Pricing Options on Ghanaian Stocks Using Black-Scholes Model

Osei Antwi¹, *, Francis Tabi Oduro²

¹Mathematics & Statistics Department, Accra Technical University, Accra, Ghana
²Department of Mathematics, Faculty of Physical and Computational Science, College of Science, Kwame Nkrumah University of Science & Technology, Kumasi, Ghana

Email address: osclantwi@yahoo.co.uk (O. Antwi), ftoduro@gmail.com (F. T. Oduro)

*Corresponding author

To cite this article:

Received: September 13, 2017; Accepted: September 30, 2017; Published: February 1, 2018

Abstract: We present a succinct new approach to derive the Black-Scholes partial differential equation and subsequently the Black-Scholes formula. We proceed to use the formula to price options using stocks listed on Ghana stock exchange as underlying assets. From one year historical stock prices we obtain volatilities of the listed stocks which are subsequently used to compute prices of three month European call option. The results indicate that it is possible to use the Black Scholes formula to price options on the stocks listed on exchange. However, it was realised that most call option prices tend to zero either due to very low volatilities or very low stock prices. On the other hand put options were found to give positive prices even for stocks with very low volatilities or low stock prices.

Keywords: Option Price, Volatility, Stochastic Process, Brownian Motion, Geometric Brownian Motion, Black–Scholes Formula

1. Introduction

An option is a contract between two parties in which the option buyer or holder purchases the right to buy or sell an underlying asset at a fixed time. Options are usually traded on stock exchanges and Over the Counter markets. The two types of options are call options and put options. An option gives the holder the right to buy or sell an underlying asset but he is in no way obliged to exercise this right. This is the main feature that distinguishes an option from other instruments such as futures and contracts. Currently, options are not traded on Ghana Stock Exchange (GSE) but it is a very lucrative business so much so that in 1972 the Chicago Stock Exchange was purposely opened in the United States of America to trade solely in options. The introduction of options trading on the GSE will greatly enhance the financial sector and attract hedgers with huge foreign investments. In addition, businesses, government institutions and other establishments can reduce the inherent market and credit risk in contracts and other market variables through hedging in options. Moreover, it will also create avenues for job opportunities as financial concerns set up hedge funds and brokerages to trade options and related derivatives.

One basic reason why options are currently not traded on the Ghanaian market is that the mathematical models required to price options have not been rigorously examined among researchers in Ghana. The aim of this paper is to begin the mathematical debate on the possibility of using various models to price options on Ghanaian assets. In view of this, we believe that there is no better starting point than the Black Scholes model. Specific objectives of this paper are to

• explain and simplify the mathematical models required to evaluate an option price
• use the model to price options with stocks listed on the GSE as the underlying assets
• test the model’s behaviour when the underlying asset exhibit very low volatility or low stock price which is synonymous with the GSE.

In using the Black–Scholes model, we must emphasize here that although one may point to the inadequacies of the
formula as a pricing model we still wish to use it because we believe this paper will serve as a platform to ignite interest in researchers and financial engineers in the country to develop more robust models in pricing options. It is our hope that this paper would serve as a foundation upon which other option strategies and derivative pricing methods could develop. In addition, it is important to realise that despite its limitations the Black–Scholes model still remain the benchmark of option valuation and it is the standard to which all other pricing models are compared.

The theory of option pricing began in the 1900s when Louis Bachelier (1900) provided a valuation for stock options based on the assumption that stock prices follow a Brownian motion. Kendall (1953), Roberts (1959), Osborne (1959), Sprengle (1961) all conducted studies into stock price behaviour and concluded that the stock price processes follow the lognormal distribution instead of a normal distribution assumed by Bachelier. The lognormal assumption ruled out the possibility of negative stock prices. Boness (1964) improved the option pricing model by accounting for the time value of money. However, it was Samuelson (1965) who proposed the Geometric Brownian Motion (GBM) as the model for the underlying stock in pricing options. Samuelson option pricing model was however not very popular as it required one to compute individual risk. The lack of certainty about a measure of an individual’s risk characteristics meant that investors and sellers could not agree on a single option price. Despite these uncertainties, most financial engineers and economists have accepted the GBM as a model for the underlying stock price because it is everywhere positive as against Brownian motion which can give negative stock prices. Thus, GBM is now the most widely accepted formula for modelling stock price behaviour. Despite these early developments in option price modeling it was not until 1973 when Black and Scholes published a seminal paper in which they obtained a closed-form formula to calculate European calls that option trading took off in earnest. The introduction of the Black–Scholes formula is often regarded as the apogee of the option pricing technique in options pricing. Mehrdoust et. al. (2017) examined the Monte Carlo technique in options pricing. Mehrdoust et. al. (2017) examined the Monte Carlo option pricing under the constant elasticity of variance model.

2. Methodology

The basic idea of Black–Scholes equation was to construct a portfolio from stocks and bonds that yields the same return as a portfolio consisting only of an option. In this hedged portfolio the risky stock is modeled as a stochastic process and the riskless bond is modeled as deterministic process.

2.1. The Black–Scholes Partial Differential Equation

We begin by looking at the arguments leading to the developments of the Black–Scholes partial differential equation. The method of solution to the partial differential equation leads to the Black–Scholes formula. The Black–Scholes model prices options using stocks as the underlying assets. The diffusion of the stock price process is however captured as a Geometric Brownian Motion and as such we begin by examining GBM.

2.1.1. Geometric Brownian Motion

In a risky stock, the stock price \( S(t) \) is assumed to follow the lognormal process and is modelled by the Geometric Brownian Motion (GBM) as

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW(t)
\]  

(1)

where \( \mu \) is the return on the stock, \( \sigma \) is the standard deviation of \( \mu \) or simply the stock’s volatility and \( W(t) \) is the standard Brownian motion or the Wiener process with mean 0 and standard deviation \( t \). To determine the solution to the Geometric Brownian Motion, let

\[
f(x) = \ln S(t)
\]

\[
f'(x) = \frac{1}{S(t)}
\]

\[
f''(x) = -\frac{1}{S^2(t)}
\]
Since \( S(t) \) is a stochastic process it follows that \( f(x) \) is an Ito process and so by Ito formula

\[
d(fS(t)) = f'(S(t))dS(t) + \frac{1}{2}f''(S(t))\sigma^2 dt
\]

Hence

\[
d(\ln S(t)) = \frac{1}{S(t)}dS(t) + \frac{1}{2}\left(-\frac{1}{S^2(t)}\right)\sigma^2 S^2(t)dt
\]

\[
\frac{1}{S(t)}(\mu S(t)d(t) + \sigma S(t)dW(t)) - \frac{1}{2}\sigma^2 dt
\]

\[
d\ln S(t) = \mu - \frac{1}{2}\sigma^2 dt + \sigma dW(t)
\]

And dropping arguments

\[
S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}
\]

2.1.2 Derivation of the Black–Scholes Partial Differential Equation

The key mathematical theory underlying the Black-Scholes equation is the Itô’s lemma.

Itô’s Lemma

Assume that \( X(t) \) is a stochastic process with the stochastic differential

\[
dX(t) = \mu dt + \sigma dW(t)
\]

Thus the stochastic process followed by \( V(S(t), t) \) in Equation 5 is also an Ito process with drift

\[
\left(\mu S(t) - \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V(S(t), t)}{\partial S^2(t)} + \frac{\partial V(S(t), t)}{\partial t}\right)
\]

and variance

\[
\left(\sigma S \frac{\partial V(S(t), t)}{\partial S(t)} \right)^2
\]

The Hedging Argument to Create a Riskless Portfolio

Now construct a portfolio in which we buy 1 option \( \uparrow \)

\[
d\pi = \left(\mu S(t) - \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V(S(t), t)}{\partial S^2(t)} + \frac{\partial V(S(t), t)}{\partial t}\right) dt + \left(\sigma S(t) \frac{\partial V(S(t), t)}{\partial S(t)} \right) dW(t)
\]

\[
d\pi = dV(S(t), t) - \Delta dS(t)
\]

We also realise that the stochastic process for the hedged portfolio is an Ito process with drift parameter

\[
\left(\mu S(t) - \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 V(S(t), t)}{\partial S^2(t)} + \frac{\partial V(S(t), t)}{\partial t} - \mu \Delta S(t)\right)
\]

and variance

\[
\left(\sigma S(t) - \frac{\partial V(S(t), t)}{\partial S(t)} - \Delta \sigma S(t)\right)^2
\]

where \( \mu \) and \( \sigma \) are adapted processes. Let \( f(X(t), t) \) be an \( \uparrow \)

Ito process and a twice differentiable function, then \( f(X(t), t) \) has the stochastic differential

\[
df(X(t), t) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dW(t)
\]

The derivation of the Black Scholes model can be summarized in three key arguments:

- Itô formula application to the value of a replicating portfolio of the option
- The hedging argument to create a riskless portfolio
- The no arbitrage arguments of a risk free return of the portfolio

Ito Formula Application to the Value of a Replicating Portfolio

Consider an asset whose price process \( S(t) \) follows the Geometric Brownian Motion such that

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)
\]

Equation (4) is an Ito process with mean \( \mu S(t) \) and \( \text{Var}(S(t)) = [\sigma S(t)]^2 \).

Consider a contingent claim on \( S(t) \) whose value \( V(S(t), t) \) depends on the \( S(t) \) and \( t \). By Itô lemma of Equation (3), the change in \( V(S(t), t) \) is given by Equation (5).

\[
dV(S(t), t) = \mu S(t)dt + \sigma S(t)dW(t)
\]

Equation (5) consist of two parts: a deterministic part given by

\[
d\pi = dV(S(t), t) - \Delta dS(t)
\]
\[
\left(\mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 V(S(t), t)}{\partial S(t)^2} + \frac{\partial V(S(t), t)}{\partial t} - \mu \Delta S(t)\right) dt
\]
and a stochastic part given by
\[
\left(\sigma S(t) \frac{\partial V(S(t), t)}{\partial S(t)} - \sigma \Delta S(t)\right) dW(t)
\]
(8)

To make the portfolio completely riskless the stochastic part i.e. Equation (8) must vanish. In other words we should have
\[
d\pi = \left(\mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 V(S(t), t)}{\partial S(t)^2} + \frac{\partial V(S(t), t)}{\partial t} - \mu \Delta S(t)\right) dt
\]
(10)

Replacing \(\Delta\) in Equation (10) by \(\frac{\partial V(S(t), t)}{\partial S(t)}\) then we have
\[
d\pi = \left(\mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 V(S(t), t)}{\partial S(t)^2} + \frac{\partial V(S(t), t)}{\partial t} - \mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)}\right) dt
\]
(11)

It follows that to completely hedge the portfolio we must purchase \(\frac{\partial V(S(t), t)}{\partial S(t)}\) of the underlying asset. It means that to have a riskless portfolio we must purchase an amount of stocks that is equal to the ratio of how much the option value changes relative to the change in value of the stock. However, this situation is only valid in a small time interval and so we must continuously change the amount of stocks purchased to rebalance \(\frac{\partial V(S(t), t)}{\partial S(t)}\).

The no Arbitrage Arguments and a Risk Free Return of the Portfolio

The change in the portfolio’s value is \(d\pi\). So what is the return of this riskless portfolio in a small time step \(dt\). Black and Scholes suggested that the return must be the risk free rate \(r\) otherwise there will be arbitrage opportunities. If this is the case then owning \(\pi\) amount of the portfolio would provide a return of \(r\pi dt\) in a small time interval \(dt\). Consequently,

\[
d\pi = r\pi dt
\]

Replacing \(d\pi\) by \(r\pi dt\) in Equation 11 we have
\[
r\pi = \left(V(S(t), t) - \Delta S(t)\right) = \left(V(S(t), t) - S(t) \frac{\partial V}{\partial S(t)}\right) and so
\]
\[
r \left(V(S(t), t) - S(t) \frac{\partial V(S(t), t)}{\partial S(t)}\right) = \left(\mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 V(S(t), t)}{\partial S(t)^2} + \frac{\partial V(S(t), t)}{\partial t} - \mu S(t) \frac{\partial V(S(t), t)}{\partial S(t)}\right)
\]
(12)

At maturity, the price of the option \(C(S(t), t)\) is equal to the value of the hedged portfolio and so \(C(S(t), t) = V(S(t), t)\). Hence Equation (12) is rewritten as

\[
\frac{\partial C(S(t), t)}{\partial t} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 C(S(t), t)}{\partial S(t)^2} + rS(t) \frac{\partial C(S(t), t)}{\partial S(t)} - rC(S(t), t) = 0
\]
(13)

Equation is the Black Scholes partial differential equation. For a European call option the boundary conditions are
\[
C(0, t) = 0, \quad C(S(T), T) = \max(S(T) - K, 0), \quad t \geq 0
\]

2.2. The Black–Scholes Formula

Theorem 1
Let \(f(S(t), t)\) be a differentiable function satisfying the partial differential equation
\[
\frac{\partial f(S(t), t)}{\partial t} + \mu(X(t), t) \frac{\partial f(S(t), t)}{\partial S(t)} + \frac{1}{2} \sigma(X(t), t)^2 \frac{\partial^2 f(S(t), t)}{\partial S(t)^2} - r(X(t), t) f(S(t), t) = 0
\]

with boundary condition \( f(X(T), T) = g(x) \). Then \( C(S(t), t) \) is the solution and

\[
C(S(t), t) = \mathbb{E}_Q \left( \left( e^{-r(T-t)^2} \right) g(S(T)) \right |_{\mathcal{F}_t}
\]

Theorem 1 asserts that if \( f(S(t), t) \) satisfies the Black-Scholes partial differential equation then \( C(S(t), t) \) can be represented as an expectation. It follows that the option price can be considered as the discounted value of the expected option payoff under the martingale measure \( Q \) such that

\[
C(S(t), T) = e^{-r(T-t)} \mathbb{E}_Q \left[ \max(S(T) - K, 0) \right]
\]

where \( \mathbb{E}_Q \) is the expectation taken under the equivalent martingale measure \( Q \), \( S(T) \) is the terminal stock price given by

\[
S(T) = S_0 \left( e^{\left( r - \frac{\sigma^2}{2} \right) T + \sigma W(T)} \right)
\]

The price of the expected payoff of the option is now given by

\[
C(S(t), t) = e^{-r(T-t)} \int_0^\infty f(x) \max(x - K, 0) \, dx
\]

Where \( f(x) \) is the density of the lognormal random variable \( X \) given by

\[
f(x) = \frac{1}{x \sigma \sqrt{2\pi}} \exp \left( - \frac{(\mu - \ln x)^2}{2 \sigma^2} \right)
\]

Where \( \mu = \mathbb{E}(\ln S(T)) \) = mean stock price and \( \sigma^2 = \text{Var}(\ln S(T)) \) the variance of the return. Now if \( S(T) < K \), the option will not be exercised and so \( \max(S(T) - K, 0) \) will be 0. We are therefore interested in the price distribution when \( S(T) > K \). So we can write

\[
C(S(t), t) = e^{-r(T-t)} \int_k^\infty f(x)(x - K) \, dx
\]

\[
C(S(t), t) = e^{-r(T-t)} \int_k^\infty S_0 f(x) dx - Ke^{-r(T-t)} \int_k^\infty f(x) dx
\]

Now the last integral \( \int_k^\infty f(x) dx \) is the probability of the event that \( S(T) > K \). So the last integral is equivalent to the statement \( P(S(T)) > K \).

Now \( S(t) = S_0 e^{\left( r - \frac{\sigma^2}{2} \right) T + \sigma W(T)} \) and so

\[
P(S(T)) > K = P \left( S_0 e^{\left( r - \frac{\sigma^2}{2} \right) T + \sigma W(T)} > K \right)
\]

\[
P(S(T)) > K = P \left( e^{\left( r - \frac{\sigma^2}{2} \right) T + \sigma W(T)} > \frac{K}{S_0} \right)
\]

\[
P(S(T)) > K = P \left( r - \frac{\sigma^2}{2} T + \sigma W(T) > \ln \left( \frac{K}{S_0} \right) \right)
\]

We now compute the first integral

\[
\int_k^\infty xf(x) dx
\]

Let \( I = \int_k^\infty xf(x) dx \)

But \( f(x) = \frac{1}{x \sigma \sqrt{2\pi}} \exp \left( - \frac{(\mu - \ln x)^2}{2 \sigma^2} \right) \)

Hence

\[
I = \int_k^\infty x \frac{1}{x \sigma \sqrt{2\pi}} \exp \left( - \frac{(\mu - \ln x)^2}{2 \sigma^2} \right) dx
\]

\[
I = \frac{1}{\sigma \sqrt{2\pi}} \int_k^\infty \exp \left( - \frac{(\mu - \ln x)^2}{2 \sigma^2} \right) dx
\]

Now the first natural change of variables is \( \ln x = s \), \( x = e^s \), and \( dx = e^s ds \) and this gives

\[
I = \frac{1}{\sigma \sqrt{2\pi}} \int_k^\infty \exp \left( - \frac{(\mu - s)^2}{2 \sigma^2} \right) e^s ds
\]
\[
I = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(\mu - s)^2}{2\sigma^2} + s\right) ds
\]

Now completing the square we have

\[
-\frac{(\mu - s)^2}{2\sigma^2} + s = \frac{(s - (\mu + \sigma^2))^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu
\]

Hence

\[
I = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(s - (\mu + \sigma^2))^2}{2\sigma^2} + \frac{\sigma^2}{2} + \mu\right] ds
\]

\[
l = \exp\left(\frac{\sigma^2}{2} + \mu\right) \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(s - (\mu + \sigma^2))^2}{2\sigma^2}\right] ds
\]

The expression under the integrand is the density function of a normal variable with mean \(\mu = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T\) and variance \(\sigma^2 T\).

Now \(l = \exp\left(\frac{\sigma^2}{2} + \mu\right) - 1 - N(\ln K; \mu + \sigma^2, \sigma^2)\) and \(S_0 e^{\left(\frac{\sigma^2}{2} + \mu\right)T + \sigma tW(T)}\)

Hence we replace \(\sigma^2\) in equation (15) by \(\sigma^2 T\) and \(\mu\) by \(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T\).

\[
l = \exp\left(\frac{\sigma^2}{2} + \mu\right) - 1 - N(\ln K; \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T)
\]

\[
l - N(x) = 1 - N(\ln K; \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T)
\]

\[
l = -\frac{\ln\left(\frac{\ln S_0 - \left(r + \frac{\sigma^2}{2}\right) T}{\sigma^2 T}\right)}{\sigma \sqrt{T}}
\]

and so

\[
l = S_0 e^{\left(r - \frac{\sigma^2}{2}\right) T} \left[-N\left(\ln\left(\frac{\ln S_0 - \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)\right)\right]
\]

\[
l = S_0 e^{\left(r - \frac{\sigma^2}{2}\right) T} N\left(-\frac{\ln K - \ln S_0 - \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)
\]

\[
l = S_0 e^{\left(r - \frac{\sigma^2}{2}\right) T} N\left(-\frac{\ln K + \ln S_0 + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)
\]

\[
l = S_0 e^{\left(r - \frac{\sigma^2}{2}\right) T} N\left(\frac{\ln S_0 + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)
\]

\[
\frac{\ln S_0 + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}
\]

Let \(d_1 = \left(\frac{\ln S_0 + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right)\)

Then

\[
l = S_0 e^{rT} N(d_1)
\]

Now from Equation (13) we can write

\[
C(S(t), t) = e^{-r(t-t)} - Ke^{-r(t-t)}P(S(T) > K)
\]

Replace \(T\) by \(T - t\) in Equation (17) and substitute Equations (14) and (17) into Equation (18) we have

\[
C(S(t), t) = e^{-r(t-t)} - Ke^{-r(t-t)}P(S(T) > K)
\]

\[
= e^{-r(t-t)} S_0 e^{r(t-t)} N(d_1)
\]

\[
= e^{-r(t-t)} N(d_2)
\]

\[
C(S(t), t) = S_0 N(d_1) - Ke^{-r(t-t)} N(d_2)
\]

Replacing \(S_0\) by \(S(t)\) we obtain the Black–Scholes formula for the price of an European call option as

\[
C(S(t), t) = S(t) N(d_1) - Ke^{-r(t-t)} N(d_2)
\]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(t-t)}{\sigma \sqrt{t-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{t-t} = \frac{\ln(S_0/K) + (r - \sigma^2/2)(t-t)}{\sigma \sqrt{t-t}}
\]

and

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt
\]

\(N(x)\) is the cumulative distribution function of a standard normal variable.

In order to arrive at the formula Black and Scholes made the following assumptions on the stock price and the market:

- There are no dividends payment on the stock during the option's life
- The model assumes European-style
- Markets are assumed to be efficient
- There are no transaction costs in buying or selling the asset or the option, no barriers to trading and no taxes.
- Interest rates remain constant and equal to the risk-free rate
- Returns on the underlying stock are lognormally distributed
- The price of the underlying asset is divisible so that we can trade any fractional share of assets

2.3. Interest Rate and Volatility Models of a Stock Price

Return processes are called interest rates and they can either be a constant or following a stochastic model. Examples of stochastic interest rate models include the Vasicek model with a return process

\[
dR(t) = (\alpha - \beta R(t)) dt + \sigma dW(t)
\]

\(\alpha, \beta\) and \(\sigma\) are positive constants. The solution to the Vasicek model is the process

\[
R(t) = R(0) e^{-\beta(t-t)} + \frac{\alpha - \beta R(0)}{\beta} + \sigma \sqrt{\frac{t-t}{2\beta}} Z(t-t)
\]

where \(Z(t-t)\) is a standard normal random variable.
\[ R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s) \]

The other most generally used stochastic interest rate model is the Cox–Ross–Ingersoll model whose return rate process is given by
\[ dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t) \tag{24} \]
where \( \alpha > 0, \beta > 0 \) and \( \sigma > 0 \) are all constants.

Other commonly used stochastic interest rate models include the Hull–White model, also called the extended Vasicek model, the Ho-Lee model and the Black–Karasinski model. The most popular stochastic volatility model is the Heston model where the stock prices process is modeled as
\[ dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t) \tag{25} \]
with
\[ d\nu(t) = \xi Vdt + \alpha \sqrt{\nu(t)}dW_2(t) \]
- \( \nu \) is the variance, which follows a square–root process.
- \( dW_1 \) and \( dW_2 \) are two Wiener processes having correlation \( \rho \).
- \( \mu \) is constant risk-free rate and both \( \mu, \xi \) may depend on \( \nu \) and \( t \).

Other stochastic volatility models include the constant elasticity of variance model and the Generalized Autoregressive conditional Heteroskedasticity (GARCH) model.

For the purpose of this study we will assume a constant interest rate and constant volatility for the option. We compute a one year historical volatility and use it to compute a three month call and put option prices for the stocks listed on GSE. Data of stock prices for (2015) on GSE are used to compute the volatilities.

The Ghana government 90-day Treasury bill rate is used as the constant risk-free interest rate. A time interval of three months is selected as the lifetime of the option. The closing stock price of January 4, 2016 is assumed as the initial stock price \( S_0 \) of the option. The volatility \( \sigma \) of a stock is calculated as
\[ \sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} \mu^2 - \frac{1}{n(n-1)}(\sum_{i=1}^{n} \mu)^2} \tag{26} \]
where \( n \) is the number of trading days in a particular year and
\[ \mu = \frac{1}{n} \sum_{i=1}^{n} \ln \left( \frac{S(t)}{S(t-i)} \right) \tag{27} \]

The summary of volatility values for all listed stocks on GSE can be found in Table 3.

3. Results

We now evaluate the price of option sold on the stocks using the Black-Scholes formula, i.e. Equation (23). Let’s first consider a theoretical example.

Example 1
The stock price 6 months from expiry of an option is \( \varepsilon 42 \).

The strike price of the option is \( \varepsilon 40 \). The risk free interest rate is 10% per annum and the volatility is 20% per annum. The price of an option written on this stock is given by
\[ C(S(t),t) = S_0 N(d_1) - Ke^{-r(T-t)}N(d_2) \]
where
\[ d_1 = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \]
\[ d_2 = d_1 - \sigma\sqrt{T - t} = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \]

Now \( S_0 = 42, \; K = 40, \; r = 0.1, \; T - t = 0.5, \; \sigma = 0.2 \)
\[ d_1 = \frac{\ln(\frac{42}{40}) + (0.1 + \frac{0.2^2}{2})(0.5)}{0.2\sqrt{0.5}} = 0.7693 \]
\[ N(d_1) = 0.7791 \]
\[ d_2 = \frac{\ln(\frac{42}{40}) + (0.1 - \frac{0.2^2}{2})(0.5)}{0.2\sqrt{0.5}} = 0.6278 \]
\[ N(d_2) = 0.7349 \]
\[ C(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \]
\[ = 42(0.7791) - 40e^{-0.1\times0.5}(0.7349) \]
\[ = 42(0.7791) - 38.0492(0.7349) \]
\[ = 32.7222 - 27.9622 \]
\[ C(S(t),t) = 4.76 \]

Thus, a call option sold on this stock would cost \( \varepsilon 4.76p \).

The put price is given by
\[ P(S(t),t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) \]
\[ N(-d_1) = N(-0.7693) = 0.2209 \]
\[ N(-d_2) = N(-0.6278) = 0.2651 \]
\[ P(S(t),t) = 40e^{-0.1\times0.5}(0.2651) - 42 \times (0.2209) \]
\[ = 38.049 \times 0.2651 - 42 \times (0.2209) \]
\[ = 0.8089 \]
\[ P(S(t),t) = 0.81 \]

Thus, a put option sold on this stock would cost \( \varepsilon 0.81p \).

We use Microsoft Excel to evaluate the option price and the results are presented in Figure 1. In Excel, we add form/active control to all the parameters; Stock Price, Strike Price, Volatility, Risk Free Rate and Time. This allows us to evaluate the option price at different parameter levels for any stock.
We present here the call and put option price for one of the listed stocks-Tullow Oil. The closing stock price for Tullow Oil for January 4, 2016 was ₵28 and the annual volatility of its stock price is 10%. The Ghana government risk free rate is 23% and we choose a strike price of ₵30. The price of a three month European call option sold on Tullow stocks is computed as follows:

\[
C(S(t), t) = S(t)N(d_1) + Ke^{-r(T-t)}N(d_2)
\]

where \(d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}\)

\[
d_2 = d_1 - \sigma \sqrt{T-t} = \frac{\ln(S_0/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
\]

Now \(S_0 = 28, \ K = 30, \ r = 0.23, \ T-t = \Delta t = 3 \text{ months} = 0.25, \ \sigma = 0.1\)

\[
d_1 = \frac{\ln(28/30) + (0.23 + \frac{0.1^2}{2}) \times (0.25)}{0.1 \times \sqrt{0.25}} = -0.2049\]

\(N(d_1) = 0.4188\)

\[
d_2 = \frac{\ln(28/30) + (0.1 - \frac{0.1^2}{2}) \times (0.25)}{0.1 \times \sqrt{0.25}} = -0.2549\]

\(N(d_2) = 0.3994\)

\[
C(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)
\]

\[
= 28(0.4188) - 30e^{-0.23 \times 0.25}(0.3994)
\]

\[
= 28(0.4188) - 28.3237(0.3994)
\]

\(C(S(t), t) = 0.4139\)

The price of a 3-month European call option sold on Tullow stocks would cost 41p.

The put price is given by

\[
P(S(t), t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1)
\]

\(N(-d_1) = N(-(-0.2049)) = 0.5812\)

\(N(-d_2) = N(-(-0.2549)) = 0.6006\)

\[
P(S(t), t) = 30e^{-0.23 \times 0.25}(0.6006) - 28 \times (0.5812)
\]

\(P(S(t), t) = 0.7384\)

The put option price sold on Tullow stocks is 74p. The results in are presented in Table 2.

### Table 1. Option Price Results for Example 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(d_1)</th>
<th>(N(d_1))</th>
<th>(d_2)</th>
<th>(N(d_2))</th>
<th>(Ke^{-rt})</th>
<th>Call Price</th>
<th>(N(-d_1))</th>
<th>(N(-d_2))</th>
<th>Put Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price</td>
<td>42</td>
<td>0.7693</td>
<td>0.7791</td>
<td>0.6278</td>
<td>0.7349</td>
<td>38.0492</td>
<td>0.2209</td>
<td>0.2651</td>
<td>0.8086</td>
</tr>
<tr>
<td>Strike Price</td>
<td>40</td>
<td>0.7693</td>
<td>0.7791</td>
<td>0.6278</td>
<td>0.7349</td>
<td>38.0492</td>
<td>0.2209</td>
<td>0.2651</td>
<td>0.8086</td>
</tr>
<tr>
<td>Volatility</td>
<td>20</td>
<td>0.7693</td>
<td>0.7791</td>
<td>0.6278</td>
<td>0.7349</td>
<td>38.0492</td>
<td>0.2209</td>
<td>0.2651</td>
<td>0.8086</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>10</td>
<td>0.7693</td>
<td>0.7791</td>
<td>0.6278</td>
<td>0.7349</td>
<td>38.0492</td>
<td>0.2209</td>
<td>0.2651</td>
<td>0.8086</td>
</tr>
<tr>
<td>Time to Expiry</td>
<td>6</td>
<td>0.7693</td>
<td>0.7791</td>
<td>0.6278</td>
<td>0.7349</td>
<td>38.0492</td>
<td>0.2209</td>
<td>0.2651</td>
<td>0.8086</td>
</tr>
</tbody>
</table>

### Table 2. Results of Option Pricing on Tallow Stock.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(d_1)</th>
<th>(N(d_1))</th>
<th>(d_2)</th>
<th>(N(d_2))</th>
<th>(Ke^{-rt})</th>
<th>Call Price</th>
<th>(N(-d_1))</th>
<th>(N(-d_2))</th>
<th>Put Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price</td>
<td>28</td>
<td>0.2049</td>
<td>0.4188</td>
<td>0.2549</td>
<td>0.3994</td>
<td>28.3237</td>
<td>0.4146</td>
<td>0.5812</td>
<td>0.6006</td>
</tr>
<tr>
<td>Strike Price</td>
<td>30</td>
<td>0.4188</td>
<td>0.2549</td>
<td>0.3994</td>
<td>28.3237</td>
<td>0.4146</td>
<td>0.5812</td>
<td>0.6006</td>
<td>0.7384</td>
</tr>
</tbody>
</table>
price, the stock volatility as well as the call and put price. We realise that stocks on the exchange are characterized by the Black-Scholes formula. The risk free interest rate used is 2.3% per annum.

4. Discussion

Table 3 shows the values of initial stock price, the strike price, the stock volatility as well as the call and put price obtained for listed stocks on GSE using the Black-Scholes formula. The risk free interest rate used is 2.3% per annum. We realise that stocks on the exchange are characterized by low volatilities and low stock prices. For these reasons we examine the effect of low volatilities and low stock prices on an option price.

Effect of low volatility on an option price

When volatility approaches zero the stock is almost virtually riskless and so at maturity time T its price will grow at a rate r to \( S_0 e^{rT} \). The payoff from the option is \( \max(S_0 e^{rT} - K, 0) \).

Discounting at a rate of r, the value of the call today is

\[
e^{-rT} \max (S_0 e^{rT} - K, 0) = \max (S_0 - Ke^{-rT}, 0)
\]

Now consider the Black-Scholes formula

\[
\text{Put Price} = \max(S_0 - K - p, 0)
\]

\[
\text{Call Price} = \max(p - (S_0 - K), 0)
\]
\[ C(S(t), t) = S(t)N(d_1) + Ke^{-r(T-t)}N(d_2) \]

with \[ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(\frac{r+\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(\frac{r-\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \]

As \( \sigma \to 0 \), \( d_1 \) and \( d_2 \) \( \to \infty \) so that \( N(d_1) \) and \( N(d_2) \) \( \to 1 \) and

\[ C(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)} \]

approaches \( S(t) - Ke^{-r(T-t)} \).

Thus \( \lim_{\sigma \to 0} C(S(t), t) = S(t) - Ke^{-r(T-t)} \).

The call price in the limit as volatility approaches zero is

\[ \lim_{\sigma \to 0} C(S(t), t) = \max(S_0e^{-rT} - K, 0). \]

A graph of stock volatility and call option price shows that call option prices are close to zero for most of the stocks. See Figure 1.

**Figure 1.** Call option price against volatility.

The put option price as stock volatility approaches zero does not tend to zero but rather to \( max(Ke^{-rT} - S_0, 0) \). That is

\[ \lim_{\sigma \to 0} P(S(t), t) = max(Ke^{-rT} - S_0, 0) \]

**Figure 2.** Put Option Price against volatility.

Figure 2 shows the graph of put option price against the stock volatility. We realise that the put price do not tend to zero as volatility decreases to zero.

**Effect of low stock price on an option price**

When stock price \( S_0 \) become very large both \( d_1 \) \( \to \infty \) and \( d_2 \) \( \to \infty \) and both \( N(d_1) \) and \( N(d_2) \) \( \to 1 \). The price of a European call option is given by

\[ C(S(t), t) = S(t)N(d_1) - Ke^{-rT} \]

The call option is then almost certain to be exercised as \( S(t) \gg Ke^{-rT} \). Note that a call option is likely to be exercised if and only if \( S(t) > K \).

On the other-hand, if stock price \( S_0 \) become very small both \( d_1 \) and \( d_2 \) \( \to -\infty \) and \( N(d_1) \) and \( N(d_2) \) \( \to 0 \). The price of a call option therefore approaches zero. i.e.

\[ \lim_{S_0 \to 0} C(S(t), t) = 0. \]

If \( S(t) \ll Ke^{-rT} \) then it is almost certain the call option will not to be exercised. Note that a call option is likely to be exercised if and only if \( S(t) > K \). This is consistent with our results. Figure 1 shows a graph of stock price against the call option price. We realise that as stock price decreases the call price also decreases to zero.
We realise from Figure 3 that the call option price for most of the low priced stocks approaches zero. This makes practical sense, of course, if the probability of exercising the option in the future is very small then its current price will certainly approach zero.

For a put option, if the stock price $S_0$ approaches zero the parameters $d_1 \to -\infty$ and $d_2 \to -\infty$ and consequently $N(-d_1)$ and $N(-d_2) \to 0$. The price of a put option then approaches zero. That is

$$\lim_{S_0 \to 0} P(S(t), t) = 0$$

In the Black-Scholes formula to calculate put option price, as the stock price $S_0$ approaches zero the parameters $d_1$ and $d_2$ approaches $+\infty$ and consequently $N(-d_1)$ and $N(-d_2)$ approaches 1. In this case the price of a put option

$$P(S(t), t) = Ke^{-rt}N(-d_2) - S_0N(-d_1) \to Ke^{-rt} - S_0$$

i.e. $\lim_{S_0 \to 0} P(S(t), t) = Ke^{-rt} - S_0$

If $S_0 \ll K$ then the put price can be approximated by $Ke^{-rt}$. The behaviour of put prices in the presence of low stock prices is shown in figure 3. We clearly observe that put prices do not necessarily tend to zero for low stock prices.

5. Conclusion

We have examined the possibility of pricing European options on Ghanaian stocks using the Black-Scholes formula. The results presented here shows that we can conveniently price options on Ghanaian stocks using the Black-Scholes model. However, it is realised that for stocks with very low volatilities and low initial stock price the call option price is zero in most cases. On the otherhand, we determined that put options do not tend to zero for low stock volatilities and low stock prices. The formula gives positive put prices for most of the listed stocks. This study suggest that if a broker wish to open trading in options stocks on GSE as underlying asset then he must concentrate on selling put options until the situation on the exchange improve in regards to stock volatility and stock prices.

References


