Moments of continuous bi-variate distributions: An alternative approach

Oyeka ICA\textsuperscript{1}, Okeh UM\textsuperscript{2}

\textsuperscript{1}Department of Statistics, Nnamdi Azikiwe University Awka, Nigeria
\textsuperscript{2}Department of Industrial Mathematics and Applied Statistics, Ebonyi State University Abakaliki, Nigeria

Email address: uzomaokley@gmail.com(Okeh UM)

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Abstract: We propose a method of obtaining the moment of some continuous bi-variate distributions with parameters \(\alpha\) and \(\beta\) in finding the nth moment of the variable \((X^c+\lambda)^n\), the nth moment of expected value of the t distribution of the cth power of X and dth power of Y about the constant \(\lambda\). These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed method is illustrated with some continuous bi-variate distributions and is shown to be easy to use even when the powers of the random variables being considered are non-negative real numbers that need not be integers. The results obtained using \(g_{n,c,d}\) are the same as results obtained using other methods such as moment generating functions when they exist.

Keywords: Moment Generating Functions, Bivariate Distributions, Continuous Random Variables, Joint Pdf

1. Introduction

Sometimes a researcher’s interest may be in finding the nth moment of the variable \((X^c+\lambda)^n\), where X and Y are continuous random variables having the joint pdf, \(f(x,y)\). Here we find the so-called \(g_{n,c,d}\) defined \(g_{n,c,d}=E\left(X^c+\lambda\right)^n\), the nth moment of the joint distribution of the cth power of X and dth power of Y about the constant \(\lambda\). That is, \(g_{n,c,d}=E\left(X^c+\lambda\right)^n\) (1)

Now, \(g_{n,c,d}=E\left(X^c+\lambda\right)^n\)

\(g_{n,c,d}=E\left(X^c+\lambda\right)^n=\int \int (x^c+\lambda)^n f(x,y)dydx\)

\(=\int \int \sum_{n=0}^{\infty} \binom{n}{t} \lambda^{n-t} x^c y^d f(x,y)dydx\)

\(\therefore g_{n,c,d}=\sum_{n=0}^{\infty} \binom{n}{t} \lambda^{n-t} \mu_{c,d}^{x,y}\) (2)

where \(\mu_{c,d}^{x,y}\) is the t-th moment of the joint distribution of \(X^c\) and \(Y^d\) about zero.

Note: that as expected \(g_{0,0}^{(0,0)}=1\).

Also,

\(g_{1,1}^{(1)}=E(x,y)+\lambda=\mu_{c,d}^{x,y}\) (3)

Where \(\mu_{c,d}^{x,y}\) is the mean of joint probability distribution of X and Y about zero.
\[ g_1^{(1)}(\lambda) = 0 \text{ when } \lambda = -\mu_{xy} \] (4)

The variance of joint distribution of \(X\) and \(Y\) is given by
\[ p_{n,o}^2 = g_2^{(1,1)} - \epsilon_1^{(1,1)^2} = g_2^{(0,0)} \text{ if } \lambda = -\mu_{xy} \] (5)

It is easily seen from equations 1 and 2 that,
\[ x g_n^{(c)}(x) = g_n^{(c,0)} \text{ and } y g_n^{(d)} = g_n^{(0,d)} \] (6)

1.1. To Illustrate

Suppose \(X\) and \(Y\) have bi-variate exponential distribution with parameters \(\beta_1\) and \(\beta_2\) that is
\[ f(x, y) = \frac{1}{\beta_1 \beta_2} e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} \]

Then
\[ g_n^{(c,d)} = E \left( X^c Y^d + \lambda \right)^n = \sum_{t=0}^{n} \left( \frac{n}{t} \right) \lambda^{n-t} \int_0^\infty \int_0^\infty x^t y^d e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} dx dy \] (7)

For instance
\[ g_2^{(2,1,1)} = \sum_{t=0}^{n} \left( \frac{2}{t} \right) \lambda^{2-t} \beta_1^2 \beta_2 \int_0^\infty \int_0^\infty x^2 y e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} dx dy \]

For \(t=0,1,2\) for example for \(t=0\), we have
\[ g_2^{(2,1,1)} = \lambda^{2-0} \times 1 = \lambda^2 \]

For \(t=1\);
\[ \left( \frac{2}{1} \right) \lambda^{2-1} \beta_1^2 \beta_2 \int_0^\infty \int_0^\infty x^2 y e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} dx dy = 2\lambda \beta_1^2 \beta_2 \times 2 = 2\lambda \beta_1 \beta_2 \]

For \(t=2\);
\[ \left( \frac{2}{2} \right) \lambda^{2-2} \beta_1^2 \beta_2 \int_0^\infty \int_0^\infty x^2 y e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} dx dy = \beta_1^2 \beta_2^2 \times (4! \times 17,280) = 17,280 \beta_1 \beta_2 \]

If \(X\) and \(Y\) have bi-variate exponential distribution with parameters \(\beta_1\) and \(\beta_2\), so that
\[ f(x, y) = \frac{1}{\beta_1 \beta_2} e^{-\left(\frac{x}{\beta_1} + \frac{y}{\beta_2}\right)} ; x > 0, y > 0 \]

Then
\[ g_{n,0}^{(c,d)} = \sum_{t=0}^{n} \left( \frac{n}{t} \right) \lambda^{c-t} \beta_1^c \beta_2 \left( \beta_1 c + 2 \right) \beta_1 c + 2 \beta_2 c + 2 \beta_1 \beta_2 c + 2 \beta_1 \beta_2 \] (12)

It is easy to calculate from equation 12 that the mean and variance of the joint distribution in equation 11 are from equations 4 and 5, \(2\beta_1\beta_2\) and \(8\beta_1^2\beta_2\) respectively.
third moment about the mean of distribution is $88 \beta_1^4 \beta_2^6$ while the fourth moment is $2016 \beta_1^4 \beta_2^4$.

Evaluating

$$\lambda^4 + 4 \lambda^3 + 1 = 16 \beta_1^4 \beta_2^4$$
$$\lambda^4 + 4 \lambda^3 (2 \beta_1 \beta_2) = -16 \beta_1^4 \beta_2^4$$
$$+ 6 \lambda (1 + 4 \beta_1^2 \beta_2^2) = +288 \beta_1^4 \beta_2^4$$
$$+ 4 \lambda (1 + 152 \beta_1^2 \beta_2^2) = -1152 \beta_1^4 \beta_2^4$$
$$= 2880 \beta_1^4 \beta_2^4$$

The moment generating function for Equation 11 is

$$\frac{\beta_1 + \beta_2 - \beta_1 \beta_2 (t_1 + t_2)}{(\beta_1 + \beta_2)(1 - \beta_1 t_1)(1 - \beta_2 t_2)}$$

Which is clearly difficult to use in obtaining the above moments than using the proposed $g_{(c,d)}^{(c,d)}$.

Note that the moments of distribution of $X$ can be easily obtained by evaluating $g_{(0)}^{(0)}$ got from equation 12 as

$$g_{(0)}^{(0)} = \frac{\sum (n) \lambda^m \beta^n \beta_{ct}^m + 2 + \beta_{ct}^m + 1}{\beta + \beta_2}$$

Similarly for $g_{(n,d)}^{(n,d)} \rightarrow g_{(n,d)}^{(n,d)}$

$$g_{(c,d)}^{(c,d)} = \frac{\sum (n) \lambda^m \beta^n \beta_{ct}^m + 2 + \beta_{ct}^m + 2}{\beta + \beta_2}$$

These are also easier and faster to use in evaluating the moments of the distributions of $X$ and $Y$ respectively of equation 11 then using the corresponding moment generating functions.

2. Bi-Variate MomenT Generating Function

$$g_{(c,d)}^{(c,d)} = E \left( \lambda_1 X^c + \lambda_2 Y^d \right)^t$$

$$= \int \left[ \lambda_1 X^c + \lambda_2 Y^d \right]^t f(X,Y) dY dX$$

$$= \int \sum_{t=0}^{n} \left( \lambda_1 \lambda_2 \lambda_{ct}^m \right)^t f(X,Y) dY dX$$

$$= \sum_{t=0}^{n} \left( \lambda_1 \lambda_2 \lambda_{ct}^m \right)^t \int X^c Y^d \left[ x^c y^d f(x,y) \right] dY dX$$

$$= \sum_{t=0}^{n} \left( \lambda_1 \lambda_2 \lambda_{ct}^m \right)^t \mu_{ct}^{(m)}$$

where $\mu_{ct}^{(m)} = \int X^c Y^d f(x,y) dy dx$

Is the joint $ct$ and $(n-t) \times d$ th moment of the distribution of $X,Y$ about the origin.

Note that

$$g_{(1,1)}^{(1,1)} = E \left( \lambda_1 X + \lambda_2 Y \right)^t$$

$$= \lambda_1 \mu_1 + \lambda_2 \mu_2$$

If $\lambda_1 = \lambda_2 = 1$.

Then $g_{(1,1)}^{(1,1)} = \mu_1 + \mu_2$.

If $d = 0$ and $\lambda_1 = 1$,

we have that

$$g_{(c,0)}^{(c,0)} = g_{(c)}$$

Note also that $g_{(0,0)}^{(0,0)} = 1$.

Example.

Let

$$f(x,y) = \beta_1 \beta_2 x y e^{-(\beta_1 x + \beta_2 y)}$$

then

$$g_{(c,d)}^{(c,d)} = E \left( \lambda_1 X^c + \lambda_2 Y^d \right)^t$$

$$= \sum_{t=0}^{n} \left( \lambda_1 \lambda_2 \lambda_{ct}^m \right)^t \int X^c Y^d e^{-((\beta_1 x + \beta_2 y)^t)} f(x,y) dY dX$$

Then

$$\beta_1 \beta_2 x y e^{-(\beta_1 x + \beta_2 y)} dx dy$$

$$= \beta_1 \beta_2 \sum_{t=0}^{n} \lambda_1 \lambda_2 \lambda_{ct}^m \int X^c Y^d e^{-((\beta_1 x + \beta_2 y)^t)} dx dy$$

$$= \sum_{t=0}^{n} \lambda_1 \lambda_2 \lambda_{ct}^m \beta_1 \beta_2 \lambda_{ct}^{(m)} \int x^c y^d \left[ x^c y^d f(x,y) \right] dY dX$$

$$= \sum_{t=0}^{n} \lambda_1 \lambda_2 \lambda_{ct}^m \beta_1 \beta_2 \lambda_{ct}^{(m)} \int x^c y^d \left[ x^c y^d f(x,y) \right] dY dX$$

Finding

$$g_{(1,1)}^{(1,1)} = E \left( \lambda_1 X + \lambda_2 Y \right)^t$$

$$= \sum_{t=0}^{n} \lambda_1 \lambda_2 \lambda_{ct}^m \beta_1 \beta_2 \lambda_{ct}^{(m)} \mu_{ct}^{(m)}$$

Evaluating for $t=0$

$$\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \lambda_1 \lambda_2 \beta_1 \beta_2 \beta_1 \beta_2 \left( \begin{array}{cc} 2 \\ 3 \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \lambda_1 \lambda_2 \beta_1 \beta_2 \left( \begin{array}{c} 2 \\ 3 \end{array} \right) \left( \begin{array}{c} 2 \\ 3 \end{array} \right) = \frac{2 \lambda_1 \beta_2}{\beta_1}$$

Evaluating for $t=1$

$$\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \lambda_1 \lambda_2 \beta_1 \beta_2 \lambda_{ct}^m \left( \begin{array}{c} 2 \\ 3 \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \lambda_1 \lambda_2 \beta_1 \beta_2 \left( \begin{array}{c} 2 \\ 3 \end{array} \right) \left( \begin{array}{c} 2 \\ 3 \end{array} \right) = \frac{2 \lambda_1}{\beta_1}$$

Suppose $c=1,d=1$ and $n=2$

Evaluating for $t=2$
\[ g_x^{(1)} = E \left( \lambda_i X + \lambda_j Y \right)^2 = \sum_{t=0}^{\infty} \binom{n}{t} \left( \lambda_i \lambda_j \beta_1 \beta_2 (t+1) \right)^2 = \frac{6 \lambda_i^2 \lambda_j^2}{\beta_1^2 \beta_2^2} \]

For \( t = 0 \)
\[ \binom{2}{0} \lambda_i^2 \lambda_j^2 \beta_1^2 \beta_2^2 (t+1)^2 = \frac{6 \lambda_i^2}{\beta_1^2} \]
\[ \text{For } t = 1 \]
\[ \binom{2}{1} \lambda_i^2 \lambda_j^2 \beta_1 \beta_2^2 (t+1)^3 = \frac{6 \lambda_i \lambda_j}{\beta_1 \beta_2} \]
\[ \text{For } t = 2 \]
\[ \binom{2}{2} \lambda_i^2 \lambda_j^2 \beta_1 \beta_2^2 (t+1)^4 = \frac{6 \lambda_i^2 \lambda_j^2}{\beta_1^2 \beta_2^2} \]
\[ \therefore g_x^{(1-i)} = \frac{6 \lambda_i^2}{\beta_1^2} + \frac{6 \lambda_i \lambda_j}{\beta_1 \beta_2} + \frac{6 \lambda_j^2}{\beta_2^2} \]

The variance of the joint distribution of \( X, Y \)
\[ g_{xy}^{(n)} = E \left( X^2 Y^2 + \lambda \right)^n \]
\[ = \sum_{t=0}^{\infty} \binom{n}{t} \lambda_i^2 \lambda_j^2 \beta_1^2 \beta_2^2 (t+1)^2 = \frac{6 \lambda_i^2 \lambda_j^2}{\beta_1^2 \beta_2^2} \]
\[ \therefore g_{xy}^{(1-i)} = \frac{6 \lambda_i^2}{\beta_1^2} + \frac{6 \lambda_i \lambda_j}{\beta_1 \beta_2} + \frac{6 \lambda_j^2}{\beta_2^2} \]

\[ = \frac{2 \beta_1^2 + 2 \beta_2^2}{\beta_1^2} \]

\[ g_{g_x^{(c)}} = g_x^{(c:0)} \text{ for when } \lambda_i = 1, d = 0 \]
\[ g_{x_1} = g_{x_1}^{(1:0)} = \sum \binom{n}{t} \lambda_i^2 \beta_1^2 (t+1)^2 \]
\[ = \left( \frac{1}{0} \right) \lambda_i^2 \beta_1^2 \sum t + \left( \frac{1}{1} \lambda_i^2 \beta_1^2 \right)^2 = \frac{2 \lambda_i^2}{\beta_1^2} \]

\[ g_{x_0} = g_{x_0}^{(0:1)} \text{ for when } \lambda_i = 1, c = 0 \]
\[ g_{x_0} = \sum \binom{n}{t} \lambda_i^2 \beta_2^2 (n-t)^2 + 2 \]
\[ \text{Note that; with } \lambda_i = 1 \]
\[ g_{x_1} = g_{x_1}^{(1:0)} = \sum \binom{n}{t} \lambda_i^2 \beta_1^2 (t+1)^2 \]
\[ = \left( \frac{1}{0} \right) \lambda_i^2 \beta_1^2 \sum t + \left( \frac{1}{1} \lambda_i^2 \beta_1^2 \right)^2 = \frac{2 \lambda_i^2}{\beta_1^2} \]

\[ g_{x_0} = g_{x_0}^{(0:1)} \text{ for with } \lambda_i = 1 \]
\[ g_{x_0} = \sum \binom{n}{t} \lambda_i^2 \beta_2^2 (n-t)^2 + 2 \]
\[ = \left( \frac{1}{0} \right) \lambda_i^2 \beta_2^2 \sum t + \left( \frac{1}{1} \lambda_i^2 \beta_2^2 \right)^2 = \frac{2 \lambda_i^2}{\beta_2^2} \]
For the bi-variate exponential distributions

\[
g_{y^n}^{(d)} = g_{x}^{(n,d)}
= \sum_{t=0}^{n} \left( \frac{1}{t^d} \right) \beta_1 \beta_2^{(n-t)d} (n-t)d + 1 \left( \lambda_2 \right)^{1}
= \sum_{t=0}^{n} \left( \frac{2}{t} \right) \lambda_1 \beta_2^{(2-t)^d} (3 - (2-t) - 1)
\]

For the bi-variate exponential distributions with
\[
g_{y^n}^{(d)} = g_{x}^{(n,d)}
= \sum_{t=0}^{n} \left( \frac{1}{t^d} \right) \beta_1 \beta_2^{(n-t)d} (n-t)d + 1 \left( \lambda_2 \right)^{1}
= \sum_{t=0}^{n} \left( \frac{2}{t} \right) \lambda_1 \beta_2^{(2-t)^d} (3 - (2-t) - 1)
\]

For \( t = 0 \), we have

\[
\left( \frac{2}{0} \right) \lambda_1 \beta_2 \times \sqrt{t} = 720 \lambda_2^6
\]

For \( t = 1 \), we have

\[
\left( \frac{2}{1} \right) \lambda_1 \beta_2 \times \sqrt{4} = 12 \lambda_2 \beta_2^3
\]

For \( t = 2 \), we have

\[
\left( \frac{2}{2} \right) \lambda_1 \beta_2 \times \sqrt{4} = 6 \lambda_2 \beta_2^3
\]

For \( t = 0 \), we have

\[
\left( \frac{1}{0} \right) \lambda_1 \beta_2 \times \sqrt{4} = 2 \lambda_2 \beta_2^3
\]

For \( t = 1 \), we have

\[
\left( \frac{1}{1} \right) \lambda_1 \beta_2 \times \sqrt{4} = 2 \lambda_2 \beta_2^3
\]

For \( t = 2 \), we have

\[
\left( \frac{1}{2} \right) \lambda_1 \beta_2 \times \sqrt{4} = 2 \lambda_2 \beta_2^3
\]

Similarly:

\[
g_{y^n}^{(1)} = \lambda_1 \beta_2 \times \sqrt{4} = 6 \lambda_2 \beta_2^3
\]

For the bi-variate Gamma distribution with
\[
f(x, y) = \frac{1}{\beta_1 \beta_2} (\beta_1 + \beta_2)^{x+y} e^{-(\frac{x+y}{\beta_1 \beta_2})}, x > 0, y > 0
\]

Suppose \( X \) and \( Y \) have a joint bivariate Gamma distribution with parameters \( \alpha_1 \) and \( \beta_1 \) and \( \alpha_2 \) and \( \beta_2 \) with pdf

\[
f(x, y) = \frac{1}{\beta_1 \beta_2} (\beta_1 + \beta_2)^{x+y} e^{-(\frac{x+y}{\beta_1 \beta_2})}, x > 0, y > 0
\]
Then the nth moment of the joint distribution of $X^c$ and $Y^d$ about zero is

$$
\mu_{n}(c, d) = \frac{1}{\beta^c \beta^d} \left( \alpha_1 + \beta \beta^d \right) \left( c_n + c d_n + 1 \right) \alpha_2
$$

Similarly the nth moment of the marginal distribution of $Y^d$ about zero is obtained by setting $c=0$ in Equation 16 to obtain

$$
\mu_{n}(d) = \mu_{n}(0, d) = \frac{\beta^d \beta^{d+n} \left( c_n + 1 \right)}{\beta \beta^d \left( \alpha_1 + \beta \beta^d \right) \alpha_2}
$$

Note that if in Equation 15 we set $\alpha_1 = \frac{k_1}{2}, \alpha_2 = \frac{k_2}{2}$ and $\beta_1 = \beta_2 = 2$, we have a bi-variate chi-square distribution then the corresponding nth moment of the joint distribution of $X^c$ and $Y^d$ is obtained by setting $\alpha_1 = \frac{k_1}{2}, \alpha_2 = \frac{k_2}{2}$ and $\beta_1 = \beta_2 = 2$ in Equation 16 which yields,

$$
\mu_{n}(c, d) = \frac{2^{\frac{c+1}{2}} \Gamma \left( \frac{c+n+1}{2} \right) \Gamma \left( \frac{d+n+1}{2} \right)}{\frac{2^{\frac{c}{2}} \Gamma \left( \frac{c}{2} + 1 \right) \Gamma \left( \frac{d}{2} + 1 \right)}{\alpha_2}}
$$

or

$$
\mu_{n}(c) = \mu_{n}(c, 0) = \frac{2^{\frac{c+1}{2}} \Gamma \left( \frac{c+n+1}{2} \right) \Gamma \left( \frac{d+n+1}{2} \right)}{\frac{2^{\frac{c}{2}} \Gamma \left( \frac{c}{2} + 1 \right) \Gamma \left( \frac{d}{2} + 1 \right)}{\alpha_2}}
$$

The nth moment of the marginal distribution of $X^c$ about zero, which is

$$
\mu_{n}(c) = \mu_{n}(c, 0) = \frac{2^{\frac{c+1}{2}} \Gamma \left( \frac{c+n+1}{2} \right)}{\frac{2^{\frac{c}{2}} \Gamma \left( \frac{c}{2} + 1 \right)}{\alpha_2}}
$$

The corresponding nth moment of the marginal distribution of $Y^d$ about zero is similarly obtained by setting $c=0$ in Equation 19. If in Equation 15, we set $\alpha_1 = \alpha_2 = 1$, we have the bi-variate exponential distribution. We obtain the corresponding nth moment of the joint distribution of $X^c$ and $Y^d$ given this bi-variate exponential distribution by setting $\alpha_1 = \alpha_2 = 1$ in Equation 16 as

$$
\mu_{n}(c, d) = \frac{\beta \beta^d \beta^{d+n} \left( c_n + 1 \right) d_n + 1 + \beta \beta^d \beta^{d+n} \left( c_n + 1 \right) d_n + 1}{\beta \beta^d \beta^{d+n} \left( c_n + 1 \right) d_n + 1}
$$

The nth moment of the marginal distribution of $X^c$ about zero, given this joint exponential distribution is obtained by setting $d=0$ in Equation 21 and is given as

$$
\mu_{n}(c) = \left( \beta^c \right)^{c+n+1} c n + 1
$$

Note that if in Equation 18, we set $c=d=1$, we have the nth moment of the bi-variate Gamma distribution of Equation 15 about zero, which is

$$
\mu_{n}(1,1) = \frac{\beta \beta^d \beta^{d+n} \left( n \alpha_1 n + 1 \right) \Gamma \left( n + 1 \right)}{\beta \beta^d \beta^{d+n} \left( n + 1 \right)}
$$

that is

$$
\mu_{n}(1,1) = \frac{\beta \beta^d \beta^{d+n} \left( n \alpha_1 n + 1 \right)}{\beta \beta^d \beta^{d+n} \left( n + 1 \right)}
$$
The corresponding nth moment of the bi-variate chi-squared distribution about zero is obtained by setting c=d=1 in Equation 19 yielding

$$
\mu_n(1,1) = \left(\frac{k_1}{2} n + \frac{1}{2} + 2\frac{k_1}{2} n + 1\right)^n
$$

(24)

The nth moment of the marginal distribution of X (which is now chi-squared distributed) about zero is obtained as

$$
\mu_n(c) = \mu_n(c,0)
$$

$$
= 2^{2n+1} \left(\frac{k_1}{2} n + \frac{1}{2} + 2\frac{k_1}{2} n + 1\right)^n
$$

(25)

Also, setting c=d=1 in Equation 21 we obtain the nth moment of the bi-variate exponential distribution about zero as

$$
\mu_n(1,1) = \beta_1^n \beta_2^n \left(\frac{cn + 1}{2}\right)^2
$$

(26)

The corresponding nth moment of the marginal distribution of X (which is now exponentially distributed) about zero is obtained by setting d=0 in Equation 21 which yields

$$
\mu_n(1) = \mu_n(1,0) = \beta_1^n \left(\frac{1}{2}\right)^n
$$

(27)

The mgf of the bi-variate Gamma distribution of Equation 25 is easily obtained as

$$
M(t_1,t_2) = \beta_1 \beta_2 \left(\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2} \right)^n
$$

(28)

Equation 23 is easier and quicker to use in finding the nth moment of the bi-variate Gamma distribution of Equation 15 than differentiating n times the corresponding mgf given in Equation 28 with respect to \( t_1 \) and \( t_2 \) evaluating the result at \( t_1 = t_2 = 0 \). Similarly, the Equations for the nth moment of the indicated marginal distributions are easier and quicker to use than the corresponding marginal mgf in finding these moments. To illustrate further, if \( \alpha_1 = \alpha_2 = 2 \), we have the pdf

$$
f(x, y) = \frac{(x + y) e^{-\frac{1}{2}(\frac{x^2}{\alpha_1^2} + \frac{y^2}{\alpha_2^2})}}{\beta_1 \beta_2 \left(\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2} \right)^n}, \quad x > 0, \; y > 0
$$

(29)

Then \( \mu(c,d) = \mu(c,d,0) = \frac{4\alpha_1 \alpha_2 \sqrt{\alpha_1 \alpha_2}}{\sqrt{\pi} (\alpha_1 \sqrt{\alpha_2} + \alpha_2 \sqrt{\alpha_1})} (x+y)e^{-(\frac{x^2}{\alpha_1^2} + \frac{y^2}{\alpha_2^2})}, \; x > 0, \; y > 0

$$
\mu(c,d) = \frac{4\alpha_1 \alpha_2 \sqrt{\alpha_1 \alpha_2}}{\sqrt{\pi} (\alpha_1 \sqrt{\alpha_2} + \alpha_2 \sqrt{\alpha_1})} \int_0^\infty x^n y^m (x+y)e^{-(\frac{x^2}{\alpha_1^2} + \frac{y^2}{\alpha_2^2})} dx dy
$$

(30)

the moment of the marginal distribution of \( X^c \) about zero is obtained by setting \( d=0 \) in Equation 30 is that

$$
\mu_n(c) = \mu_n(c,0) = \frac{\alpha_1 \alpha_2 \sqrt{\alpha_1 \alpha_2}}{\sqrt{\pi} (\alpha_1 \sqrt{\alpha_2} + \alpha_2 \sqrt{\alpha_1})} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right)^n \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right)^n \left(\frac{\alpha_1 + \alpha_2}{n + 1} \right)^n
$$

(31)

If we set \( c=d=1 \), in Equation 30 we obtain the nth of the bi-variate Releigh distributions of Equation 29 about zero as

$$
\mu_n(1,1) = \frac{\alpha_1 \alpha_2 \sqrt{\alpha_1 \alpha_2}}{\sqrt{\pi} (\alpha_1 \sqrt{\alpha_2} + \alpha_2 \sqrt{\alpha_1})} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right)^n \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right)^n \left(\frac{\alpha_1 + \alpha_2}{n + 1} \right)^n
$$

(32)

that is
Suppose \( X \) and \( Y \) have the bivariate normal distribution

\[
\mu_{\alpha, \beta} = \frac{\alpha \beta}{\sqrt{\pi}} \left( \alpha \beta + \alpha + \beta \right)
\]

The marginal of \( X \) is obtained from Equation 30 by setting \( c=1 \) and \( d=0 \) in Equation 31 as

\[
\mu_{\alpha, \beta} = \frac{\alpha \beta}{\sqrt{\pi}} \left( \alpha + \beta \right)
\]

Moment of continuous bi-variate distribution for normal

For \( t, s=0, 2, \ldots \), that is provided \( t \) and \( s \) are even

\[
\frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = (2\sigma)^t \frac{1}{\sqrt{\pi}} = 0
\]

For all odds values of \( t \) and \( s \) since for example with \( v = \frac{x-\mu}{\sigma} \), we have that \( \frac{1}{2\pi} \int_{-\infty}^{\infty} v' e^{-\frac{v^2}{2}} dv = 0 \), for all odds values of \( t \) that is for \( t=1,3,5, \ldots \), may easily be verified.

### 3. Summary and Conclusion

We have presented in this paper method of obtaining the moment of some continuous bi-variate distributions with parameters \( \alpha, \beta \). The proposed methods were the so called \( g_{\alpha, \beta} \) defined \( g_{\alpha, \beta} = E \left( X^t Y^s + \lambda \right) \), the nth moment of expected value of the t distribution of the cth power of \( X \) and dth power of \( Y \) about the constant \( \lambda \). These moments are obtained by the use of bi-variate moment generating functions, when they exist. The proposed \( g_{\alpha, \beta} \) exists for all continuous probability distributions unlike some of its competitors such as factorial moments of moment generating function which do not always exist. The results obtained using \( g_{\alpha, \beta} \) are the same as results obtained using such other methods as moment generating functions of available. The proposed method is available and easy to use without the need for any modifications even when the powers of the random variable being considered are non-negative real numbers that do not need to be integers. The results obtained using \( g_{\alpha, \beta} \) are the same as results obtained using other methods such as moment generating functions when they exist.

### References


