On the Construction of Molaei’s Generalized Hypergroups

Nosratollah Shajareh Poursalavati

Department of Pure Mathematics, Faculty of Mathematics and Computer, Mahani Mathematical Research Center, Shahid Bahonar University of Kerman, Kerman, Iran

Email address:
salavati@uk.ac.ir

To cite this article:

Received: February 17, 2017; Accepted: April 20, 2017; Published: June 1, 2017

Abstract: The purpose of this paper is making a construction and generalization of Molaei’s generalized groups by using construction of the Rees matrix semigroup over a polygroup and a matrix with entries in $H$. We call it “Molaei’s generalized hypergroups” and we give some examples.

Keywords: Hypergroup, Polygroup, Molaei’s Generalized Hypergroup

1. Introduction

In [10] generalized groups or completely simple semigroups is introduced as a class of algebras of interest in physics and they are an interesting generalization of groups. In [1], it is proved the generalized groups are the completely simple semigroups. In this paper we change the group to the polygroup and we obtain a new construction, by using this construction we can define “Molaei’s generalized hypergroup” and we give some examples.

Let $H$ be a non-empty set. A hyperoperation on $H$ is a function from $H \times H$ to $P^{\prime}(H)$, which $P^{\prime}(H)$ is the set of all non-empty subsets of $H$. A hypergroupoid is the couple $(H, \ast)$, where $H$ is a non-empty set and “$\ast$” is a hyperoperation on $H$, i.e., $(H, \ast): H \times H \rightarrow P^{\prime}(H)$. As usual, we use $a \ast b = * (a, b)$, for all $a$ and $b$ in $H$. If $M$ and $N$ belong to $P^{\prime}(H)$ and $a$ be an element of $H$, we define:

$M \ast N := \bigcup_{m \in M, n \in N} m \ast n, \quad M \ast a := M \ast \{a\}, \quad a \ast N := \{a\} \ast N.$

The relational notation $M \subseteq N$ is used to assert that $M$ and $N$ have an element in common, i.e., $M \cap N$ is non empty set.

We recalled the following definitions: [3, 4, 6, 9]
1) the hyperoperation $\ast$ is associative, if for every elements $a, b$ and $c$ of $H$, $(a \ast b) \ast c = a \ast (b \ast c);
2) the hypergroupoid $(H, \ast)$ is semihypergroup, if the hyperoperation $\ast$ is associative;
3) the hypergroupoid $(H, \ast)$ is quasihypergroup, if for all $a$ of $H$, $a \ast H = H \ast a = H$;
4) the hypergroupoid $(H, \ast)$ is hypergroup if it is both quasihypergroup and semihypergroup,
5) the hypergroup $(H, \ast)$ is polygroup if there exist a unique element $e$ in $H$, which for every $a$ in $H$, $e \ast a = a \ast e = a$.

As usual, this polygroup is demonstrated by $H, \ast, e, ^{-1}$.

We refer to [2, 5, 6, 7], for more details about polygroups.

Let $H, \ast, e, ^{-1}$ be a polygroup and $K$ be a non-empty subset of $H$, we denoted $K^{-1} = \{k^{-1}: k \text{ is an element of } K\}$, it is easy to show that, the following axioms hold for every $a$ and $b$ in $H$:

$(a^{-1})^{-1} = a, e^{-1} = e, e \ast (a \ast a^{-1}) \cap (a^{-1} \ast a), (a \ast b)^{-1} = b^{-1} \ast a^{-1}.$

2. Molaei’s Generalized Hypergroups

In this section, we consider a polygroup and by using the Rees matrix semigroup’s structure over polygroup, we construct a new structure and obtain three properties of this new structure. Theorem 2.1, 2.2 and 2.3 guide us inspire the definition of Molaei’s generalized hypergroups.

Let $H, \ast, e, ^{-1}$ be a polygroup and let $L, A$ be non-empty sets and $M$ be a map from $A \times I$ to $H$, by $M(\lambda, i) =$
Assume that $\mathcal{MGH}(H; I, A, M) := I \times H \times A$, we define the following hyper-composition:

\[
\circ : \mathcal{MGH}(H; I, A, M) \times \mathcal{MGH}(H; I, A, M) \to P^*(\mathcal{MGH}(H; I, A, M))
\]

\[
((i, x, \lambda), (j, y, \mu)) \mapsto (i, x, \lambda) \circ (j, y, \mu),
\]

which for all $i$ and $j$ in $I$, for all $x$ and $y$ in $H$ and for all $\lambda$ and $\mu$ in $A$,

\[
(i, a, \lambda) \circ ((j, b, \mu) \circ (k, c, \nu)) = (i, a, \lambda) \circ \bigcup_{s \in b \ast m_{\lambda j}^{-1} \ast a} (j, s, \nu) \circ (k, c, \nu)
\]

Therefore, $\mathcal{MGH}(H; I, A, M)$ is a semihypergroup.

Theorem 2.1. $\mathcal{MGH}(H; I, A, M)$ is a semihypergroup.

Proof. Let $i$ and $j$ in $I$, $\lambda$ and $\mu$ in $A$, and $a, b, c$ in $H$. Since $(a \ast m_{\lambda j} \ast b)$ is a non-empty subset of $H$, so

\[
\{i\} \neq (i, x, \lambda) \circ (j, y, \mu) \in P^*(\mathcal{MGH}(H; I, A, M)).
\]

Therefore, $\circ$ is a hyperoperation. Now we check the associative property of hyperoperation $\circ$.

We have the following equations:

\[
(i, a, \lambda) \circ ((j, b, \mu) \circ (k, c, \nu)) = (i, a, \lambda) \circ \bigcup_{s \in b \ast m_{\lambda j}^{-1} \ast a} (j, s, \nu) \circ (k, c, \nu)
\]

Therefore, $\mathcal{MGH}(H; I, A, M)$ is a semihypergroup.

Theorem 2.2. For every element $(i, a, \lambda) \in \mathcal{MGH}(H; I, A, M)$, there is a unique non-empty subset $E(i, a, \lambda) \subseteq \mathcal{MGH}(H; I, A, M)$, such that for every element $(j, b, \mu)$ of $E(i, a, \lambda)$, implies $(i, a, \lambda) \in (i, a, \lambda) \circ (j, b, \mu) \cap (j, b, \mu) \circ (i, a, \lambda)$. Moreover,

\[
E(i, a, \lambda) = \{i\} \times (m_{\lambda}^{-1} \ast a \ast i \ast a) \cap (a \ast a^{-1} \ast m_{\lambda}) \times \{\lambda\}.
\]

Proof. Since $m_{\lambda}$ is an element of polygroup $H$, there exist $m_{\lambda}^{-1}$, such that $e \in \{m_{\lambda}^{-1} \ast m_{\lambda} \cap (m_{\lambda} \ast m_{\lambda})^{-1}\}$. Now, we have:

\[
(i, a, \lambda) \circ (i, m_{\lambda}^{-1}, \lambda) = \{i\} \times (a \ast m_{\lambda} \ast m_{\lambda}^{-1}) \times \{\lambda\}
\]

Also,

\[
(i, a, \lambda) \circ (i, a, \lambda) = \{i\} \times \{a\} \times \{\lambda\} = \{i\} \times (a \ast a) \times \{\lambda\}
\]

Therefore, $(i, m_{\lambda}^{-1}, \lambda)$ is an element of $E(i, a, \lambda)$.

If $(j, b, \mu)$ be an arbitrary element of $E(i, a, \lambda)$, then we have:

\[
(i, a, \lambda) \in (i, a, \lambda) \circ (j, b, \mu) \cap (j, b, \mu) \circ (i, a, \lambda)
\]

\[
= \{i\} \times (a \ast m_{\lambda} \ast b) \times \{\mu\} \cap \{j\} \times (b \ast m_{\lambda} \ast a) \times \{\lambda\}.
\]
Therefore, \( j = i \) and \( \mu = \lambda \) and \( a \in (a * m_{ij} * b) \cap (b * m_{ji} * a) \). Since, 
\[
a \in (a * m_{ij} * b) \iff b \in (a * m_{ji})^{-1} * a = m_{ji}^{-1} * a^{-1} * a,
\]
\[
a \in (b * m_{ji} * a) \iff b \in a * (m_{ji})^{-1} = a * a^{-1} * m_{ji}^{-1},
\]
therefore, \( b \in \{m_{ji} * a^{-1} * a\} \cap (a * a^{-1} * m_{ji}^{-1}) \).

Conversely if \( b \in \{m_{ij} * a^{-1} * a\} \cap (a * a^{-1} * m_{ij}^{-1}) \), then \( (i, b, \lambda) \) is an element of \( E \) \((i, a, \lambda) \) and the proof is complete.

Theorem 2.3. For every element \((i, a, \lambda) \in \mathcal{MGH} \((H; I, \Lambda, M)\)\), there is a non-empty subset \( \mathcal{J}(i, a, \lambda) \in \mathcal{MGH} \((H; I, \Lambda, M)\)\), such that for all \( \beta \) in \( \mathcal{J}(i, a, \lambda) \)

\[
[\{i, a, \lambda\} \cap \{\beta \cap (i, a, \lambda)\}] = E \((i, a, \lambda)\).
\]

Proof. Assume that \( c = m_{ji} \), by choosing \( \mathcal{J}(i, a, \lambda) = \{i\} \times (c^{-1} * a^{-1} * c^{-1}) \times \{\lambda\} \). which, it is a non-empty subset of \( \mathcal{MGH} \((H; I, \Lambda, M)\)\), we show that it satisfies the condition of Theorem. Since \( H \) is a polygroup, hence, \((a) = a * e \) and \( e \in c^{-1} * c^{-1} \) and \( e \in a * a^{-1} \) then we have:

\[
e \in a * a^{-1} = (a * e) * a^{-1} = (a * (c * c^{-1} * a^{-1})) = (a * c * c^{-1} * a^{-1}).
\]

Then \( c^{-1} \subseteq \{c^{-1}\} = e \subseteq (c * a) * (c * a) = (c * c) = c * a \).

Similarly,

\[
e \in a^{-1} * a = (a^{-1} * e) * a = (a^{-1} * (c * c * a) = (a^{-1} * c * c * a).\]

Then \( c^{-1} \subseteq \{c^{-1}\} = c * e \subseteq (a^{-1} * c^{-1} * c^{-1}) = (c * a) * (c * a) \).

Also, \( c^{-1} \subseteq (c^{-1} * a^{-1} * a) \cap (a * a^{-1} * c^{-1}) \), hence, \( c^{-1} \) is an element of the following set

\[
[(a * c) * (c^{-1} * a^{-1} * c^{-1}) \cap \{(c^{-1} * a^{-1} * c^{-1}) * (c * a) \cap \{(c^{-1} * a^{-1} * a) \cap \{(a * a^{-1} * c^{-1})],
\]

then,

\[
(i, m_{ji}^{-1}, \lambda) \in [\{i, a, \lambda\} \cap \{\mathcal{J}(i, a, \lambda) \cap (i, a, \lambda) \cap E (i, a, \lambda)\]
\]

and the proof is complete.

Theorem 2.1, 2.2 and 2.3 guidance us to follows for definition of a generalization of Molaei’s generalized group.

Definition 2.1. A semihypergroup \((\mathcal{H}, \odot)\) is called Molaei’s generalized hypergroup, if it satisfies in the following conditions:

\[(MGH) \forall h \in \mathcal{H}, \exists! \mathcal{E} (h) \subseteq \mathcal{H}, \text{ such that for every element } \alpha \in \mathcal{E} (h), h \in [h \odot \alpha] \cap \{a \odot h\}.
\]

\[(MGH1) \forall h \in \mathcal{H}, \exists! \mathcal{J} (h) \subseteq \mathcal{H}, \text{ such that for every element } \beta \in \mathcal{J} (h), [h \in \beta \cap \{\beta \cap h \} = \mathcal{E} (h).
\]

(The symbol \( \exists! \) means there is a unique.)

Example 2.1. If \(<H, *, \cdot>\) is a polygroup and let \(I, \Lambda\) be non-empty sets and \(M\) be a map from \(A \times I\) to \(H\) by \(M (i, j) = m_{ji}\). Then, by use Theorems 2.1, 2.2 and 2.3, \(\mathcal{MGH} \((H; I, \Lambda, M)\)\), with hyperoperation \(\odot\) is a Molaei’s generalized hypergroup.

Example 2.2. Every polygroup is a Molaei’s generalized hypergroup. If \(<H, *, \cdot>\) be a polygroup, it is semihypergroup and for every element \(h \in H\),

\[
\mathcal{E} (h) = (h * h^{-1}) \cap (h^{-1} * h) \text{ and } \mathcal{J} (h) = \{h^{-1}\}.
\]

Example 2.3. Every Molaei’s generalized group is a Molaei’s generalized hypergroup. If \(G\) is a Molaei’s generalized group, we consider the hyperoperation \(x * y = \{xy\}\), then \((G, *)\) is a semihypergroup and for every \(h \in G\),

\[
\mathcal{E} (h) = \{e(h)\} \text{ is unique and } \mathcal{J} (h) = \{h^{-1}\}.
\]

3. Conclusion

This paper deal with one of the newest construction of a generalization of hypergroups. We changed the group to the polygroup in the structure of Rees matrix semigroup and we obtained a new construction. By using this construction we defined “Molaei’s generalized hypergroup” and we gave some examples.

References


