Existence of Coupled Solutions of BVP for φ-Laplacian Impulsive Differential Equations

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Abstract: In this paper, we study the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with φ-Laplacian operator. Based on a pair of coupled lower and upper solutions and appropriate Nagumo condition, we prove the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with φ-Laplacian operator.

Keywords: Boundary Value Problems, Coupled Solutions, Impulsive Differential Equations, φ-Laplacian Operator

1. Introduction

In recent years, the study boundary value problems (BVPs for short) with p-Laplacian operator has been emerging as an important area and obtained a considerable attention. Since p-Laplacian operator appears in the study of flow through porous media (p = 3/2), nonlinear elasticity (p ≥ 2), glaciology (1 ≤ p ≤ 4/3) and so on, there are many works about existence of solutions for differential equations with p-Laplacian operator [24, 25]. Usually, p-Laplacian operator is replaced by abstract and more general version φ-Laplacian operator, which lead to clearer expositions and a better understanding of the methods which ware employed to derive the existence results [12, 22, 23].

Moreover, impulsive differential equations have become an important aspect in some mathematical models of real processes and phenomena in science. There has a significant development in impulsive differential equations and impulse theory (see [2, 3, 14]). Moreover, p-Laplacian operator arises in turbulent filtration in porous media, non-Newtonian fluid flows and in many other application areas [10, 12].

Furthermore, the study of anti-periodic problem for nonlinear evolution equations is closely related to the study of periodic problem which was initiated by Okochi [17]. Anti-periodic problem which is a very important area of research has been extensively studied during the past decades, such as anti-periodic trigonometric polynomials [11] and anti-periodic wavelets [4]. Moreover, anti-periodic boundary conditions also appear in physics in a variety of situations (see [1, 13]) and difference and differential equations (see [6, 8, 19, 20]). The anti-periodic problem is a very important area of research.

In addition, we known that every T-anti-periodic solution gives rise to a 2T-periodic solution if the nonlinearity f satisfy some symmetry condition. Indeed, the periodic and anti-periodic boundary value problems have attracted many researchers great interest (see [6, 8, 9, 15, 16, 19, 20] and references therein). Recently, Guo and Gu [22] study a class of nonlinear impulsive differential equation with anti-periodic boundary condition:

\[
(\phi(u(t)))' = f(t, u(t), u'(t)) \quad a.e. \ t \in [0, T], P, \tag{1}
\]

\[
I_k(u(t_k), u(t_k^+)) = 0, \quad k = 1, 2, \ldots, p, \tag{2}
\]

\[
M_k(u(t_k), u(t_k^+), u'(t_k), u'(t_k^+), u) = 0, \quad k = 1, 2, \ldots, p, \tag{2}
\]

\[
u(0) = -u(T), \quad u'(0) = -u'(T). \tag{3}
\]

where \( \phi \) is an increasing homeomorphism from \( R \) to \( R \), \( f : [0, T] \times R^2 \to R \) is a Carathéodory function. \( P = \{ t_1, \ldots, t_p : 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T \} \), \( I_k \in C^0(R^2), M_k \in C^0(R^4 \times C'_p), k = 1, \ldots, p \) are impulsive functions. \( C'_p \) will be given later. In [22], the authors obtained the existence of solution for anti-periodic boundary value problems (1)-(3)
for impulsive differential equations with \( \phi \)-Laplacian operator. In this paper, we will continue to consider the existence of coupled solutions for boundary value problems (1)-(3).

This paper is organized as follows: In section 2, we will state some preliminaries that will be used throughout the paper. In section 3, we will obtain the existence of coupled solutions for anti-periodic \( \phi \)-Laplacian impulsive differential equations boundary value problems (1)-(3).

## 2. Preliminaries

In this section, we will introduce some definitions and preliminaries which are used throughout this paper.

For a given Banach space \( E \), let \( C^0(E) \) be the set of all continuous functions \( f : E \to R \). Let \( C^m(I) \) be the set of functions \( u \) which are \( m \) times continuously differentiable on \( I \) with finite norm

\[
\| u \|_{C^m(I)} = \max_{k=0,\ldots,m} \| u^{(k)} \|_E.
\]

For \( 1 \leq q \leq \infty \), let \( L^q(I) \) be the set of Lebesgue measurable functions \( u \) on \( I \) such that \( \| u \|_q \) is finite. \( AC(I,q) \) denotes the set of absolutely continuous functions \( u \) on \( I \) satisfying \( u^q \in L^q(I) \). \( W^{m,q}(I) \) denotes the set of functions \( u \in C^{m-1}(I) \) and \( u^{(m-1)} \in AC(I,q) \) with finite norm

\[
\| u \|_{W^{m,q}(I)} = \max_{k=0,\ldots,m} \| u^{(k)} \|_q.
\]

It is easy to see that \( C^m(I) \) and \( W^{m,q}(I) \) are Banach spaces and \( W^{m,q} \) is a usual Sobolev space.

Let \( p \in N \). A finite subset \( P \) of the interval \([0,T]\) and defined by

\[
P = \{ t_0, \ldots, t_p : 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T \}.
\]

Let \( J_0 = [0,t_1) \) and \( J_k = (t_{k-1},t_k] \) for all \( k = 1, \ldots, p \). For \( m \in N \cup \{0\} \) and \( 1 \leq q \leq \infty \), we denote

\[
C^m_p = \{ u : [0,T] \to R : \text{for all} \ k = 0, \ldots, p, u \in C^m(J_k), \ \text{there exist} \ u^{(k)}(t_i), \ k = 1, \ldots, p \ \text{and} \ u^{(l)}(t_i) = u^{(l)}(t_{i-1}), \ k = 1, \ldots, p-1, l = 0, \ldots, m \},
\]

\[
W^{m,q}_p = \{ u : [0,T] \to R : u_{j_k} \in W^{m,q}(J_k), k = 0, \ldots, p \}.
\]

It is easy to verify that the spaces \( C^m_p \) and \( W^{m,q}_p \) are Banach spaces with the norms

\[
\| u \|_{C^m_p} = \max_{k=0,\ldots,p} \| u_{j_k} \|_{C^m(J_k)} \ \text{and} \ \| u \|_{W^{m,q}_p} = \max_{k=0,\ldots,p} \| u_{j_k} \|_{W^{m,q}(J_k)}.
\]

We say that \( f : [0,T] \times S \to R \) satisfies the restricted Carathéodory conditions on \([0,T] \times S\) if

i. for each \( x \in S \) the function \( f(\cdot,x) \) is measurable on \([0,T]\);

ii. the function \( f(t,\cdot) \) is continuous on \( S \) for a.e. \( t \in [0,T] \);

iii. for every compact set \( K \subset S \), there exists a nonnegative function \( \mu \in L^1(0,T) \) such that

\[
| f(t,x) | \leq \mu(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in K.
\]

In this paper, we use \( \text{Car}( [0,T] \times S ) \) to denote the set of functions satisfying the restricted Carathéodory conditions on \([0,T] \times S \). In what follows, \( D^x \) and \( D_z \) denote the Dini derivatives.

**Definition 1.** The functions \( \alpha, \beta \in W^{\infty}_p \) such that \( \alpha \leq \beta \) are called a pair of coupled lower and upper solutions of problem (1)-(3) if \( \alpha, \beta \) satisfy the following conditions:

(i) \( D_+ \alpha(t) \leq D_+ \beta(t) \) for all \( t \in [0,T], P \). Moreover, if \( \tau \in [0,T], P \) such that \( D_+ \beta(\tau) = D_+ \beta(\tau) \), then there exists \( \varepsilon > 0 \) such that

\[
\alpha \in C^1([\tau-\varepsilon, \tau+\varepsilon]), \quad \phi \circ D_\alpha \in AC([\tau, \tau+\varepsilon])
\]

and

\[
(\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \text{ a.e. } t \in [\tau, \tau+\varepsilon].
\]

(ii) \( D_+ \beta(t) \geq D_+ \beta(t) \) for all \( t \in [0,T], P \). Moreover, if \( \tau \in [0,T], P \) such that \( D_+ \beta(\tau) = D_+ \beta(\tau) \), then there exists \( \varepsilon > 0 \) such that

\[
\beta \in C^1([\tau-\varepsilon, \tau+\varepsilon]), \quad \phi \circ D_\beta \in AC([\tau, \tau+\varepsilon])
\]

and

\[
(\phi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t)) \text{ a.e. } t \in [\tau, \tau+\varepsilon].
\]

(iii) For all \( k = 1, \ldots, p \), \( l_i(\alpha(t_i), \bullet, \bullet) \) are injective and there exist \( D^0 \alpha(t_i), D^1 \alpha(t_i), D^2 \beta(t_i), D^2 \beta(t_i) \) in \( R \) such that

\[
I_1(\alpha(t_i), \alpha(t_i)) = 0 \leq M_1(\alpha(t_i), \alpha(t_i), D \alpha(t_i), D^2 \alpha(t_i), \alpha), \quad I_2(\beta(t_i), \beta(t_i)) = 0 \geq M_2(\beta(t_i), \beta(t_i), D^2 \beta(t_i), D \beta(t_i), \beta).
\]

and there exist \( D^0 \alpha(0), D^1 \alpha(T), D^2 \beta(0), D^2 \beta(T) \) in \( R \) such that

\[
\alpha(0) + \beta(T) = 0 \leq D^0 \alpha(0) + D^2 \beta(T), \quad \alpha(T) + \beta(0) = 0 \geq D^0 \alpha(T) + D^2 \beta(T).
\]

**Definition 2.** Given a function \( u \in C^1_p \) is called a solution of the problem (1)-(3) if \( \phi \circ u' \in W^{\infty}_p \) and \( u \) satisfies (1) and fulfills conditions (2) and (3).

**Definition 3.** Assume that \( f \in \text{Car}([0,T] \times R^2) \) and
\( \alpha, \beta \in W^{1}_{+} \) satisfying \( \alpha(t) \leq \beta(t) \) for \( \forall t \in [0, T] \). We say that \( f \) satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \) if, for \( k = 1, \ldots, p \), there exist \( \phi_{k} \in C[0, \infty) \) and \( w \in L^{0}(0, T), 1 \leq q \leq \infty \), such that \( \phi_{k} > 0 \) on \( [0, \infty) \),

\[
| f(t, u, v) | \leq w(t) \phi_{k}(| v |) \quad \text{on} \quad J_{k} \times [\alpha(t), \beta(t)] \times R.
\]

Moreover, there exists a constant \( K = K(\alpha, \beta) \) with \( K > \max \{ t_{k}, \| \alpha \|, \| \beta \| \} \), such that

\[
\int_{t_{k}(\alpha)}^{t_{k}(\beta)} \phi_{k}(| f(x) |)^{\frac{1}{q}} dx > \| w \|_{L^{q}} \phi_{k}^{| \frac{1}{q} } , \quad \text{or}
\]

\[
-\int_{t_{k}(\beta)}^{t_{k}(\alpha)} \phi_{k}(| f(x) |)^{\frac{1}{q}} dx > \| w \|_{L^{q}} \phi_{k}^{| \frac{1}{q} } ,
\]

where \( \phi_{k} = \sup \{ \beta(t) - \inf \alpha(t) \mid t \in \mathbb{R} \} \) and \( r_{k} = \frac{1}{t_{k}(\alpha) - t_{k}(\beta)} \max \{ \beta(t_{k}) - \alpha(t_{k}), \beta(t_{k}^{+}) - \alpha(t_{k}^{-}) \} \). Any constant such \( K > \max \{ t_{k}, k = 0, \ldots, p \} > 0 \) will be called a Nagumo constant.

Throughout this paper, we impose the following hypotheses:

(H.1) The function \( \phi: R \rightarrow R \) is a continuous and strictly increasing function.

(H.2) The BVP (1)-(3) has a pair of coupled lower and upper solutions \( \alpha \) and \( \beta \).

(H.3) \( f \in \text{Car}(0, T) \times \mathbb{R}^{2} \) and satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \).

(H.4) The functions \( I_{k} \in C^{0}(\mathbb{R}^{2}) \) are non-decreasing in the first variable for \( k = 1, \ldots, p \), and the functions \( M_{k} \in C^{0}(\mathbb{R}^{2}) \) are non-increasing in the third variable and non-decreasing in the fourth and fifth variables.

3. Existence Results of Coupled Solutions

This section is devoted to proving the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with \( \phi \)-Laplacian operator. Firstly, we state the following existence and uniqueness result.

\( \alpha, \beta \in W_{+}^{1} \) satisfying \( \alpha(t) \leq \beta(t) \) for \( \forall t \in [0, T] \). We say that \( f \) satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \) if, for \( k = 1, \ldots, p \), there exist \( \phi_{k} \in C[0, \infty) \) and \( w \in L^{0}(0, T), 1 \leq q \leq \infty \), such that \( \phi_{k} > 0 \) on \( [0, \infty) \),

\[
| f(t, u, v) | \leq w(t) \phi_{k}(| v |) \quad \text{on} \quad J_{k} \times [\alpha(t), \beta(t)] \times R.
\]

Moreover, there exists a constant \( K = K(\alpha, \beta) \) with \( K > \max \{ t_{k}, \| \alpha \|, \| \beta \| \} \), such that

\[
\int_{t_{k}(\alpha)}^{t_{k}(\beta)} \phi_{k}(| f(x) |)^{\frac{1}{q}} dx > \| w \|_{L^{q}} \phi_{k}^{| \frac{1}{q} } , \quad \text{or}
\]

\[
-\int_{t_{k}(\beta)}^{t_{k}(\alpha)} \phi_{k}(| f(x) |)^{\frac{1}{q}} dx > \| w \|_{L^{q}} \phi_{k}^{| \frac{1}{q} } ,
\]

where \( \phi_{k} = \sup \{ \beta(t) - \inf \alpha(t) \mid t \in \mathbb{R} \} \) and \( r_{k} = \frac{1}{t_{k}(\alpha) - t_{k}(\beta)} \max \{ \beta(t_{k}) - \alpha(t_{k}), \beta(t_{k}^{+}) - \alpha(t_{k}^{-}) \} \). Any constant such \( K > \max \{ t_{k}, k = 0, \ldots, p \} > 0 \) will be called a Nagumo constant.

Throughout this paper, we impose the following hypotheses:

(H.1) The function \( \phi: R \rightarrow R \) is a continuous and strictly increasing function.

(H.2) The BVP (1)-(3) has a pair of coupled lower and upper solutions \( \alpha \) and \( \beta \).

(H.3) \( f \in \text{Car}(0, T) \times \mathbb{R}^{2} \) and satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \).

(H.4) The functions \( I_{k} \in C^{0}(\mathbb{R}^{2}) \) are non-decreasing in the first variable for \( k = 1, \ldots, p \), and the functions \( M_{k} \in C^{0}(\mathbb{R}^{2}) \) are non-increasing in the third variable and non-decreasing in the fourth and fifth variables.

u(t) = A_{k} + \int_{t_{k}}^{t} \phi_{k}^{-1}(\int_{t_{k}}^{\tau_{k}} f(s) ds + \tau_{k}) d\tau_{k}, \quad t \in J_{k}, \quad k = 0, \ldots, p,
\]

where \( \tau_{k} \) is the unique solution of the equation

\[
B_{k} - A_{k} = \int_{t_{k}}^{\tau_{k}} \phi_{k}^{-1}(\int_{t_{k}}^{\tau_{k}} f(s) ds + \tau_{k}) d\tau_{k}.
\]

Next, let us consider the following functions

\[
\delta_{k}(y) = \min \{ K, \max \{ y, -K \} \} \quad \text{for} \quad y \in R,
\]

where \( K \) is the constant introduced in definition 2.3,

\[
\rho(t, u) = \min \{ \beta(t), \max \{ u, \alpha(t) \} \} \quad \text{for} \quad (t, u) \in [0, T] \times R
\]

coupled with functionals \( A_{k}, B_{k} : C_{p} \rightarrow R \) given by

\[
A_{k}(u) = \rho(0, -u(T)),
\]

\[
B_{k}(u) = \rho(T, u(T) - u'(0) - u'(T)).
\]

Moreover, for each \( u \in C_{p} \), we consider a function \( \tilde{f}_{u} : [0, T] \rightarrow R \) defined by

\[
\tilde{f}_{u}(t) = f(t, \rho(t, u(t))), \quad \delta_{k}(\frac{d}{dt} \rho(t, u(t))).
\]

The function \( \tilde{f}_{u} \) is well defined according to the following result (by redefining function \( \frac{d}{dt} \rho(t, u(t)) \) as zero when it does not exist). It can be proved in a similar way to Lemma 2 in [24].

\[
\text{Lemma 2. For given} \quad u, u_{n} \in C_{p} \quad \text{such that} \quad u_{n} \rightarrow u \quad \text{in} \quad C_{p}, \quad \text{then}
\]

\[ \text{(i)} \quad \frac{d}{dt} \rho(t, u(t)) \text{ exists for a.e.} \quad t \in [0, T], \quad P; \]

\[ \text{(ii)} \quad \frac{d}{dt} \rho(t, u_{n}(t)) \rightarrow \frac{d}{dt} \rho(t, u(t)) \quad \text{for a.e.} \quad t \in [0, T], \quad P. \]

Now, we can define a strictly increasing homeomorphism \( \hat{\phi} : R \rightarrow R \) by:

\[
x \in R \rightarrow \hat{\phi}(x) = \begin{cases} \phi(x), & | x | \leq K, \\ \frac{\phi(K) - \phi(-K)}{2} x - \frac{1}{2} (\phi(K) + \phi(-K)), & | x | > K. \end{cases}
\]

In the following, we are in a position to prove the existence theorem for our considering problems.

\[ \text{Lemma 3. (Theorem 3.3 of [22]) Assume that} \quad (H_{1})-(H_{4}) \quad \text{hold. Then there exists at least one solution} \quad u \quad \text{of the problem} \]

\[ (1)-(3) \quad \text{such that} \]

\[
\alpha(t) \leq u(t) \leq \beta(t)
\]

and

\[
| u'(t) | \leq K, \quad t \in [0, T].
\]
where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

Next, we are devoted to the existence of coupled solutions. We first introduce the following definition.

**Definition 4.** The functions $x, y$ are called coupled solutions of problems (1)-(3) if $x, y \in C^1_p$ and satisfy (1)-(2) and

$$
\begin{align*}
  x(0) &= -y(T), \\
  x'(0) &= -y'(T), \\
  y(0) &= -x(T), \\
  y'(0) &= -x'(T).
\end{align*}
$$

**Remark** If the coupled solutions $x$ and $y$ of problem (1)-(3) satisfy $x = y$, the $x = y$ is a solution of problem (1)-(3).

Next, we give the existence of coupled solutions for problems (1)-(3).

**Theorem 5.** Assume hypotheses (H$_1$)-(H$_4$) hold. Then there exists at least a pair of coupled solutions $x, y \in C^1_p$ of the impulsive differential equations boundary value problem (1)-(3) such that

$$
\begin{align*}
  x \in [\alpha, \beta] = \{u : \alpha(t) \leq u(t) \leq \beta(t), t \in [0, T]\},
  y \in [\alpha, \beta] = \{v : \alpha(t) \leq v(t) \leq \beta(t), t \in [0, T]\},
  |x'(t)| \leq K, \quad |y'(t)| \leq K, \quad \text{for } t \in [0, T],
\end{align*}
$$

where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

**Proof.** Let us define $\rho, A_k, B_k, \text{ for each } k = 1, \ldots, p$ in the same way as above, and construct a modified problem $(P')$ similar to the proof of Lemma 3, that is

$$
\begin{align*}
  (\mathcal{F}(x'(t)))' &= \tilde{f}_x(t), \quad \text{a.e. } t \in [0, T], \\
  (\mathcal{F}(y'(t)))' &= \tilde{f}_y(t), \quad \text{a.e. } t \in [0, T], \\
  x(t_k) &= B_{k, x}(x), \quad y(t_k) = B_{k, y}(y), \quad k = 1, 2, \ldots, p, \\
  x(t'_k) &= A_{k, x}(x), \quad y(t'_k) = A_{k, y}(y), \quad k = 1, 2, \ldots, p, \\
  x(0) &= A_0(x), \quad y(0) = A_0(y), \\
  x(T) &= B_p(x), \quad y(T) = B_p(y),
\end{align*}
$$

where

$$
\begin{align*}
  A_{k, x}(x) &= \rho(0, -y(T)), \\
  B_{p, x}(x) &= \rho(T, x(T) - y'(0) - x'(T)), \\
  A_{k, y}(y) &= \rho(0, -x(T)), \\
  B_{p, y}(y) &= \rho(T, y(T) - x'(0) - y'(T)).
\end{align*}
$$

From the proof of the Lemma 3, there exists a couple of solutions $x, y \in C^1_p$ such that

$$
\begin{align*}
  x \in [\alpha, \beta] = \{u : \alpha(t) \leq u(t) \leq \beta(t), t \in [0, T]\}, \\
  y \in [\alpha, \beta] = \{v : \alpha(t) \leq v(t) \leq \beta(t), t \in [0, T]\}, \\
  |x'(t)| \leq K, |y'(t)| \leq K, \quad \text{for } t \in [0, T],
\end{align*}
$$

and

$$
\begin{align*}
  \alpha(0) &\leq -y(T) \leq \beta(0), \\
  \alpha(0) &\leq -x(T) \leq \beta(0), \\
  \alpha(T) &\leq x(T) - y'(0) - x'(T) \leq \beta(T), \\
  \alpha(T) &\leq y(T) - x'(0) - y'(T) \leq \beta(T).
\end{align*}
$$

Furthermore, $x, y$ satisfy the condition (2). Now, to prove that (5)-(8) is verified, it suffices to prove that

$$
\begin{align*}
  x(0) &= -\beta(0), \\
  y(0) &= \beta(0), \\
  x(T) &= \beta(T), \\
  y(T) &= \beta(T).
\end{align*}
$$

Firstly, we will prove (10), by contradiction, if $\alpha(0) > -y(T)$, then by $\alpha \leq y \leq \beta$, we have $\alpha(0) > -y(T) \geq -\beta(T)$, which contradict to $\alpha(0) + \beta(0) = 0$. Moreover, $-y(T) \leq \beta(0)$ can be proved similarly.

As the same way, we can obtain that the inequality (10) is holds. Thus we have

$$
\begin{align*}
  x(0) &= -y(T), \\
  y(0) &= -x(T).
\end{align*}
$$

Assume that the first inequality if (11) isn’t holds, as a consequence, we have

$$
\begin{align*}
  x(T) &= \alpha(T), \\
  y(T) &= \beta(T).
\end{align*}
$$

From (14) and $\alpha(T) + \beta(0) = 0$, we have

$$
\begin{align*}
  y(0) &= -x(T) = -\alpha(T) = \beta(0).
\end{align*}
$$

From these facts and the relation $\alpha \leq x, y \leq \beta$, we have

$$
\begin{align*}
  x'(T) \leq D, y'(T) \leq D, \alpha(0), \\
  y'(0) + x'(T) < 0.
\end{align*}
$$

From (14) and $\alpha(T) + \beta(0) = 0$, we have

$$
\begin{align*}
  y(0) &= -x(T) = -\alpha(T) = \beta(0).
\end{align*}
$$

From these facts and the relation $\alpha \leq x, y \leq \beta$, we have

$$
\begin{align*}
  x'(T) \leq D, y'(T) \leq D, \alpha(0), \\
  y'(0) + x'(T) < 0.
\end{align*}
$$

Thus

$$
0 < y'(0) + x'(T) \leq D, \alpha(0) + D, \beta(0) \leq 0.
$$

It is a contradiction. Moreover, the inequality in (13) be obtain in a similar way. Hence inequalities (11)-(12) are hold, that is to say $x, y$ satisfy (5)-(8).

Therefore, the functions $x, y$ is a coupled solutions of the problem (1)-(3), which completes the proof.
4. Conclusion

In this paper, we mainly discuss the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with $\phi$-Laplacian operator. To give the existence results of coupled solutions for the problem (1)-(3), we first introduce a pair of coupled lower and upper solutions (see Definition 1). Then, we provide and prove the existence results of coupled solutions for anti-periodic $\phi$-Laplacian impulsive differential equations boundary value problems based on a pair of coupled lower and upper solutions and appropriate Nagumo condition (Theorem 5).

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References