Existence of Coupled Solutions of BVP for ϕ -Laplacian Impulsive Differential Equations

Xiufeng Guo

College of Sciences, Hezhou University, Hezhou, China

Email address:

llgxf88@163.com

To cite this article:

Xiufeng Guo. Existence of Coupled Solutions of BVP for ϕ -Laplacian Impulsive Differential Equations. *Science Journal of Applied Mathematics and Statistics*. Vol. 4, No. 6, 2016, pp. 298-302. doi: 10.11648/j.sjams.20160406.18

Received: November 4, 2016; **Accepted:** November 25, 2016; **Published:** December 14, 2016

Abstract: In this paper, we study the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with ϕ -Laplacian operator. Based on a pair of coupled lower and upper solutions and appropriate Nagumo condition, we prove the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with ϕ -Laplacian operator.

Keywords: Boundary Value Problems, Coupled Solutions, Impulsive Differential Equations, ϕ -Laplacian Operator

1. Introduction

In recent years, the study boundary value problems (BVPs for short) with p -Laplacian operator has been emerging as an important area and obtained a considerable attention. Since p -Laplacian operator appears in the study of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$), glaciology ($1 \leq p \leq 4/3$) and so on, there are many works about existence of solutions for differential equations with p -Laplacian operator [24, 25]. Usually, p -Laplacian operator is replaced by abstract and more general version ϕ -Laplacian operator, which lead to clearer expositions and a better understanding of the methods which were employed to derive the existence results [12, 22, 23].

Moreover, impulsive differential equations have become an important aspect in some mathematical models of real processes and phenomena in science. There has a significant development in impulsive differential equations and impulse theory (see [2, 3, 14]). Moreover, p -Laplacian operator arises in turbulent filtration in porous media, non-Newtonian fluid flows and in many other application areas [10, 12].

Furthermore, the study of anti-periodic problem for nonlinear evolution equations is closely related to the study of periodic problem which was initiated by Okochi [17]. Anti-periodic problem which is a very important area of research has been extensively studied during the past decades, such as anti-periodic trigonometric polynomials [11] and

anti-periodic wavelets [4]. Moreover, anti-periodic boundary conditions also appear in physics in a variety of situations (see [1, 13]) and difference and differential equations (see [6, 8, 19, 20]). The anti-periodic problem is a very important area of research.

In addition, we know that every T -anti-periodic solution gives rise to a $2T$ -periodic solution if the nonlinearity f satisfy some symmetry condition. Indeed, the periodic and anti-periodic boundary value problems have attracted many researchers great interest (see [6, 8, 9, 15, 16, 19, 20, 21] and references therein). Recently, Guo and Gu [22] study a class of nonlinear impulsive differential equation with anti-periodic boundary condition:

$$(\phi(u'(t)))' = f(t, u(t), u'(t)) \quad a.e. \quad t \in [0, T], P, \quad (1)$$

$$\begin{cases} I_k(u(t_k), u(t_k^+)) = 0, \\ M_k(u(t_k), u(t_k^+), u'(t_k), u'(t_k^+), u) = 0, \end{cases} \quad k = 1, 2, \dots, p, \quad (2)$$

$$u(0) = -u(T), u'(0) = -u'(T), \quad (3)$$

where ϕ is an increasing homeomorphism from R to R , $f: [0, T] \times R^2 \rightarrow R$ is a Carathéodory function. $P = \{t_1, \dots, t_p : 0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T\}$, $I_k \in C^0(R^2)$, $M_k \in C^0(R^4 \times C_p^1)$, $k = 1, \dots, p$ are impulsive functions. C_p^1 will be given later. In [22], the authors obtained the existence of solution for anti-periodic boundary value problems (1)-(3)

for impulsive differential equations with ϕ -Laplacian operator. In this paper, we will continuous to consider the existence of coupled solutions for boundary value problems (1)-(3).

This paper is organized as follows: In section 2, we will state some preliminaries that will be used throughout the paper. In section 3, we will obtain the existence of coupled solutions for anti-periodic ϕ -Laplacian impulsive differential equations boundary value problems (1)-(3).

2. Preliminaries

In this section, we will introduce some definitions and preliminaries which are used throughout this paper.

For a given Banachspace E , let $C^0(E)$ be the set of all continuous functions $f: E \rightarrow R$. Let $C^m(I)$ be the set of functions u which are m times continuously differentiable on I with finite norm

$$\|u\|_{C^m(I)} = \max_{k=0, \dots, m} \|u^{(k)}\|_{\infty}.$$

For $1 \leq q \leq \infty$, Let $L^q(I)$ be the set of Lebesgue measurable functions u on I such that $\|u\|_q$ is finite.

$AC(I, q)$ denotes the set of absolutely continuous functions u on I satisfy $u' \in L^q(I)$. $W^{m, q}(I)$ denotes the set of functions $u \in C^{m-1}(I)$ and $u^{(m-1)} \in AC(I, q)$ with finite norm

$$\|u\|_{W^{m, q}(I)} = \max_{k=0, \dots, m} \|u^{(k)}\|_q.$$

It is easy to see that $C^m(I)$ and $W^{m, q}(I)$ are Banach spaces and $W^{m, q}(I)$ is a usual Sobolev space.

Let $p \in N$. A finite subset P of the interval $[0, T]$ defined by

$$P = \{t_1, \dots, t_p : 0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T\}.$$

Let $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ for all $k = 1, \dots, p$. For $m \in N \cup \{0\}$ and $1 \leq q \leq \infty$, we denote

$$\begin{aligned} C_p^m &= \{u : [0, T] \rightarrow R : \text{for all } k = 0, \dots, p, u \in C^m(J_k), \text{ there exist } u^{(l)}(t_k^+), \\ &\quad k = 1, \dots, p \text{ and } u^{(l)}(t_k^-) = u^{(l)}(t_k), k = 1, \dots, p+1; l = 0, \dots, m\}, \\ W_p^{m, q} &= \{u : [0, T] \rightarrow R : u_{|J_k} \in W^{m, q}(J_k), k = 0, \dots, p\}. \end{aligned}$$

It is easy to verify that the spaces C_p^m and $W_p^{m, q}$ are Banach spaces with the norms

$$\|u\|_{C_p^m} = \max_{k=0, \dots, p} \|u_{|J_k}\|_{C^m(J_k)} \quad \text{and} \quad \|u\|_{W_p^{m, q}} = \max_{k=0, \dots, p} \|u_{|J_k}\|_{W^{m, q}(J_k)}.$$

We say that $f : [0, T] \times S \rightarrow R$ ($S \subset R^2$) satisfies the restricted Carathéodory conditions on $[0, T] \times S$ if

- for each $x \in S$ the function $f(\cdot, x)$ is measurable on $[0, T]$;

- the function $f(t, \cdot)$ is continuous on S a.e. $t \in [0, T]$;
- for every compact set $K \subset S$, there exists a nonnegative function $\mu_K \in L^1(0, T)$ such that

$$|f(t, x)| \leq \mu_K(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x \in K.$$

In this paper, we use $\text{Car}([0, T] \times S)$ to denote the set of functions satisfying the *restricted Carathéodory conditions* on $[0, T] \times S$. In what follows, D^+ and D_- denote the Dini derivatives.

Definition 1. The functions $\alpha, \beta \in W_p^{1, \infty}$ such that $\alpha \leq \beta$ are said to be a pair of coupled lower and upper solutions of problem (1)-(3) if α, β satisfy the following conditions:

- $D_- \alpha(t) \leq D^+ \alpha(t)$ for all $t \in [0, T], P$. Moreover, if $\tau \in [0, T], P$ such that $D_- \alpha(\tau) = D^+ \alpha(\tau)$, then there exists $\varepsilon > 0$ such that

$$\alpha \in C^1([\tau - \varepsilon, \tau + \varepsilon]),$$

$$\phi \circ \alpha' \in AC([\tau, \tau + \varepsilon])$$

and

$$(\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \quad \text{a.e. } t \in [\tau, \tau + \varepsilon].$$

- $D_- \beta(t) \geq D^+ \beta(t)$ for all $t \in [0, T], P$. Moreover, if $\tau \in [0, T], P$ such that $D_- \beta(\tau) = D^+ \beta(\tau)$, then there exists $\varepsilon > 0$ such that

$$\beta \in C^1([\tau - \varepsilon, \tau + \varepsilon]),$$

$$\phi \circ \beta' \in AC([\tau, \tau + \varepsilon])$$

and

$$(\phi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t)) \quad \text{a.e. } t \in [\tau, \tau + \varepsilon].$$

- For all $k = 1, \dots, p$, $I_k(\alpha(t_k), \cdot)$ are injective and there exist $D^+ \alpha(t_k)$, $D_- \alpha(t_k)$, $D_+ \beta(t_k)$, $D_- \beta(t_k) \in R$ such that

$$\begin{aligned} I_k(\alpha(t_k), \alpha(t_k^+)) &= 0 \leq M_k(\alpha(t_k), \alpha(t_k^+), D_- \alpha(t_k), D^+ \alpha(t_k), \alpha), \\ I_k(\beta(t_k), \beta(t_k^+)) &= 0 \geq M_k(\beta(t_k), \beta(t_k^+), D^- \beta(t_k), D_+ \beta(t_k), \beta). \end{aligned}$$

and there exist $D^+ \alpha(0)$, $D_- \alpha(T)$, $D_+ \beta(0)$, $D^- \beta(T) \in R$ such that

$$\alpha(0) + \beta(T) = 0 \leq D^+ \alpha(0) + D^- \beta(T),$$

$$\alpha(T) + \beta(0) = 0 \geq D_- \alpha(T) + D_+ \beta(0).$$

Definition 2. Given a function $u \in C_p^1$ is called a solution of the problem (1)-(3) if $\phi \circ u' \in W_p^{1, 1}$ and u satisfies (1) and fulfills conditions (2) and (3).

Definition 3. Assume that $f \in \text{Car}([0, T] \times R^2)$ and

$\alpha, \beta \in W_p^{1,\infty}$ satisfying $\alpha(t) \leq \beta(t)$ for $\forall t \in [0, T]$. We say that f satisfies a Nagumo condition with respect to α and β if, for $k=1, \dots, p$, there exist $\phi_k \in C[0, \infty)$ and $w \in L^q(0, T)$, $1 \leq q \leq \infty$, such that $\phi_k > 0$ on $[0, \infty)$,

$$|f(t, u, v)| \leq w(t)\phi_k(|v|) \text{ on } J_k \times [\alpha(t), \beta(t)] \times R.$$

Moreover, there exists a constant $K = K(\alpha, \beta)$ with $K > \max\{r_k, \|\alpha'\|_\infty, \|\beta'\|_\infty\}$, such that

$$\begin{aligned} \int_{\phi(r_k)}^{\phi(K)} \frac{(\phi^{-1}(x))^{(q-1)/q}}{\phi_k(\phi^{-1}(x))} dx &> \|w\|_{J_k, q} \eta_k^{(q-1)/q}, \text{ or} \\ -\int_{\phi(-K)}^{\phi(-r_k)} \frac{(\phi^{-1}(x))^{(q-1)/q}}{\phi_k(-\phi^{-1}(x))} dx &> \|w\|_{J_k, q} \eta_k^{(q-1)/q}, \end{aligned} \quad (4)$$

where $\eta_k = \sup_{t \in J_k} \beta(t) - \inf_{t \in J_k} \alpha(t)$ and

$$r_k = \frac{1}{t_{k+1} - t_k} \max\{\beta(t_{k+1}^-) - \alpha(t_k^+), \beta(t_k^+) - \alpha(t_{k+1}^-)\}. \text{ Any}$$

constant such $K > \max\{r_k : k=0, \dots, p\} > 0$ will be called a Nagumo constant.

Throughout this paper, we impose the following hypotheses:

(H₁) The function $\varphi : R \rightarrow R$ is a continuous and strictly increasing.

(H₂) The BVP (1)-(3) has a pair of coupled lower and upper solutions α and β .

(H₃) $f \in \text{Car}([0, T] \times R^2)$ and satisfies a Nagumo condition with respect to α and β .

(H₄) The functions $I_k \in C^0(R^2)$ are non-decreasing in the first variable for $k=1, \dots, p$, and the functions $M_k \in C^0(R^4 \times C_p^1)$ are non-increasing in the third variable and non-decreasing in the fourth and fifth variables.

3. Existence Results of Coupled Solutions

This section is devoted to proving the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with ϕ -Laplacian operator. Firstly, we state the following existence and uniqueness result.

Lemma 1. (Lemma 7 of [23]) Assume that $\tilde{f} \in L^1[0, T]$ and $A_k, B_k \in R$ for each $k=0, \dots, p$. Suppose that $\bar{\varphi} : R \rightarrow R$ is a strictly increasing function satisfies $\bar{\varphi}(R) = R$. Then the non homogeneous impulsive Dirichlet problem

$$\begin{cases} (\bar{\varphi}(u'(t)))' = \tilde{f}(t) \text{ a.e. } t \in [0, T], \\ u(t_k) = B_{k-1}, u(t_k^+) = A_k, \quad k=1, 2, \dots, p, \\ u(0) = A_0, u(T) = B_p, \end{cases}$$

has a unique solution u , which can be written in the form

$$u(t) = A_k + \int_{t_k}^t \bar{\varphi}^{-1}(\int_{t_k}^z \tilde{f}(s) ds + \tau_k) dz, \quad t \in J_k, k=0, \dots, p,$$

where τ_k is the unique solution of the equation

$$B_k - A_k = \int_{t_k}^{t_{k+1}} \bar{\varphi}^{-1}(\int_{t_k}^z \tilde{f}(s) ds + \tau_k) dz.$$

Next, let us consider the following functions

$$\delta_K(y) = \min\{K, \max\{y, -K\}\} \text{ for all } y \in R,$$

where K is the constant introduced in definition 2.3,

$$\rho(t, u) = \min\{\beta(t), \max\{u, \alpha(t)\}\} \text{ for } (t, u) \in [0, T] \times R,$$

coupled with functionals $A_k, B_k : C_p^1 \rightarrow R$ given by

$$A_0(u) = \rho(0, -u(T)),$$

$$B_p(u) = \rho(T, u(T) - u'(0) - u'(T)),$$

$$A_k(u) = \rho(t_k^+, u(t_k^+) + I_k(u(t_k), u(t_k^+))), \quad k=1, \dots, p,$$

$$B_{k-1}(u) = \rho(t_k, u(t_k) + M_k(u(t_k), u(t_k^+), u'(t_k), u'(t_k^+), u)), \quad k=1, \dots, p.$$

Moreover, for each $u \in C_p^1$ we consider a function $\tilde{f}_u : [0, T] \rightarrow R$ defined by

$$\tilde{f}_u(t) = f(t, \rho(t, u(t)), \delta_K(\frac{d}{dt} \rho(t, u(t)))).$$

The function $\tilde{f}_u$ is well defined according to the following result (by redefining function $\frac{d}{dt} \rho(t, u(t))$ as zero when it does not exist). It can be proved in a similar way to Lemma 2 in [24].

Lemma 2. For given $u, u_n \in C_p^1$ such that $u_n \rightarrow u$ in C_p^1 , then

- (i) $\frac{d}{dt} \rho(t, u(t))$ exists for a.e. $t \in [0, T], P$;
- (ii) $\frac{d}{dt} \rho(t, u_n(t)) \rightarrow \frac{d}{dt} \rho(t, u(t))$ for a.e. $t \in [0, T], P$.

Now, we can define a strictly increasing homeomorphism $\bar{\varphi} : R \rightarrow R$ by:

$$x \in R \rightarrow \bar{\varphi}(x) = \begin{cases} \varphi(x), & |x| \leq K, \\ \frac{\varphi(K) - \varphi(-K)}{2K} x - \frac{1}{2}(\varphi(K) + \varphi(-K)), & |x| > K. \end{cases}$$

In the following, we are in a position to prove the existence theorem for our considering problems.

Lemma 3. (Theorem 3.3 of [22]) Assume that (H₁)-(H₄) hold. Then there exists at least one solution u of the problem (1)-(3) such that

$$\alpha(t) \leq u(t) \leq \beta(t)$$

and

$$|u'(t)| \leq K, \quad t \in [0, T],$$

where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

Next, we are devoted to the existence of coupled solutions. We first introduce the following definition.

Definition 4. The functions x, y are called coupled solutions of problems (1)-(3) if $x, y \in C_p^1$ and satisfy (1)-(2) and

$$x(0) = -y(T), \quad (5)$$

$$x'(0) = -y'(T), \quad (6)$$

$$y(0) = -x(T), \quad (7)$$

$$y'(0) = -x'(T). \quad (8)$$

Remark If the coupled solutions x and y of problem (1)-(3) satisfy $x = y$, the $x = y$ is a solution of problem (1)-(3).

Next, we give the existence of coupled solutions for problems (1)-(3).

Theorem 5. Assume hypotheses (H_1) -(H_4) hold. Then there exists at least a pair of coupled solutions $x, y \in C_p^1$ of the impulsive differential equations boundary value problem (1)-(3) such that

$$x, y \in [\alpha, \beta] = \{u : \alpha(t) \leq u(t) \leq \beta(t), t \in [0, T]\}, \quad (9)$$

and

$$|x'(t)| \leq K \text{ for } t \in [0, T],$$

$$|y'(t)| \leq K \text{ for } t \in [0, T],$$

where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

Proof. Let us define ρ, A_k, B_{k-1} for each $k = 1, \dots, p$ in the same way as above, and construct a modified problem (P^*) similar to the proof of Lemma 3, that is

$$\begin{cases} (\bar{\varphi}(x'(t)))' = \tilde{f}_x(t), & a.e. t \in [0, T], P, \\ (\bar{\varphi}(y'(t)))' = \tilde{f}_y(t), & a.e. t \in [0, T], P, \\ x(t_k) = B_{k-1}(x), & y(t_k) = B_{k-1}(y), \quad k = 1, 2, \dots, p, \\ x(t_k^+) = A_k(x), & y(t_k^+) = A_k(y), \quad k = 1, 2, \dots, p, \\ x(0) = A_0(x), & y(0) = A_0(y), \\ x(T) = B_p(x), & y(T) = B_p(y), \end{cases}$$

where

$$A_0(x) = \rho(0, -y(T)),$$

$$B_p(x) = \rho(T, x(T) - y'(0) - x'(T)),$$

$$A_0(y) = \rho(0, -x(T)),$$

$$B_p(y) = \rho(T, y(T) - x'(0) - y'(T)).$$

From the proof of the Lemma 3, there exists a couple of

solutions $x, y \in C_p^1$ such that

$$\alpha \leq x \leq \beta,$$

$$\alpha \leq y \leq \beta,$$

and

$$|x'(t)| \leq K, |y'(t)| \leq K \text{ for } t \in [0, T].$$

Furthermore, x, y satisfy the condition (2). Now, to prove that (5)-(8) is verified, it suffices to prove that

$$\alpha(0) \leq -y(T) \leq \beta(0), \quad (10)$$

$$\alpha(0) \leq -x(T) \leq \beta(0), \quad (11)$$

$$\alpha(T) \leq x(T) - y'(0) - x'(T) \leq \beta(T), \quad (12)$$

$$\alpha(T) \leq y(T) - x'(0) - y'(T) \leq \beta(T). \quad (13)$$

Firstly, we will prove (10), by contradiction, if $\alpha(0) > -y(T)$, then by $\alpha \leq y \leq \beta$, we have

$$\alpha(0) > -y(T) \geq -\beta(T),$$

which contradict to $\alpha(0) + \beta(T) = 0$. Moreover, $-y(T) \leq \beta(0)$ can be proved similarly.

As the same way, we can obtain that the inequality (10) is holds. Thus we have

$$x(0) = -y(T), \quad y(0) = -x(T). \quad (14)$$

Assume that the first inequality if (11) isn't holds, as a consequence, we have

$$x(T) = \alpha(T)$$

and

$$y'(0) + x'(T) > 0.$$

From (14) and $\alpha(T) + \beta(0) = 0$, we have

$$y(0) = -x(T) = -\alpha(T) = \beta(0).$$

From these facts and the relation $\alpha \leq x, y \leq \beta$, we have

$$x'(T) \leq D_- \alpha(T), \quad y'(0) \leq D_+ \beta(0),$$

thus

$$0 < y'(0) + x'(T) \leq D_+ \beta(0) + D_- \alpha(T) \leq 0.$$

It is a contradiction. Moreover, the inequality in (13) be obtain in a similar way. Hence inequalities (11)-(12) are hold, that is to say x, y satisfy (5)-(8).

Therefore, the functions x, y is a coupled solutions of the problem (1)-(3), which completes the proof.

4. Conclusion

In this paper, we mainly discuss the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with ϕ -Laplacian operator. To give the existence results of coupled solutions for the problem (1)-(3), we first introduce a pair of coupled lower and upper solutions (see Definition 1). Then, we provide and prove the existence results of coupled solutions for anti-periodic ϕ -Laplacian impulsive differential equations boundary value problems based on a pair of coupled lower and upper solutions and appropriate Nagumo condition (Theorem 5).

Acknowledgments

The work was partially supported by NNSF of China Grants No.11461021, NNSF of Guangxi Grant No. 2014GXNSFAA118028, the Scientific Research Foundation of Guangxi Education Department No. KY2015YB306, the Scientific Research Project of Hezhou University Nos. 2015ZZZK16, 2016HZXYSX07, and Guangxi Colleges and Universities Key Laboratory of Symbolic Computation and Engineering Data Processing.

References

- [1] C. Ahn, C. Rim, Boundary flows in general coset theories, *J. Phys. A* 32 (1999) 2509-2525.
- [2] D. Bainov, V. Covachev, *Impulsive Differential Equations With a Small Parameter*, World Scientific, Singapore, 1994.
- [3] M. Benchohra, J. Henderson, S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, New York, 2006.
- [4] H. L. Chen, Antiperiodic wavelets, *J. Comput. Math.* 14 (1996) 32-39.
- [5] A. Cabada, D. R. Vivero, Existence and uniqueness of solutions of higher-order antiperiodic dynamic equations, *Adv. Difference Equ.* 4 (2004) 291-310.
- [6] A. Cabada, The method of lower and upper solutions for periodic and anti-periodic difference equations, *Electron. Trans. Numer. Anal.* 27 (2007) 13-25.
- [7] A. Cabada, An overview of the lower and upper solutions method with nonlinear boundary value conditions, *Bound. Value Probl.* (2011) 18. Art. ID 893753.
- [8] Y. Chen, J. J. Nieto, D. O'Regan, Anti-periodic solutions for fully nonlinear first-order differential equations, *Math. Comput. Model.* 46 (2007) 1183-1190.
- [9] Y. Chen, J. J. Nieto, D. O'Regan, Anti-periodic solutions for evolution equations associated with maximal monotone mappings, *Appl. Math. Lett.* 24 (2011) 302-307.
- [10] E. N. Dancer, On the Dirichlet problem for weakly non-linear elliptic partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 76 (1977) 283-300.
- [11] F. J. Delves, L. Knoche, Lacunary interpolation by anti-periodic trigonometric polynomials, *BIT* 39 (1999) 439-450.
- [12] X. Guo, L. Lu, Z. Liu, BVPs for higher-order integro-differential equations with ϕ -Laplacian and functional boundary conditions, *Adv. Differ. Equ.* 2014:285 (2014) 1-13.
- [13] H. Kleinert, A. Chervyakov, Functional determinants from Wronski Green function, *J. Math. Phys.* 40 (1999) 6044-6051.
- [14] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [15] S. P. Lu, Periodic solutions to a second order p -Laplacian neutral functional differential system, *Nonlinear Anal.* 69 (2008) 4215-4229.
- [16] Z. Luo, J. J. Nieto, New results of periodic boundary value problem for impulsive integro-differential equations, *Nonlinear Anal.* 70 (2009) 2248-2260.
- [17] H. Okochi, On the existence of periodic solutions to nonlinear abstract parabolic equations, *J. Math. Soc. Japan* 40(3) (1988) 541-553.
- [18] K. Perera, R. P. Agarwal, D. O'Regan, *Morse Theoretic Aspects of p -Laplacian Type Operators*, American Mathematical Society, Providence, Rhode Island, 2010.
- [19] W. Wang, J. Shen, Existence of solutions for anti-periodic boundary value problems, *Nonlinear Anal.* 70 (2009) 598-605.
- [20] R. Wu, The existence of T -anti-periodic solutions, *Appl. Math. Lett.* 23 (2010) 984-987.
- [21] M. P. Yao, A. M. Zhao, J. R. Yan, Anti-periodic boundary value problems of second order impulsive differential equations, *Comp. Math. Appl.* 59 (2010) 3617-362.
- [22] X. F. Guo, Y. Gu, Anti-periodic Boundary Value Problems of ϕ -Laplacian Impulsive Differential Equations, *Appl. Comput. Math.* 5(2) (2016) 91-96.
- [23] A. Cabada, J. Tomecek, Extremal solutions for nonlinear functional ϕ -Laplacian impulsive equations, *Nonlinear Anal.* 67(2007)827-841.
- [24] M. Wang, A. Cabada, J. J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Caratheodory functions, *Ann. Polon. Math.* 58(3) (1993) 221-235.
- [25] J. F. Xu, Z. L. Yang, Positive solutions for a fourth order p -Laplacian boundary value problem, *Nonlinear Anal.* 74 (2011) 2612-2623.