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Methodology Article

A Review of Fractals Properties: Mathematical Approach

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Abstract: In this article, we will discuss some spectacularly beautiful images known as Fractals such as Sierpiński Triangle, Koch Curve, Dragon Curve, Koch Island, H Fractal, The Levy Curve Fractal, Box Fractal etc. We will investigate and calculate the area, perimeter and self-similar dimension of fractals. Observing the results we see some similarities about the said properties for some fractals those are generated by particular method. Our attention is restricted to find the mathematical behavior of Fractals so that we can establish mathematical formulas concerning the fractals.

Keywords: Fractals, Iterations, Area, Perimeter, Fractal Dimension

1. Introduction

In this article we will describe some of the wonderful new ideas in the area of mathematics known as fractal geometry. Today Fractal geometry is completely new area of research in the field of computer science and engineering. It has wide range of applications. Fractals in nature are so complicated and irregular that it is hopeless to model them by simply using classical geometry objects. Benoit Mandelbrot, the father of fractal geometry, from his book The Fractal Geometry of Nature, 1982. This paper explor various concepts of fractal i.e. fractal dimension, various techniques to generate fractal, their characteristics and their application in real life [14]. As we can see, fractals are incredibly complicated and often quite beautiful geometric shapes that can be generated by simple rules. We have tried to find out the mathematics behind these incredible geometric shapes called fractals. We have measured their shapes (perimeter & area) and the fractal dimension to predict the behavior of similar types of fractal.

1.1. Background

Fractal geometry is a branch of mathematics concerned with irregular patterns made of parts that are in some way similar to the whole. The images that we call fractals have been known in mathematics for well over a century. Objects such as the Cantor set, the c triangle, and the Koch curve have appeared often in the mathematical literature over the past hundred years. However, these objects were once regarded as almost pathological shapes mainly of interest in mathematical research [13, 18].

All of this has changed in the last 20 years. Two events occurred in that period that brought fractal geometry into the mainstream of contemporary science and mathematics. The first was the observation by the mathematician *Benoit Mandelbrot* that fractals are not just mathematical curiosities, but rather the geometry of nature. He observed that many objects in the natural world were fractal in appearance [15, 17]. Ferns, clouds, trees, coastlines, and many other "irregular shapes" could best be understood using fractal geometry rather than Euclidean geometry. Indeed, while the straight lines, triangles, and circles of Euclidean geometry are important for humans to build bridges, houses, roads, and the like, nature seems to construct its objects differently [8]. Natural objects are often more complicated and have a richer geometrical property. As we will see, they can often be modeled with fractals.

The second event that brought fractal geometry into the limelight was the availability of computers. Before people had access of computers and computer graphics, fractals could only be envisioned in the mind. They were often too complicated for a human to draw and also difficult to explain to others. The computer changed this dramatically. When mathematicians could see the structure of the objects they were working with, they realized that the objects were much more interesting and beautiful than they had previously thought.

Moreover, the computer allowed mathematicians to discover many more exciting examples of fractals that no one had imagined before. This includes the gorgeous images known as the Mandelbrot and Julia sets.

1.2. Application

People are now use and study fractals in many more areas than just mathematics. Fractals arise in medicine: Cancerous tumors, human lungs, and vascular systems are all examples of fractals. A large number of fractal antenna designs have been proposed. The purpose of this paper is to show various applications and remarkable growth of fractal antenna in the fields of wireless communication [12]. Art historians use fractals to date early Chinese paintings. Seismologists use fractals to study the fissures caused by earthquakes. Computer programmers use fractal techniques to encode large sets of data efficiently. Fractals even occur in Broadway plays (such as Tom Stoppard's Arcadia) and in films (such as Jurassic Park), where they are used to create extraterrestrial planet-escapes and other special effects.

1.3. Definitions

The word is related to the Latin verb “frangere”, which means “to break” [6]. In the Roman mind, frangere may have evoked the action of breaking a stone; since the adjective derived it combines the two most obvious properties of broken stones, irregular and fragmentation [6, 17]. This adjective is fractus, which lead to fractal. The etymological kinship with “fraction” is also significant if ones interprets “fraction” as a number that lies between integers. Indeed, a fractal set can be considered as lying between the shapes of Euclid.

Many object in the nature can be created by applying the concept of classical geometry like- lines, circles, conic sections, polygons, spheres, quadratic surface and so on. There are various objects of nature which can not be modeled by applying Euclidean geometry, hence there is need to deal with such complicated and irregular object which can only be constructed by fractal geometry [14].

A [fractal is a] rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole [3].

Mathematical Definitions: A fractal is a subset of R^n which is self-similar and whose fractal dimension exceeds its topological dimension [1].

1.4. An Examples of Famous Fractal

The Sierpinski Triangle or Gasket:

The Sierpinski triangle is created by replacing an equilateral triangle of unit size, E_0, by three triangles of half its size, leaving the middle region empty, giving E_1, see Fig-1. E_2 is created by replacing each of the three triangles of E_1 by three half-sized triangles, leaving the middle region empty as before, see Fig-1. Thus, applying the rules on E_{k-1}, we obtain E_k, and when k tends to infinity, we get the Sierpinski triangle of Figure.

We see that the set E_k consists of $(3)^k$ triangles, each with side length 2^{-k}. Thus, the total area of the Sierpinski triangle is $(3)^k \,.\, (2^{-k})^2 \,.\, \sqrt{3}/4$ which tends to zero when $k \to \infty$, i.e.

$$\lim_{k\to\infty} \tfrac{\sqrt{3}}{4}\left(\tfrac{3}{4}\right)^k = 0$$

We note that at everysteps of iteration, we always keep the line segments that constitute the boundary of the triangles from the lastiteration, and we always get new line segments from the new triangles. Starting with three line segments, we get one new for each triangle of the k^{th} iteration. Thus, in the k^{th} iteration we have $3+\sum_{k=1}^{\infty} 3^k$ line segments. This goes to ∞ as $k \to \infty$, which means that the length of the Sierpinski triangle is infinite.

We have displayed the results of randomly iterating this procedure 6 times. (We have not pictured the first 50 points on this orbit so that only the eventual behavior is shown). We note the intricate shape of the resulting orbit. This figure, called the Sierpinski triangle or gasket, is a classical example of a fractal.

Fig. 1. *Sierpinski Triangle or Gasket (after 6 iteration).*

Area of the Sierpinski Triangle

As we know, at each level, one quarter of the triangle is removed. That is, three quaters of the area of the original triangle is left after the first iteration. Thus, it is not hard to infer that after n iterations, the area of the Sierpinski's Triangle would be $(0.75)^n$ times the area of the original triangle. So after an infinite number of iterations, you would find there was no area at all. [5]

Number of Triangles Grow

After observing the number of triangles pointing down for several iterations, we have the following table:

***Table 1.** Number of Triangles Grow (up to 5 iteration).*

Iteration	No. of Triangles pointing down
1	1
2	4
3	13
4	40
5	121

From the table, we can come up with a general formula to predict the number of triangles being removed for any iteration:

At nth iteration, the number of triangles being removed.

2. Methodology

2.1. Iteration Rules

In Fractal geometry the geometrical fractal set should be considered as an infinite ordered series of geometrical objects defined on a metric space [13]. We begin the study of fractals by introducing one of the main process by which fractals are generated, namely, *iteration*. Iteration means to repeat a process over and over again [4]. There are many types of iterative process in mathematics. Most of the iteration will involve a geometric rule or construction. We begin with some geometric shape or figure called the seed. Then we perform a geometric operation on this seed. This geometric operation is called the iteration rule. The rule might involve rotating or squeezing or cutting apart the shape [2]. After we perform this operation we obtain a new figure. Then we iterate; this means we perform the same operation on the new figure to produce the next figure. We then repeat this process, continually applying the iteration rule to produce a sequence of figures.

2.2. Some Iteration Rules

2.2.1. Shrinking Iteration Rules

(i) The Seed is the Straight Line Segment Below. The Iteration Rule is to Shrink the Segment so that its Length is Half the Length of the Original one [11].

***Fig. 2.** Shrinking Iteration (after 4 iteration).*

Mathematical Approach:

Assume the length of the original straight line segment is 1 unit.

The length of the second segment is 1/2

The length of the second segment is 1/4

The length of the segments on further iterations is like 1/8 → 1/16 → 1/32

Fate: The fate of the orbit is a single point but the length of the orbit's segments tends to zero since the series of its length 1→1/2 →1/4 → 1 / 8 → 1 / 16 → 1 / 32converges to zero.

(ii) The Seed is a Square Whose Sides have Length 1.

Shrink the square so that each side is half as long.

We can picture the orbit as a sequence of squares. Each succeeding square has linear dimensions equal to half that of the preceding square.

***Fig. 3.** Shrinking Iteration (after 4 iteration).*

Mathematical Approach:

Assume the length of each side of the original square is 1

The side length is decreasing by a scale factor of 1 / 2

The perimeter is decreasing by a scale factor of 1 / 2

The area is decreasing by a scale factor of 1 / 4

Perimeter = 4• Side Length

Square Area = (Length)^2

The table below represented the Side length, Perimeter and Area of the iterations

***Table 2.** Calculation for Shrinking Iteration (up to 4 iteration).*

	Side length (as a fraction)	Perimeter (in units)	Area (in square units)
Original square	1	4.1 = 4	$1^2 = 1$
First iteration	½	4. ½ = 4/2 = 2	$(½)^2 = 1/2^2 = 1/4$
Second iteration	¼	4. ¼ = 1	$(1/4)^2 = 1/4^2 = 1/16$
Third iteration	1/8	4. 1/8 = ½	$(1/8)^2 = 1/8^2 = 1/64$
Fourth iteration	1/16	4. 1/16 = ¼	$(1/16)^2 = 1/16^2 = 1/256$

Fate: The limiting shape or the fate of this orbit is a single point and the side length, perimeter and area tends to zero.

2.2.2. Replacement Iteration Rules

The seed is the circle below and the iteration rule is to replace a circle by two smaller copies of itself (lined up side by side) whose diameters are each one half of the original[11].

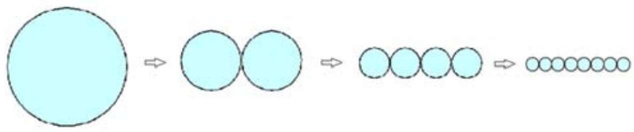


***Fig. 4.** Replacement Iteration (after 4 iteration).*

Mathematical Approach:

Assume that diameter's length of the original circle is 2 units.

Remember:

Circumference = 2 • π• radius = π• diameter Circle Area = π• r^2.

Table 3. *Calculation for Replacement Iteration (up to 4 iteration).*

	Diameter	Radius	Perimeter(in units)	Area (in square units)
Original quare	2	1	$\pi.2 = 2\,\pi$	$\pi.1^2 = \pi.\ 1 = \pi$
First iteration	1	½	$\pi\cdot 1 = \pi$	$\pi\,(1/2)^2 = \pi/4$
Second iteration	1/2	¼	$\pi\cdot 1 / 2 = \pi/ 2$	$\pi\cdot (1/4)^2 = \pi/4^2 = \pi/16$
Third iteration	1/4	1/8	$\pi\cdot 1 / 4 = \pi/ 4$	$\pi.\ (1/8)^2 = \pi/8^2 = \pi/64$
Fourth iteration	1/8	1/16	$\pi\cdot 1 / 8 = \pi/ 8$	$\pi.\ (1/16)^2 = \pi/16^2 = \pi/256$

Fate: The limiting shape or the fate of this orbit is a single point and the side length, perimeter and area tends to zero.

3. Fractal Dimension

It is a fascinating fact that certain geometric images have fractional dimension. The Sierpinski triangle provides an easy way to explain why this must be so.

To explain the concept of fractal dimension, it is necessary to understand what we mean by dimension in the first place. Obviously, a line has dimension 1, a plane has dimension 2, and a cube has dimension 3.It is interesting to struggle to enunciate why these facts are true. What is the dimension of the Sierpinski triangle?

A line has dimension 1 because there is only 1 way to move on a line. Similarly, the plane has dimension 2 because there are 2 directions in which to move. There are really 2 directions in a line -- backward and forward -- and infinitely many in the plane. Actually there are 2 linearly independent directions in the plane [9]. Of course, it is right. But the notion of linear independence is quite sophisticated and difficult to articulate. We often say that the plane is two-dimensional because it has ``two dimensions," meaning length and width. Similarly, a cube is three-dimensional because it has ``three dimensions," length, width, and height [7]. Again, this is a valid notion, though not expressed in particularly rigorous mathematical language.

Another pitfall occurs when try to determine the dimension of a curve in the plane or in three-dimensional space. An interesting debate occurs between teachers and students when a teacher suggests that these curves are actually one-dimensional. But they have 2 or 3 dimensions, the students object.

So why a line is one-dimensional and the plane is two-dimensional? Note that both of these objects are self-similar. We may break a line segment into 4 self-similar intervals, each with the same length, and each of which can be magnified by a factor of 4 to yield the original segment [1, 16]. We can also break a line segment into 7 self-similar pieces, each with magnification factor 7, or 20 self-similar pieces with magnification factor 20. In general, we can break a line segment into N self-similar pieces, each with magnification factor N.

A square is different. We can decompose a square into 4 self-similar sub-squares, and the magnification factor here is 2. Alternatively, we can break the square into 9 self-similar pieces with magnification factor 3, or 25 self-similar pieces with magnification factor 5. Clearly, the square may be broken into N^2 self-similar copies of itself, each of which must be magnified by a factor of N to yield the original figure. See Fig-5. Finally, we can decompose a cube into N^3 self-similar pieces, each of which has magnification factor N.

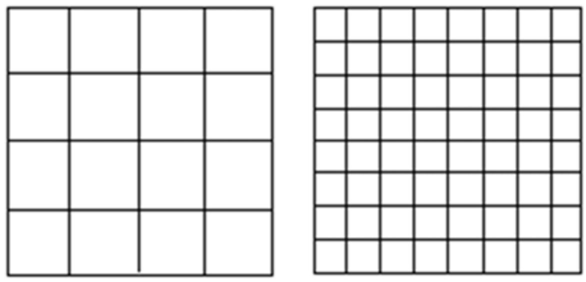


Fig. 5. *A square may be broken into N^2 self-similar pieces, each with magnification factor N.*

Now we see an alternative way to specify the dimension of a self-similar object: The dimension is simply the exponent of the number of self-similar pieces with magnification factor N into which the figure may be broken.

So what is the dimension of the Sierpinski triangle? How do we find the exponent in this case? For this, we need logarithms. Note that, for the square, we have N^2 self-similar pieces, each with magnification factor N [10, 16]. So we can write

$$\text{Dimension} = \frac{\log(\mathit{Number\ of\ self\ \ ?\ similar\ pieces})}{\mathit{log}\left(\mathit{magnification\ factor}\ \right)}$$

$$= \frac{\log N^2}{\log N} = \frac{2\log N}{\log N} = 2$$

Similarly, the dimension of a cube is

$$\text{Dimension} = \frac{\log(\mathit{Number\ of\ self\ \ ?\ similar\ pieces})}{\mathit{log}\left(\mathit{magnification\ factor}\ \right)}$$

$$= \frac{\log N^3}{\log N} = \frac{3\log N}{\log N} = 3$$

Thus, we take as the *definition* of the fractal dimension of a self-similar object

$$\text{Fractal dimension} = \frac{\log(\mathit{Number\ of\ self\ \ ?\ similar\ pieces})}{\mathit{log}\left(\mathit{magnification\ factor}\ \right)}$$

Now we can compute the dimension of S. For the Sierpinski triangle consists of 3 self-similar pieces, each with magnification factor 2. So the fractal dimension is

$$\frac{\log(Number\ of\ self\ \ ?\ similar\ pieces)}{log\left(magnification\ factor\ \right)}=\frac{\log 3}{\log 2}\approx 1.58$$

so the dimension of S is somewhere between 1 and 2, just as our "eye" is telling us. But wait a moment; **S** also consists of 9 self-similar pieces with magnification factor 4. No problem -- we have

$$\text{Fractal dimension } =\frac{\log 9}{\log 4}=\frac{2\log 3}{2\log 2}=\frac{\log 3}{\log 2}\approx 1.58$$

as before. Similarly, S breaks into 3^N self-similar pieces with magnification factors 2^N, so we again have

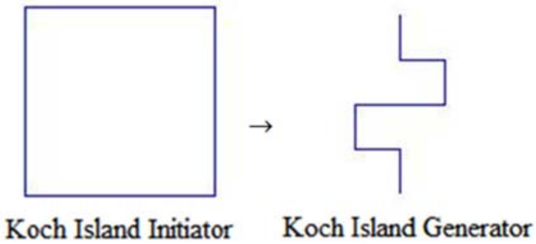
$$\text{Fractal dimension } =\frac{\log 3^N}{\log 2^N}=\frac{N\log 3}{N\log 2}=\frac{\log 3}{\log 2}=1.58$$

Fractal dimension is a measure of how "complicated" a self-similar figure is. In a rough sense, it measures "how many points" lie in a given set. A plane is "larger" than a line, while **S** sits somewhere in between these two sets [1].

4. Calculation Method

4.1. The Koch Island Fractal

The Koch Island has the following base and generator:



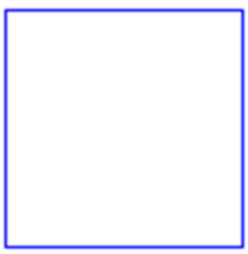
Here are the steps that we will need to follow in order to draw each level of the Koch Island Fractal.

This fractal will be completed using the “Copies of Copies” method.

We will copy the generator shape using a "contraction factor" of 1/4 (reducing each segment in the Generator shape by 1/4) and replacing each segment in the current level with the smaller copy of the generator.

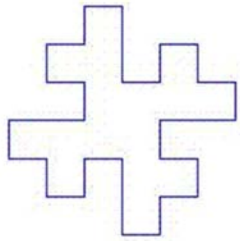
Level 1

On a large piece of construction paper, draw a large square with sides that are 1 foot in length. (Fill in the perimeter and area for level 1 into the chart below. We will say that each side of the initial triangle has length s and therefore has a perimeter of 4s. Also, we can say that the original area is 1A.)



Level 2

Now, replace all the sides (segments) of the square with the Generator shape. The generator shape has 8 small segments to it (the horizontal segment in the middle has two regular segments put together). Each of those smaller segments is 1/4 the size of the original side of the square that we replaced. (Fill in the perimeter and area for level 2 into the chart below.)



Level 3.

Now it gets a little more challenging. Replace each of the small segments from level 2 with the generator shape (smaller version). Each of those segments is 1/4 the size of the segments used in the previous level. (Fill in the perimeter and area for level 3 into the chart below.)

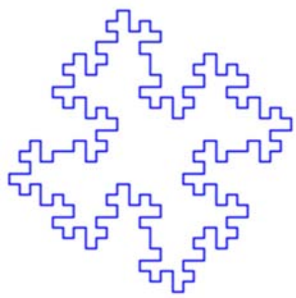


Table 4. *Table for Koch Island Fractal.*

Level	Segments	Segment Length	Perimeter	Area
1	4	1s	4s	A
2	32	(1/4)s	8s	A
3	256	(1/16)s	16s	A
n	$(8^n)/2$	$(1/4)^{n-1}$s	$4(2)^{n-1}$s	A

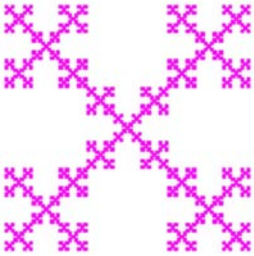
The perimeter of the Koch Island fractal is infinite but the total area stayed constant at each level.

We know the self- similarity Dimension = $\frac{\log(Number\ of\ self\ \ ?\ similar\ pieces)}{log\left(magnification\ factor\ \right)}$

Therefore the self- similarity dimension of Koch Island = = 1.5

4.2. The Box Fractal

This fractal is made by using the "method of successive removals."

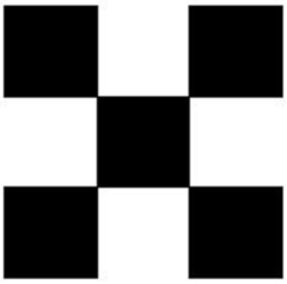


Level 1

On a large piece of construction paper, draw a square that has a side length of 1 foot. Shade in the entire square.

Level 2.

Lightly draw a square again that has a side length of 1 foot. Lightly divide that square up into 9 smaller squares. Shade in all of the squares except the middle squares on each side. We should have shaded in 5 out of 9 of the squares.



Level 3

Lightly draw a square again that has a side length of 1 foot. Lightly divide that square up into 81 smaller squares. Level 2 drawing had 4 shaded squares in the corners and 1 shaded square in the middle. Copy that drawing except this time do not shade the middle (3x3) squares from each side of the 5 previously shaded squares.

Table 5. *Table for Box Fractal.*

Level	Line of Segments	Length of Each Segment	Perimeter	Area
1	4	1s	4s	A
2	20	(1/3)s	6.67s	(5/9)A
3	100	(1/9)s	11.11s	(25/81)A
n	N	$(1/3)^{n-1}$s	$(n)(1/3)^{n-1}$s	$(5/9)^{n-1}$A

The perimeter increases at each successive level and as the level n approaches infinity, the perimeter will approach infinity.

The area decreases at each successive level and as the level n approaches infinity, the area will approach 0.

We know the self- similarity Dimension =

$$\frac{\log(\textit{Number of self ? similar pieces})}{\log\left(\textit{magnification factor}\right)}$$

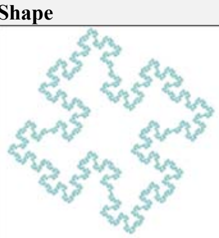
Self-similarity dimension of Cantor square fractal == 1.46

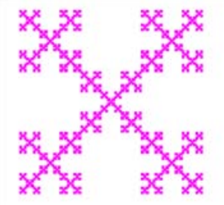
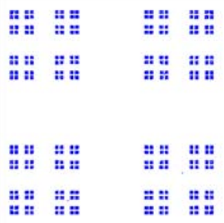
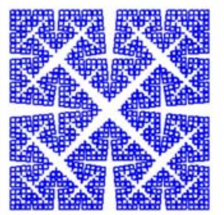
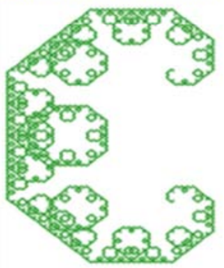
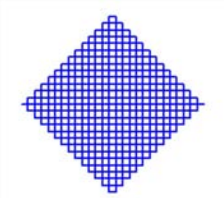
5. Experiments

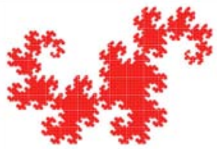
Using the formulas and techniques discussed above we now calculate the mathematical properties (area, perimeter & dimension) of some fractals. We determine the ultimate fate of the fractals by analyzing the data calculated. Finally we tabulated all the data to observe them and to come a decision concerning the fractals.

We tabulated the result of all the mathematical approaches of fractals that we have studied. We can visualize all the properties together so that we can come into a decision to generalize them.

Table 6. *Table for Some Fractal.*

Fractal	Shape	Method	Perimeter	Area	Dimension
The Koch Island Fractal		Copies of Copies	Infinite	Constant	1.5

Fractal	Shape	Method	Perimeter	Area	Dimension
The Box Fractal		Successive Removals	Infinite	Zero	1.46
The Cantor Square Fractal		Successive Removals	Infinite	Zero	1.26
The Cesaro Curve Fractal		Copies of Copies	Infinite	Converging	1.691
The Levy Curve Fractals		Replacing generator	Infinite	Converging	2
The Peano Curve Fractal		Copies of Copies	Infinite	Filling the space	2
The Sierpinski Arrowhead Fractal		Copies of Copies	Infinite	Converging	1.58
The H fractal		Copies of Copies	Infinite	Filling the space	1.46
The Sierpinski Carpet Fractal		Successive Removals	Infinite	Zero	1.89

Fractal	Shape	Method	Perimeter	Area	Dimension
The Dragon Curve Fractal		Replacing generator or Paper folding	Infinite	Converging	2

Observing the above fractals we can say that the fractals made by any method, their perimeter always approaches to infinity.

Only those fractals which are made by method of successive removals has the area tends to zero.

6. Conclusion

Here we try to understand fractal with their mathematical properties. A mathematical approach is done for some of the known fractals. We calculate the area, perimeter and self-similarity dimension of several fractals. We observe that there is a similarity in fractals about the said property. The fractals generated by similar method have same mathematical property. Perimeter always approaches to infinity for the studied fractals and the area of fractals generated under certain method becomes zero. This result can be general for fractals. Further research is needed to come into a decision.

References

[1] Prof. Emma Carberry, Alan Dunn, 18.091 Introduction to Dynamical Systems, Lecture 11: Fractals and Dimension, Spring 2005.

[2] Peitgen, H.-O., Jürgens, H., Saupe, D.: Chaos and Fractals, Springer-Velag New York, inc., 1992

[3] Peitgen, H.-O., Richter, P.: The Beauty of Fractals, Springer-Verlag, Heidelberg, 1986

[4] Peitgen, H.-O., Saupe, D.: The Science of Fractal Images, Springer-Verlag, New York, 1988

[5] http://www.math.nus.edu.sg/aslaksen/gem-projects/maa/World_of_Fractal.pdf

[6] https://en.wiktionary.org/wiki/frangere.

[7] *Robert L. Devaney, Fractal Dimension, Sun Apr 2 14:31:18 EDT 1995.*

[8] wikipedia.org/wiki/Fractal.

[9] Bahman Zohuri, Dimensional Analysis and Self-Similarity Methods for Engineers and Scientists, Springer International Publishing Switzerland 2015.

[10] Laxmi Narayan Padhy, Fractal Dimension of Gray Scale & Colour Image, ijarcsse, © 2015.

[11] The GEOMETRY OF NATURE:FRACTALS,Josefina Dexeus, IES Severo Ochoa.

[12] Rupleen Kaur, A Review of Various Fractal Geometries for Wireless Applications, Int. Journal of Electrical & Electronics Engg.

[13] Gabriele A. Losa, From Fractal Geometry to Fractal Analysis, Applied Mathematics, 2016, 7, 346-354 Scientific Research Publishing Inc.

[14] Ankit Garg, A Review on Natural Phenomenon of Fractal Geometry, International Journal of Computer Applications (0975 – 8887) Volume 86 – No 4, January 2014.

[15] Dr. Vyomesh Pant, FRACTAL GEOMETRY: AN INTRODUCTION, Journal of Indian Research, vol:1, No:2, 66-70, April-June.

[16] T Pant, Effect of Noise in Estimation of Fractal Dimension of Digital Images, International Journal of Signal Processing, Image Processing and Pattern Recognition Vol.6, No.5 (2013), pp.101-116.

[17] Harry A. Schmitz, On the Role of the Fractal Cosmos in the Birth and Origin of Universes, Journal of Theoretics, 5 Marwood Road South, Port Washington, New York 11050.

[18] GUOQIANG SHEN, Fractal dimension and fractal growth of urbanized areas, int. j. geographical information science, 2002 vol. 16, no. 5, 419± 437.