multiparameter system and the last equation.

This method allows solving the problems when the number of equations in the multiparameter system is greater than the number of parameters.

The special case of the multiparameter system is the nonlinear algebraic systems. In the case the complicated nonlinear algebraic systems with many variables with help of these two approaches we find the number of solutions and prove the reality of solutions \([13],[15],\text{etc.}\)

### 6. Conclusion

The methods of investigations of non-self-adjoint multiparameter system are stated.

### References


