

A Logarithmic Derivative of Theta Function and Implication

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Abstract: In this paper we establish an identity involving logarithmic derivative of theta function by the theory of elliptic functions. Using these identities we introduce Ramanujan's modular identities, and also re-derive the product identity, and many other new interesting identities.

Keywords: Theta Function, Elliptic Function, Logarithmic Derivative

1. Introduction and Definitions

Assume throughout this paper that $q = e^{\pi i \tau}$, when $\Im \tau > 0$. As usual, the classical Jacobi theta functions are defined as follow[1-3],

$$\theta_1(z | \tau) = -iq^{\frac{1}{4}} \sum_{n=-\infty}^{+\infty} (-1)^n q^{n(n+1)} e^{(2n+1)iz} \quad (1.1)$$

$$\theta_2(z | \tau) = q^{\frac{1}{4}} \sum_{n=-\infty}^{+\infty} (-1)^n q^{n(n+1)} e^{(2n+1)iz} \quad (1.2)$$

$$\theta_3(z | \tau) = \sum_{n=-\infty}^{+\infty} q^{n^2} e^{2niz} \quad (1.3)$$

$$\theta_4(z | \tau) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} e^{2niz} \quad (1.4)$$

The q -shifed factorial is defined by

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$$

and some times write

$$(a_1, a_2, \dots, a_m; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \dots (a_m; q)_{\infty}$$

With above notation, the celebrated Jacobi triple product identity can be expressed as follow

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n = (q; q)_{\infty} (z; q)_{\infty} (q/z; q)_{\infty} \quad (1.5)$$

Employing the Jacobi triple product identity, we can derive the infinite product expressions for theta function

Proposition 1.1. (Infinite product representations for theta functions)

$$\theta_1(z | \tau) = 2q^{\frac{1}{4}} \sin z (q^2; q^2)_{\infty} (q^2 e^{2iz}; q^2)_{\infty} (q^2 e^{-2iz}; q^2)_{\infty}$$

$$\theta_2(z | \tau) = 2q^{\frac{1}{4}} (q^2; q^2)_{\infty} (-q^2 e^{2iz}; q^2)_{\infty} (-q^2 e^{-2iz}; q^2)_{\infty}$$

$$\theta_3(z | \tau) = (q^2; q^2)_{\infty} (-q^2 e^{2iz}; q^2)_{\infty} (-q e^{-2iz}; q^2)_{\infty}$$

$$\theta_4(z | \tau) = (q^2; q^2)_{\infty} (q e^{2iz}; q^2)_{\infty} (q e^{-2iz}; q^2)_{\infty}$$

When there is no confusion, We will use $\theta_i(z)$ for $\theta_i(z | \tau)$, $\theta'_i(z)$ for $\theta'_i(z | \tau)$ to denote the partial derivative with respect to the variable, and θ_i for $\theta_i(0 | \tau)$, $i = 1, 2, 3, 4$. From the above equations, the following facts are obvious

$$\begin{aligned}
\theta_1' &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)} = 2q^{\frac{1}{4}} (q^2; q^2)_{\infty}^3 \\
\theta_2 &= q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} q^{n(n+1)} = 2q^{\frac{1}{4}} (q^2; q^2)_{\infty} (-q^2; q^2)_{\infty}^2 \\
\theta_3 &= \sum_{n=-\infty}^{\infty} q^{n^2} = (q^2; q^2)_{\infty} (q; q^2)_{\infty}^2 \\
\theta_4 &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q^2; q^2)_{\infty} (-q; q^2)_{\infty}^2
\end{aligned} \tag{1.6}$$

With respect to the (quasi) period π and $\pi\tau$, Jacobi theta functions $\theta_i, i=1,2,3,4$ satisfy the following relations

$$\begin{aligned}
\theta_1(z+\pi|\tau) &= -\theta_1(z|\tau), & \theta_2(z+\pi|\tau) &= -\theta_2(z|\tau) \\
\theta_3(z+\pi|\tau) &= \theta_3(z|\tau), & \theta_4(z+\pi|\tau) &= \theta_4(z|\tau) \\
\theta_1(z+\pi\tau|\tau) &= -q^{-1}e^{-2z}\theta_1(z|\tau) \\
\theta_2(z+\pi\tau|\tau) &= q^{-1}e^{-2z}\theta_2(z|\tau) \\
\theta_3(z+\pi\tau|\tau) &= q^{-1}e^{-2z}\theta_3(z|\tau) \\
\theta_4(z+\pi\tau|\tau) &= -q^{-1}e^{-2z}\theta_4(z|\tau)
\end{aligned}$$

and

$$\begin{aligned}
\theta_1\left(z+\frac{\pi}{2}|\tau\right) &= \theta_2(z|\tau), & \theta_2\left(z+\frac{\pi}{2}|\tau\right) &= -\theta_1(z|\tau) \\
\theta_3\left(z+\frac{\pi}{2}|\tau\right) &= \theta_4(z|\tau), & \theta_4\left(z+\frac{\pi}{2}|\tau\right) &= \theta_3(z|\tau)
\end{aligned}$$

also have

$$\begin{aligned}
\theta_1\left(z+\frac{\pi\tau}{2}|\tau\right) &= iM\theta_4(z|\tau) \\
\theta_2\left(z+\frac{\pi\tau}{2}|\tau\right) &= M\theta_3(z|\tau) \\
\theta_3\left(z+\frac{\pi\tau}{2}|\tau\right) &= M\theta_2(z|\tau) \\
\theta_4\left(z+\frac{\pi\tau}{2}|\tau\right) &= iM\theta_1(z|\tau)
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
\theta_1\left(z+\frac{\pi\tau+\pi}{2}|\tau\right) &= M\theta_3(z|\tau) \\
\theta_2\left(z+\frac{\pi\tau+\pi}{2}|\tau\right) &= -iM\theta_4(z|\tau) \\
\theta_3\left(z+\frac{\pi\tau+\pi}{2}|\tau\right) &= iM\theta_1(z|\tau) \\
\theta_4\left(z+\frac{\pi\tau+\pi}{2}|\tau\right) &= M\theta_2(z|\tau)
\end{aligned} \tag{1.8}$$

Where $M = q^{-\frac{1}{4}}e^{-iz}$.

The following trigonometric series expressions for the logarithmic derivative with respect to z of Jacobi Theta functions will be very useful in this paper,

$$\begin{aligned}
\frac{\theta_1'}{\theta_1}(z|\tau) &= \cot z + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin 2nz \\
\frac{\theta_2'}{\theta_2}(z|\tau) &= -\tan z + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1-q^{2n}} \sin 2nz \\
\frac{\theta_3'}{\theta_3}(z|\tau) &= 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^{2n}} \sin 2nz \\
\frac{\theta_4'}{\theta_4}(z|\tau) &= 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin 2nz
\end{aligned} \tag{1.9}$$

Theorem 1.1. The sum of all the residues of an elliptic function in the period parallelogram is zero.

2. Main Theorem and Proofs

Theorem 2.1. For x and y , we have

$$\frac{\theta_4'}{\theta_4}(x) + \frac{\theta_4'}{\theta_4}(y) - \frac{\theta_4'}{\theta_4}(x+y) = \theta_2\theta_3 \frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_4(x)\theta_4(y)\theta_4(x+y)}$$

Proof. We consider the following function

$$f(z) = \frac{\theta_4(z+x)\theta_4(z+y)\theta_4(z-x-y)}{\theta_1^2(z)\theta_4(z)} \tag{2.1}$$

by the definition of $\theta_i(z|\tau)$, we can readily verify that $f(z)$ is an elliptic function with periods π and $\pi\tau$. The only poles of $f(z)$ is 0 and $\frac{\pi\tau}{2}$. Furthermore, $\frac{\pi\tau}{2}$ is

its simple pole and 0 is its pole with order two. By virtue of the residue theorem of elliptic functions, we have

$$\text{Res}\left(f; \frac{\pi\tau}{2}\right) + \text{Res}(f; 0) = 0. \tag{2.2}$$

And applying relation of θ_1 and θ_4 in (1.7-1.8) and L'Hospital' rule, we can obtain

$$\begin{aligned} \operatorname{Res}\left(f; \frac{\pi\tau}{2}\right) &= \lim_{z \rightarrow \frac{\pi\tau}{2}} \left(z - \frac{\pi\tau}{2}\right) f(z) \\ &= \lim_{z \rightarrow \frac{\pi\tau}{2}} \frac{\left(z - \frac{\pi\tau}{2}\right) \theta_4(z+x) \theta_4(z+y) \theta_4(z-x-y)}{\theta_1^2(z) \theta_4(z)} \\ &= \frac{(iB)^3 \theta_1(x) \theta_1(y) \theta_1(x+y)}{\theta_4\left(\frac{\pi\tau}{2}\right) \theta_1^2\left(\frac{\pi\tau}{2}\right)} \quad (2.3) \\ &= -\frac{\theta_1(x) \theta_1(y) \theta_1(x+y)}{\theta_1'(0) \theta_4^2(0)} \end{aligned}$$

Next we compute $\operatorname{Res}(f; 0)$,

$$\begin{aligned} \operatorname{Res}(f; 0) &= \lim_{z \rightarrow 0} \frac{d}{dz} (z^2 f(z)) \\ &= \lim_{z \rightarrow 0} z^2 f(z) \left[\frac{2}{z} + \frac{f'(z)}{f(z)} \right] \quad (2.4) \\ &= \lim_{z \rightarrow 0} z^2 f(z) \left[\frac{2}{z} + \frac{\theta_4'(z+x)}{\theta_4(z+x)} + \frac{\theta_4'(z+y)}{\theta_4(z+y)} \right. \\ &\quad \left. + \frac{\theta_4'(z-x-y)}{\theta_4(z-x-y)} - 2 \frac{\theta_1'(z)}{\theta_1(z)} - \frac{\theta_4'(z)}{\theta_4(z)} \right] \\ &= \frac{\theta_4(x) \theta_4(y) \theta_4(x+y)}{\theta_1^2(0) \theta_4(0)} \left[\frac{\theta_4'(x)}{\theta_4(x)} + \frac{\theta_4'(y)}{\theta_4(y)} - \frac{\theta_4'(x+y)}{\theta_4(x+y)} \right] \end{aligned}$$

From Theorem 1.1, substituting (2.3) and (2.4) into (2.2), by performing a little reduction we can complete the proof of Theorem 2.1.

Corollary 2.1. For x and y , we have

$$\left(\frac{\theta_4'}{\theta_4}\right)'(y) - \left(\frac{\theta_4'}{\theta_4}\right)'(x) = (\theta_1')^2 \frac{\theta_1(x+y) \theta_1(x-y)}{\theta_4^2(x) \theta_4^2(y)}$$

Proof. We differentiate the formulae of Theorem2.1 with respect to y , and then set $y = 0$, then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(0) - \left(\frac{\theta_4'}{\theta_4}\right)'(x) = (\theta_2 \theta_3)^2 \left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2 \quad (2.5)$$

Now we combine with another elementary identity [7, p.467]

$$\theta_1^2(x) \theta_4^2(y) - \theta_1^2(y) \theta_4^2(x) = \theta_4^2 \theta_1(x+y) \theta_1(x-y) \quad (2.6)$$

From formula (2.5) and (2.6), we can obtain

$$\begin{aligned} \left(\frac{\theta_4'}{\theta_4}\right)'(y) - \left(\frac{\theta_4'}{\theta_4}\right)'(x) &= (\theta_2 \theta_3)^2 \left(\left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2 - \left(\frac{\theta_1(y)}{\theta_4(y)}\right)^2 \right) \\ &= (\theta_2 \theta_3)^2 \frac{\theta_1^2(x) \theta_4^2(y) - \theta_1^2(y) \theta_4^2(x)}{\theta_4^2(x) \theta_4^2(y)} \\ &= (\theta_1')^2 \frac{\theta_1(x+y) \theta_1(x-y)}{\theta_4^2(x) \theta_4^2(y)} \end{aligned}$$

This completes the proof of Corollary 2.2.

Remark 2.1. The corollary2.1 is often written in terms of the weierstrass elliptic and sigma functions as [7, p.451]

$$\wp(y) - \wp(x) = \frac{\sigma(x+y) \sigma(x-y)}{\sigma^2(x) \sigma^2(y)}$$

Theorem 2.2. For x and y are real, we have

$$\begin{aligned} \operatorname{Im} \left[\frac{\theta_1'(z-x) - \theta_4'(z-y)}{\theta_1(z-x) - \theta_4(z-y)} \right] \\ \left| \frac{\left(\frac{\theta_1'}{\theta_1}\right)'(z-x) - \left(\frac{\theta_4'}{\theta_4}\right)'(z-y)}{\left(\frac{\theta_1'}{\theta_1}\right)'(z-x) - \left(\frac{\theta_4'}{\theta_4}\right)'(z-y)} \right| \\ = \frac{i \theta_1(z-\bar{z}) \theta_4(z+\bar{z}-x-y)}{2 \theta_1' \theta_4 | \theta_4(2z-x-y) |} \quad (2.7) \end{aligned}$$

where $\operatorname{Im} z$ denotes the imaginary part of the complex number z .

Proof. Firstly, we replace x by $z-x + \frac{\pi\tau}{2}$ and y by $x-y + \frac{\pi\tau}{2}$ in Theorem 2.1. it becomes

$$\begin{aligned} \frac{\theta_1'(z-x) + \theta_1'(x-y) - \theta_4'(z-y)}{\theta_1(z-x) + \theta_1(x-y) - \theta_4(z-y)} \\ = \theta_2 \theta_3 \frac{\theta_4(z-x) \theta_4(x-y) \theta_1(z-y)}{\theta_1(z-x) \theta_1(x-y) \theta_4(z-y)} \quad (2.8) \end{aligned}$$

Since $x-y$ is real, $\theta_i(x-y), i=1,2,3,4$ is also real valued, then we have

$$\begin{aligned} \operatorname{Im} \left[\frac{\theta_1'(z-x) - \theta_4'(z-y)}{\theta_1(z-x) - \theta_4(z-y)} \right] \\ = \theta_2 \theta_3 \frac{\theta_4(x-y)}{\theta_1(x-y)} \operatorname{Im} \left(\frac{\theta_4(z-x) \theta_1(z-y)}{\theta_4(z-y) \theta_1(z-x)} \right) \quad (2.9) \end{aligned}$$

We note that (2.9) is precisely the numerator of (2.7). We now consider its denominator. In Corollary2.1, replace y by

$z - y$ $z - x + \frac{\pi\tau}{2}$ and x by $z - y$, then obtain

$$\begin{aligned} & \left(\frac{\theta_1'}{\theta_1}\right)'(z-x) - \left(\frac{\theta_4'}{\theta_4}\right)'(z-y) \\ &= (\theta_1')^2 \frac{\theta_4(2z-x-y)\theta_4(y-x)}{\theta_1^2(z-x)\theta_4^2(z-y)} \end{aligned} \quad (2.10)$$

Now from (2.9) and (2.10), the left hand side of (2.7) becomes

$$\begin{aligned} & \frac{1}{\theta_1'\theta_4} \frac{1}{\theta_1(x-y)} \operatorname{Im} \left(\frac{\theta_4(z-x)\theta_1(z-y)}{\theta_4(z-y)\theta_1(z-x)} \right) \times \\ & \left| \frac{\theta_1^2(z-x)\theta_4^2(z-y)}{\theta_4(2z-x-y)\theta_4(2z-x-y)} \right| \\ &= \frac{\operatorname{Im}(\theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y))}{\theta_1'\theta_4\theta(x-y)|\theta_4(2z-x-y)|} \end{aligned} \quad (2.11)$$

Here we can see that it is crucial that x and y are both real, Since $\overline{\theta_i(z-x)} = \theta_i(\bar{z}-y)$. On the other hand, we can derive a different expression for the imaginary part of the above quantity. Since we note that in Corollary 2.1, replacing y by $z - y$, x by $x - z$ and z by $y - \bar{z}$, we can deduce (where $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$).

$$\begin{aligned} & 2i \operatorname{Im}(\theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y)) \\ &= \theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y) \\ & \quad - \theta_1(z-x)\theta_4(z-y)\theta_4(\bar{z}-x)\theta_1(\bar{z}-y) \\ &= \theta_4\theta_4(z+\bar{z}-x-y)\theta_1(y-x)\theta_1(z-\bar{z}) \end{aligned}$$

Substituting above equality into (2.11), we can obtain the result (2.7). This complete the proof of Theorem 2.2.

3. Implications for Square Sum

In this section, we will re-derive the Lambert series representations for $\theta_i^2, \theta_i^4, \theta_i^6$ from Theorem 2.1 easily and difference methods from [4-6].

Theorem 3.1. For Jacobi Theta function θ_3 , we have

$$\theta_3^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

Proof. We note that $\theta_4\left(x + \frac{\pi\tau + \pi}{2}\right) = q^{\frac{1}{4}} e^{-ix} \theta_2(x)$,

then $\frac{\theta_4'}{\theta_4}\left(x + \frac{\pi\tau + \pi}{2}\right) = -i + \frac{\theta_2'}{\theta_2}(x)$

In Theorem 2.1, we replace x by $x + \frac{\pi\tau + \pi}{2}$, then it becomes

$$\frac{\theta_2'}{\theta_2}(x) + \frac{\theta_4'}{\theta_4}(y) - \frac{\theta_2'}{\theta_2}(x+y) = \theta_2\theta_3 \frac{\theta_3(x)\theta_1(y)\theta_3(x+y)}{\theta_2(x)\theta_4(y)\theta_2(x+y)}$$

Next, we choose $x = 0$ and $y = \frac{\pi}{4}$ with the facts that

$$\theta_2'(0) = 0, \theta_4\left(\frac{\pi}{4}\right) = \theta_3\left(\frac{\pi}{4}\right), \theta_2\left(\frac{\pi}{4}\right) = \theta_1\left(\frac{\pi}{4}\right),$$
 then

above equality becomes

$$\theta_3^2 = \frac{\theta_4'}{\theta_4}\left(\frac{\pi}{4}\right) - \frac{\theta_2'}{\theta_2}\left(\frac{\pi}{4}\right) = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

This complete the proof of Theorem 3.1

Theorem 3.2. For Jacobi Theta function θ_3 , we have

$$\theta_3^4 = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n (1 - (-1)^n q^n)}{1 - q^{2n}}$$

Proof. We set $x = 0$ in (12), then differentiate it with respect to y and set $y = 0$, We recall (1.9) for $\frac{\theta_4'}{\theta_4}(z|\tau)$,

then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(z|\tau) = 8 \sum_{n=1}^{\infty} \cos 2nz = 4 \sum_{n=1}^{\infty} \frac{nq^n (e^{2inz} + e^{-2inz})}{1 - q^{2n}}$$

And from (1.9), we can obtain

$$\left(\frac{\theta_2'}{\theta_2}\right)'(z|\tau) = -\sec^2 z + \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n} \cos 2nz}{1 - q^{2n}}$$

Hence, we can obtain

$$\begin{aligned} \theta_3^4 &= \left(\frac{\theta_4'}{\theta_4}\right)'(0) - \left(\frac{\theta_2'}{\theta_2}\right)'(0) \\ &= 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n (1 - (-1)^n q^n)}{1 - q^{2n}} \end{aligned}$$

This complete the proof of Theorem 3.2

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