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# On One 3-Dimensional Boundary-Value Problem with Inclined Derivatives

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**Abstract:** A boundary-value problem with inclined derivatives in 3-dimensional space with the boundaries – surfaces of Liapunov type is considered in the paper. The method of investigation is based on the necessary conditions. The advantage compared to the theory of potentials is that we don't have limit passage, we use boundary values which are obtained from the principal relationships called necessary conditions. Remark that the directions of the derivatives given in the boundary conditions are arbitrary. Tangent directions may be some subset of the given directions.

**Keywords:** Inclined Derivatives, Normal Derivative, Necessary Conditions, Theory of Potentials, Fredholm Integral Equations of Second Kind

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## 1. Introduction

As it is known from [1], [2], [3], [4], [5], formulas of jumps obtained both for normal derivative of the simple layer potential and for the potential of double layer itself enable to reduce Neumann and Dirichlet boundary-value problems to Fredholm integral equations of second kind. What concerns the problems with inclined derivatives then they were completely investigated only in 2-dimensional case due to the methods of the theory of functions of complex variable [2], [5].

For 3-dimensional case as it was remarked by Bitsadze A.B. in [2] "as there isn't theory of 3-dimensional analogue of Riemann-Hilbert problem, such reduction is not very useful". This case for a general second order linear differential equation of elliptic type when the given directions on a smooth boundary of the domain makes an acute angle with the outer normals was investigated in [1], [6]. By means of the concept of a symbol J. Jiro's results [1], [6] were obtained more simple in [7], [8].

The case of a plane system of second order equations when the principal part is Laplace operator was considered in [9], [10], [2]. There was considered finiteness of the problem index, explicitly written out the necessary and sufficient condition of the problem solvability and, finally, shown that the problem character is essentially influenced by the

presence of minor terms entering the boundary condition if the coefficients at the derivative satisfy some equalities.

Further was determined that if the inclined line is tangent at some finite number of points of boundary  $\Gamma$  or in some finite number of lines on  $\Gamma$  the given inclination (direction) is tangent for this line then the kernel of the considered operator is finite dimensional, i.e. the solution of the problem is not unique [2], [11], [12]. These results of 2-dimensional problems (the start of investigations is in the monographs [13]) are generalized in [2], [14].

In 3-dimensional case the index formula for the problem with inclined derivatives of the system of Laplace equations was given without proof by A.I. Volpert [15]. Further remark that the work of a group of mathematicians [16] is devoted to the indices of boundary-value problems in abstract case.

The problem with inclined derivative in 3-dimensional Euclid space was investigated by A. Yanuschauskas [17]. There was obtained a condition of non-conditional solvability of the problem. Remark that in [11] there is given an additional condition along the line lying on the boundary for one-valued solvability and the problem is reduced to a Dirichlet problem.

At last remark that the existing, for instance in [18], limit theorems for tangent derivative of the potential of a simple

and double layer, as well as for the normal derivative of the double layer potential, unfortunately, didn't find their application for the solution of boundary-value problems for elliptic equations.

The present paper is devoted to the question of solvability of boundary-value problem with inclined derivatives in 3-dimensional space.

The problem statement belongs to Bitsadze A.V. [19], [2].

## 2. Problem Statement and Necessary Conditions

So, consider the following problem:

$$\sum_{j=1}^3 \frac{\partial^2 u}{\partial x_j^2} = 0, \quad x \in D \subset R^3, \quad (1)$$

$$\sum_{j=1}^3 \alpha_j(x) \frac{\partial u(x)}{\partial x_j} = \alpha(x), \quad x \in \Gamma = \overline{D} \setminus D, \quad (2)$$

where  $D$  is a bounded domain with boundary  $\Gamma$ .

Let there hold true the following restrictions:

$I^0$ . The boundary  $\Gamma$  of the bounded domain  $D \subset R^3$  is Liapunov surface.

$2^0$ . The coefficients  $\alpha_j(x)$  ( $j = \overline{1,3}$ ) of boundary condition (2) belong to some Holder class and  $\alpha(x)$  is a continuous function.

Then holds true

Theorem 1. On condition  $I^0$  any solution to equation (1) defined in  $D$  satisfies the relationships:

$$\begin{aligned} u(x) &= 2 \int_{\Gamma} \frac{dU(x-\xi)}{dn_{\xi}} u(\xi) d\xi - \\ & 2 \sum_{j=1}^3 \int_{\Gamma} U(x-\xi) \cdot \frac{\partial u(\xi)}{\partial \xi_j} \cos(n_{\xi}, x_j) d\xi \\ \frac{\partial u(x)}{\partial x_i} &= 2 \int_{\Gamma} \frac{dU(x-\xi)}{dn_{\xi}} \cdot \frac{\partial u(\xi)}{\partial \xi_i} d\xi + \\ & 2 \sum_{\substack{j=1 \\ j \neq i}}^3 \int_{\Gamma} \left[ \frac{\partial U(x-\xi)}{\partial \xi_i} \cos(n_{\xi}, x_j) - \frac{\partial U(x-\xi)}{\partial \xi_j} \cos(n_{\xi}, x_i) \right] \\ & \frac{\partial u(\xi)}{\partial \xi_j} d\xi, \quad (i = \overline{1,3}, x \in \Gamma); \end{aligned} \quad (3)$$

where  $U(x-\xi)$  is a fundamental solution to equation (1) and  $n_{\xi}$  is an outer normal to the boundary  $\Gamma$  at point  $\xi$ .

Really, the first of expressions (3) is known from the general course of equations of Math Physics [18]. For the proof of the second expression let us do the following. Supposing that  $u(x)$  is the solution of equation (1) and multiplying both parts of these equations written at point  $\xi$

by function  $\frac{\partial U(x-\xi)}{\partial \xi_k}$  where  $U(x-\xi)$  is a fundamental solution of Laplace equation, integrating by domain  $D$  and applying Gauss-Ostrogradsky's formula [4] we will obtain:

$$\begin{aligned} 0 &= \int_D \sum_{j=1}^3 \frac{\partial U(x-\xi)}{\partial \xi_k} \cdot \frac{\partial^2 u(\xi)}{\partial \xi_j^2} d\xi \\ &= \int_D \frac{\partial U(x-\xi)}{\partial \xi_k} \cdot \frac{\partial^2 u(x)}{\partial \xi_k^2} d\xi + \sum_{\substack{j=1 \\ j \neq k}}^3 \int_D \frac{\partial U(x-\xi)}{\partial \xi_k} \cdot \frac{\partial^2 u(\xi)}{\partial \xi_j^2} d\xi = \\ &= \int_{\Gamma} \frac{\partial U(x-\xi)}{\partial \xi_k} \cdot \frac{\partial u(\xi)}{\partial \xi_k} \cos(n_{\xi}, x_k) d\xi - \\ & \int_D \frac{\partial^2 U(x-\xi)}{\partial \xi_k^2} \cdot \frac{\partial u(\xi)}{\partial \xi_k} + \sum_{\substack{j=1 \\ j \neq k}}^3 \int_{\Gamma} \left[ \frac{\partial U(x-\xi)}{\partial \xi_k} \cdot \frac{\partial u(\xi)}{\partial \xi_j} \cos(n_{\xi}, x_j) - \right. \\ & \left. - \frac{\partial U(x-\xi)}{\partial \xi_j} \cdot \frac{\partial u(\xi)}{\partial \xi_k} \cos(n_{\xi}, x_k) + \frac{\partial U(x-\xi)}{\partial \xi_j} \cdot \frac{\partial u(\xi)}{\partial \xi_k} \cos(n_{\xi}, x_j) \right] \\ & d\xi - \sum_{\substack{j=1 \\ j \neq k}}^3 \int_D \frac{\partial^2 U(x-\xi)}{\partial \xi_j^2} \cdot \frac{\partial u(\xi)}{\partial \xi_k} d\xi \end{aligned}$$

or

$$\begin{aligned} & \int_{\Gamma} \frac{\partial U(x-\xi)}{\partial n_{\xi}} \frac{\partial u(\xi)}{\partial \xi_k} d\xi + \\ & \sum_{\substack{j=1 \\ j \neq k}}^3 \int_{\Gamma} \left[ \frac{\partial U(x-\xi)}{\partial \xi_k} \cos(n_{\xi}, x_j) - \frac{\partial U(x-\xi)}{\partial \xi_j} \cos(n_{\xi}, x_k) \right] \\ & \frac{\partial u(\xi)}{\partial \xi_j} d\xi = \int_D \sum_{j=1}^3 \frac{\partial^2 U(x-\xi)}{\partial \xi_j^2} \frac{\partial u(\xi)}{\partial \xi_k} d\xi \end{aligned}$$

From this follows the second expression from (3) if to take in to account that  $U(x-\xi)$  is fundamental solution of Laplace equation [18], [2], [3].

Thus, for unknown functions  $u(x)$ ,  $\frac{\partial u(x)}{\partial x_j}$ , ( $j = \overline{1,3}$ )

we have five relationships: boundary condition (2) and four expressions entering (3). Remark that the last three relationships contain singular integrals.

## 3. Separation of Singularities

Let us try to replace these 3 relationships with two ones where singular integrals don't enter. For this, taking in to account that fundamental solution to Laplace equation has the form [3]

$$U(x-\xi) = -\frac{1}{4\pi} \frac{1}{|x-\xi|},$$

accepting the designations

$$\begin{aligned} K_{ij}(x, \xi) &= \cos(x-\xi, x_j) \cos(n_{\xi}, x_i) \\ & - \cos(x-\xi, x_i) \cos(n_{\xi}, x_j); \quad (i, j = \overline{1,3}) \end{aligned} \quad (4)$$

let us make the following expressions:

$$\begin{aligned} \sum_{i=1}^3 \beta_i(x) \frac{\partial u(x)}{\partial x_i} &= 2 \int_{\Gamma} \frac{dU(x-\xi)}{dn_{\xi}} \times \sum_{i=1}^3 \frac{\partial u(\xi)}{\partial \xi_i} \beta_i(x) d\xi \\ &+ \frac{1}{2\pi} \int_{\Gamma} \frac{d\xi}{|x-\xi|^2} \left[ (K_{12}(x,\xi)\beta_2(x) + K_{13}(x,\xi)\beta_3(x)) \frac{\partial u(\xi)}{\partial \xi_1} + \right. \\ &+ (K_{21}(x,\xi)\beta_1(x) + K_{23}(x,\xi)\beta_3(x)) \times \\ &\left. \frac{\partial u(\xi)}{\partial \xi_2} + (K_{31}(x,\xi)\beta_1(x) + K_{32}(x,\xi)\beta_2(x)) \frac{\partial u(\xi)}{\partial \xi_3} \right] \end{aligned} \quad (5)$$

where the unknowns  $\beta_i(x)$  ( $i=\overline{1,3}$ ) are defined from the system:

$$\begin{cases} K_{12}(\xi,\xi)\beta_2(\xi) + K_{13}(\xi,\xi)\beta_3(\xi) = \alpha_1(\xi), \\ K_{21}(\xi,\xi)\beta_1(\xi) + K_{23}(\xi,\xi)\beta_3(\xi) = \alpha_2(\xi), \\ K_{31}(\xi,\xi)\beta_1(\xi) + K_{32}(\xi,\xi)\beta_2(\xi) = \alpha_3(\xi). \end{cases} \quad (6)$$

Then from representation (4) there easily follows the properly:

$$K_{ij}(x,\xi) + K_{ji}(x,\xi) = 0, \quad (i, j = \overline{1,3}),$$

by means of which the principal determinant of system (6) vanishes, i.e. the rank of the principal matrix is equal to two. For calculation of the rank of the augmented matrix let us designate the unit vectors by directions  $x-\xi$  and  $n_{\xi}$  as  $(x-\xi)^0$  and  $n_{\xi}^0$  respectively.

Then

$$\begin{aligned} (x-\xi)^0 &= (\cos(x-\xi, x_1), \cos(x-\xi, x_2), \cos(x-\xi, x_3)), \\ n_{\xi}^0 &= (\cos(n_{\xi}, x_1), \cos(n_{\xi}, x_2), \cos(n_{\xi}, x_3)). \end{aligned}$$

Consider the unit vector  $\sigma^0(x,\xi)$  defined as follows:

$$\sigma^0(x,\xi) = \left[ (x-\xi)^0, n_{\xi}^0 \right].$$

Taking into account (4), for  $\sigma^0(x,\xi)$  we shall obtain:

$$\begin{aligned} \sigma^0(x,\xi) &= \begin{vmatrix} i_1 & i_2 & i_3 \\ \cos(x-\xi, x_1) & \cos(x-\xi, x_2) & \cos(x-\xi, x_3) \\ \cos(n_{\xi}, x_1) & \cos(n_{\xi}, x_2) & \cos(n_{\xi}, x_3) \end{vmatrix} = \\ &= \left( (\cos(x-\xi, x_2)\cos(n_{\xi}, x_3) - \cos(x-\xi, x_3)\cos(n_{\xi}, x_2)), \right. \\ &- (\cos(x-\xi, x_1)\cos(n_{\xi}, x_3) - \cos(x-\xi, x_3)\cos(n_{\xi}, x_1)), \\ &\left. (\cos(x-\xi, x_1)\cos(n_{\xi}, x_2) - \cos(x-\xi, x_2)\cos(n_{\xi}, x_1)) \right) = \\ &= (K_{23}(x,\xi), K_{31}(x,\xi), K_{12}(x,\xi)). \end{aligned}$$

Thus,  $K_{ij}(x,\xi)$  are direction cosines of the unit vector lying on the tangent plane, drawn to the boundary  $\Gamma$  at point

$\xi$ . Furthermore, accepting designations:

$$\begin{aligned} \lim_{x \rightarrow \xi} (x-\xi)^0 &= \tau_{\xi}^0, \\ \lim_{x \rightarrow \xi} \sigma^0(x,\xi) &= \sigma_{\xi}^0, \end{aligned}$$

it is easy that  $(\tau_{\xi}^0, n_{\xi}^0, \sigma_{\xi}^0)$  form a right triple, i.e. they are orthonormed and directed so that their mixed product is equal to unit.

Now, let us calculate the rank of the augmented matrix of system (6):

$$\begin{pmatrix} 0 & K_{12}(\xi,\xi) & K_{13}(\xi,\xi) & \alpha_1(\xi) \\ K_{21}(\xi,\xi) & 0 & K_{23}(\xi,\xi) & \alpha_2(\xi) \\ K_{31}(\xi,\xi) & K_{32}(\xi,\xi) & 0 & \alpha_3(\xi) \end{pmatrix}.$$

As the first determinant of third order vanishes let us consider the following determinants:

$$\begin{aligned} \begin{vmatrix} 0 & K_{12} & \alpha_1 \\ K_{21} & 0 & \alpha_2 \\ K_{31} & K_{32} & \alpha_3 \end{vmatrix} &= K_{12}K_{31}\alpha_2 + K_{21}K_{32}\alpha_1 - K_{12}K_{21}\alpha_3 \\ &= K_{21}(\xi,\xi)[K_{32}(\xi,\xi)\alpha_1(\xi) + K_{13}(\xi,\xi)\alpha_2(\xi) + K_{21}(\xi,\xi)\alpha_3(\xi)] = \\ &= K_{12}(\xi,\xi)[K_{23}(\xi,\xi)\alpha_1(\xi) + K_{31}(\xi,\xi)\alpha_2(\xi) + K_{12}(\xi,\xi)\alpha_3(\xi)] \\ &= K_{12}(\xi,\xi)(\sigma_{\xi}^0, \bar{\alpha}(\xi)), \end{aligned}$$

where  $\bar{\alpha}(\xi) = (\alpha_1(\xi), \alpha_2(\xi), \alpha_3(\xi))$

$$\begin{aligned} \begin{vmatrix} 0 & K_{13} & \alpha_1 \\ K_{21} & K_{23} & \alpha_2 \\ K_{31} & 0 & \alpha_3 \end{vmatrix} &= K_{13}K_{31}\alpha_2 - K_{31}K_{23}\alpha_1 - K_{13}K_{21}\alpha_3 \\ &= K_{13}[K_{23}\alpha_1 + K_{31}\alpha_2 + K_{12}\alpha_3] = K_{13}(\xi,\xi)(\sigma_{\xi}^0, \bar{\alpha}(\xi)), \end{aligned}$$

and, at last,

$$\begin{aligned} \begin{vmatrix} K_{12} & K_{13} & \alpha_1 \\ 0 & K_{23} & \alpha_2 \\ K_{32} & 0 & \alpha_3 \end{vmatrix} &= K_{12}K_{23}\alpha_3 + K_{13}K_{32}\alpha_2 - K_{23}K_{32}\alpha_1 \\ &= K_{23}(\xi,\xi)(\sigma_{\xi}^0, \bar{\alpha}(\xi)). \end{aligned}$$

From the obtained expressions it follow that for compatibility of system (6) by Croneker-Capelli theorem the rank of the augmented matrix must be equal to two.

Then, taking in to account that vector  $\sigma_{\xi}^0$  is unit, we obtain:

$$(\sigma_{\xi}^0, \bar{\alpha}(\xi)) = 0. \quad (7)$$

### 4. Regularization

As it is seen from the problem statement, vector  $\bar{\alpha}(\xi)$  doesn't depend on us. So, relationship (7) to hold true it is sufficient for each  $x \in \Gamma$  to tend point  $\xi$  ( $\xi$  is the integral

variable) to point  $x$  by reasonable way.

Then from system (6) (this system in vector designations has the form  $[\bar{\beta}(\xi) \sigma_\xi^0] = \bar{\alpha}(\xi)$  for the vector

$$\bar{\beta}(\xi) = (\beta_1(\xi), \beta_2(\xi), \beta_3(\xi))$$

we obtain two linear-independent solutions.

Let us designate them as

$$\bar{\beta}^{(k)}(\xi) = (\beta_1^{(k)}(\xi), \beta_2^{(k)}(\xi), \beta_3^{(k)}(\xi)), (k=1,2) \quad (8)$$

Remark that if  $K_{12}(\xi, \xi) \neq 0$  then

$$\bar{\beta}^{(1)}(\xi) = \left( -\frac{\alpha_2(\xi)}{K_{12}(\xi, \xi)}, \frac{\alpha_1(\xi)}{K_{12}(\xi, \xi)}, 0 \right),$$

$$\bar{\beta}^{(2)}(\xi) = \left( K_{23}(\xi, \xi) - \frac{\alpha_2(\xi)}{K_{12}(\xi, \xi)}, \frac{\alpha_1(\xi)}{K_{12}(\xi, \xi)}, -K_{13}(\xi, \xi), K_{12}(\xi, \xi) \right)$$

are linear-independent solutions of system (6).

As  $\sigma_\xi^0$  is a unit vector then one of the component must be different from zero.

Thus, taking into account (8) from (5) we obtain:

$$\sum_{i=1}^3 \beta_i^{(k)}(x) \frac{\partial u(x)}{\partial x_i} = 2 \int_{\Gamma} \frac{dU(x-\xi)}{dn_\xi} \times$$

$$\sum_{i=1}^3 \frac{\partial u(\xi)}{\partial \xi_i} \beta_i^{(k)}(x) d\xi + \frac{1}{2\pi} \int_{\Gamma} \frac{d\xi}{|x-\xi|^2} \beta_2^{(k)}(x) +$$

$$+ K_{13}(x, \xi) \beta_3^{(k)}(x) - \eta K_{12}(\xi, \xi) \beta_2^{(k)}(\xi) -$$

$$\eta K_{13}(\xi, \xi) \beta_3^{(k)}(\xi) \frac{\partial u(\xi)}{\partial \xi_1} +$$

$$+ (K_{21}(x, \xi) \beta_1^{(k)}(x) + K_{23}(x, \xi) \beta_3^{(k)}(x) - \eta K_{21}(\xi, \xi) \beta_1^{(k)}(\xi) -$$

$$\eta K_{23}(\xi, \xi) \beta_3^{(k)}(\xi)) \frac{\partial u(\xi)}{\partial \xi_2} +$$

$$+ (K_{31}(x, \xi) \beta_1^{(k)}(x) + K_{32}(x, \xi) \beta_2^{(k)}(x) -$$

$$\eta K_{31}(\xi, \xi) \beta_1^{(k)}(\xi) - \eta K_{32}(\xi, \xi) \beta_2^{(k)}(\xi)) \frac{\partial u(\xi)}{\partial \xi_3} \Big] + \frac{1}{2\pi} \int_{\Gamma} \frac{\eta d\xi}{|x-\xi|^2} \alpha(\xi);$$

$x \in \Gamma, (k=1,2)$

where  $\eta = \eta(x-\xi), |\eta|=1$ .

In this way there is proved

Theorem 2. On conditions  $1^0-2^0$  for the solution of problem (1) – (2) there hold true the following regular relationships:

$$\sum_{i=1}^3 \beta_i^{(k)}(x) \frac{\partial u(x)}{\partial x_i} = 2 \int_{\Gamma} \frac{dU(x-\xi)}{dn_\xi}$$

$$\times \sum_{i=1}^3 \beta_i^{(k)}(x) \frac{\partial u(\xi)}{\partial \xi_i} d\xi + \frac{1}{2\pi} \int_{\Gamma} \frac{d\xi}{|x-\xi|^2} \times$$

$$\{ [(K_{12}(x, \xi) - \eta K_{12}(\xi, \xi)) \beta_2^{(k)}(x) +$$

$$+ \eta K_{12}(\xi, \xi) (\beta_2^{(k)}(x) - \beta_2^{(k)}(\xi))] +$$

$$(K_{13}(x, \xi) - \eta K_{13}(\xi, \xi)) \beta_3^{(k)}(x) + \eta K_{13}(\xi, \xi) (\beta_3^{(k)}(x) - \beta_3^{(k)}(\xi))] \}$$

$$\frac{\partial u(\xi)}{\partial \xi_1} + [(K_{12}(x, \xi) - \eta K_{21}(\xi, \xi)) \beta_1^{(k)}(x) +$$

$$+ \eta K_{21}(\xi, \xi) (\beta_1^{(k)}(x) - \beta_1^{(k)}(\xi))] +$$

$$+ (K_{23}(x, \xi) - \eta K_{23}(\xi, \xi)) \beta_3^{(k)}(x) +$$

$$\eta K_{23}(\xi, \xi) (\beta_3^{(k)}(x) - \beta_3^{(k)}(\xi))] \frac{\partial u(\xi)}{\partial \xi_2} +$$

$$+ [(K_{31}(x, \xi) - \eta K_{31}(\xi, \xi)) \beta_1^{(k)}(x) + \eta K_{31}(\xi, \xi) \times$$

$$(\beta_1^{(k)}(x) - \beta_1^{(k)}(\xi))] + (K_{32}(x, \xi) - \eta K_{32}(\xi, \xi)) \beta_2^{(k)}(x) +$$

$$+ \eta K_{32}(\xi, \xi) (\beta_2^{(k)}(x) - \beta_2^{(k)}(\xi))] \frac{\partial u(\xi)}{\partial \xi_3} \Big\} +$$

$$\frac{1}{2\pi} \int_{\Gamma} \alpha(\xi) \frac{\eta d\xi}{|x-\xi|^2}, \quad x \in \Gamma, (k=1,2). \quad (9)$$

Theorem 3. On conditions  $1^0-2^0$  if  $|\bar{\alpha}(\xi)| \neq 0$  then vectors  $\bar{\alpha}(\xi), \bar{\beta}^{(1)}(\xi)$  and  $\bar{\beta}^{(2)}(\xi)$  are linear independent.

Indeed, without loss of generality, supposing that  $K_{12}(\xi, \xi) \neq 0$ , we have:

$$(\bar{\alpha}(\xi) \bar{\beta}^{(1)}(\xi) \bar{\beta}^{(2)}(\xi)) =$$

$$\begin{vmatrix} \alpha_1(\xi) & \alpha_2(\xi) & \alpha_3(\xi) \\ -\frac{\alpha_2(\xi)}{K_{12}(\xi, \xi)} & \frac{\alpha_1(\xi)}{K_{12}(\xi, \xi)} & 0 \\ K_{23} - \frac{\alpha_2(\xi)}{K_{12}(\xi, \xi)} & \frac{\alpha_1(\xi)}{K_{12}(\xi, \xi)} - K_{13} & K_{12} \end{vmatrix}$$

$$= \alpha_1^2 - \frac{\alpha_2 \alpha_3}{K_{12}} \left( \frac{\alpha_1}{K_{12}} - K_{13} \right) - \frac{\alpha_1 \alpha_3}{K_{12}} \left( K_{23} - \frac{\alpha_2}{K_{12}} \right) + \alpha_2^2$$

$$= \alpha_1^2 + \alpha_2^2 + \frac{\alpha_3}{K_{12}} (\alpha_2 K_{13} - \alpha_1 K_{23}) =$$

$$= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 (\sigma_\xi^0, \bar{\alpha}(\xi)) = |\bar{\alpha}(\xi)|^2.$$

Remark: The statement of theorem 3 is equivalent to Lopatinsky's condition [20].

Remark: In this way there are given sufficient conditions for the validity of system (7.73) from Bitsadze A.V. book [2] and defined the coefficients of this system.

### 5. Fredholm Property

Theorem 4. On the conditions of theorem 3 if  $\alpha(x)$

belongs to some Holder class then the stated problem (1) – (2) has Fredholm property.

Indeed, it follows from theorem 3 that relationships (2) and (9) enable to obtain a normal system of Fredholm integral equations of second kind for functions  $\frac{\partial u(x)}{\partial x_i}$ , ( $i = \overline{1,3}$ ),  $x \in \Gamma$ . If to adjoin the first expression

from (3) to the obtained a compact, i.e. complete system. The obtained system is a Fredholm one of second kind and the kernel contains a weak singularity.

Remark: If  $\alpha_1(x) \equiv \alpha_2(x) \equiv 0$ ,  $\alpha_3(x) \equiv 1$  then, evidently, from (7),  $K_{12}(\xi, \xi) \equiv 0$ , and for validity of system (6)  $\beta_3(\xi) \equiv 0$ . In this case system (6) has the unique normal solution  $(K_{31}(\xi, \xi), K_{32}(\xi, \xi), 0)$ . That is why the statement of theorem 3 doesn't hold true and the stated problem is not Fredholm.

Theorem 5. If on conditions  $1^0 - 2^0$  the obtained system of Fredholm integral equations of second kind is solvable then the solution of boundary-value problem (1) – (2) is represented in the form:

$$u(x) = \int_{\Gamma} \frac{dU(x-\xi)}{dn_{\xi}} u(\xi) d\xi - \sum_{j=1}^3 \int_{\Gamma} U(x-\xi) \frac{\partial U(\xi)}{\partial \xi_j} \cos(n_{\xi}, x_j) d\xi, \quad (10)$$

$$x \in D.$$

Remark: If the statement of theorem 3 doesn't hold true along some manifold  $\Gamma_0 \subset \Gamma$  then an additional boundary condition on  $\Gamma_0$  should be given for Fredholm property of the stated problems.

By this method there were investigated various problems some of which we enumerate:

1. Boundary-value problems with non-local conditions for an equation of elliptic type [21], [22], [23].

2. Boundary-value problems for integro-differential equation with partial derivatives with nonlocal and global addends in a boundary condition [24], [25].

3. Cauchy problem and boundary-value problem for Navie-Stoks equation [26], [27], [28], [29], [30].

4. Cauchy problem and mixed problem for an equation of hyperbolic type [31], [32], [33].

5. Stephan's inverse problem [34], [35].

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