

Quantum State Evolution in C^2 and G_3^+

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Abstract: Quantum mechanical qubit states as elements of two dimensional complex Hilbert space can be generalized to elements of even subalgebra of geometric algebra over three dimensional Euclidian space. The construction critically depends on generalization of formal, unspecified, complex plane to arbitrary variable, but explicitly defined, planes in 3D, and of usual Hopf fibration to special maps of the geometric algebra elements to the unit sphere in 3D generated by arbitrary unit value bivectors. Analysis of the structure of the map of the even subalgebra to the Hilbert space demonstrates that quantum state evolution in the latter gives only restricted information compared to that in geometric algebra.

Keywords: Qubits, Geometric Algebra, Clifford Translation, Berry Phase, Topological Quantum Computing

1. Introduction

Qubits, unit value elements of the Hilbert space C^2 of two dimensional complex vectors, can be generalized to unit value elements $\alpha + I_S \beta$ of even subalgebra G_3^+ of geometric algebra G_3 over Euclidian space E_3 . I called such generalized qubits \mathcal{G} -qubits [1].

Some minimal information about G_3 and G_3^+ is necessary. Algebraically, G_3 is linear space with canonical basis $\{1, e_1, e_2, e_3, e_2e_3, e_3e_1, e_1e_2, e_1e_2e_3\}$, where 1 is unit value scalar, $\{e_i\}$ are orthonormal basis vectors in E_3 , $\{e_i e_j\}$ are oriented, mutually orthogonal unit value areas (bivectors) spanned by e_i and e_j as edges, with orientation defined by rotation e_i to e_j by angle $\frac{\pi}{2}$; and $e_1 e_2 e_3$ is unit value oriented volume spanned by ordered edges e_1 , e_2 and e_3 .

Subalgebra G_3^+ is spanned by scalar 1 and basis bivectors: $\{1, e_2 e_3, e_3 e_1, e_1 e_2\}$. Variables α and β in $\alpha + \beta I_S \in G_3^+$ are scalars, I_S is a unit size oriented area,

bivector, lying in an arbitrary given plane $S \subset E_3$. Bivector I_S is linear combination of basis bivectors $e_i e_j$, see Fig. 1a.

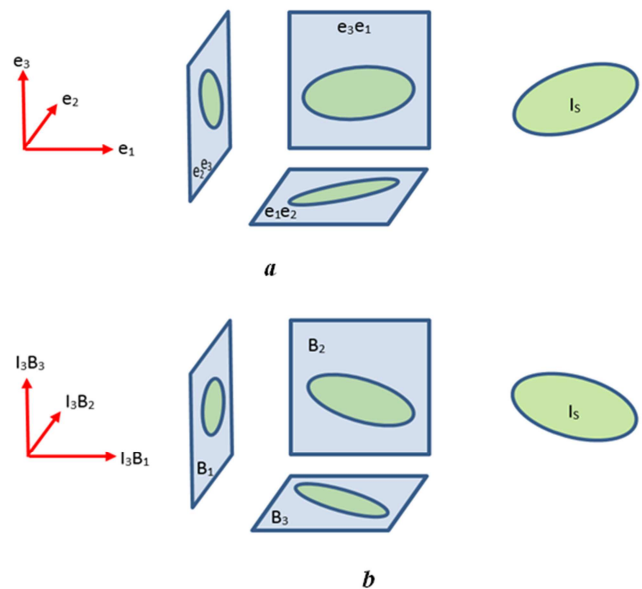


Fig. 1. Variable complex plane in different bivector bases.

It was explained in [2], [3] that elements $\alpha + I_S \beta$ only differ from what is traditionally called “complex numbers” by the fact that $S \subset E_3$ is an arbitrary, though explicitly defined, plane in E_3 . Putting it simply, $\alpha + I_S \beta$ are “complex

1 This sum bears the sense “something and something”. It is not a sum of geometrically similar elements giving another element of the same type, see [7], [9].

numbers” depending on E_2 embedded into E_3 . Traditional “imaginary unit” i is actually associated with some unspecified I_S – everything is going on in one tacit fixed plane, not in 3D world. Usually i is considered just as a “number” with additional algebraic property $i^2 = -1$. That may be a source of ambiguities, as it happens in quantum mechanics.

I will denote unit value oriented volume $e_1e_2e_3$ by I_3 . It has the property $I_3^2 = -1$. Actually, there always are two options to create oriented unit volume, depending on the order of basis vectors in the product. They correspond to the two types of handedness – left and right screw handedness. In Fig.1a the variant of right screw handedness is shown. It is geometrically obvious that $e_ie_j = -e_je_i$ (bivector flips, changes orientation to the opposite, in swapping the edge vectors). Then changing of handedness, for example by $e_1e_2e_3 \rightarrow e_2e_1e_3$, gives $e_2e_1e_3 = -e_1e_2e_3 = -I_3$.

Any basis vector is conjugate (conjugation means multiplication by I_3), or dual, to a basis bivector and inverse. For example $I_3e_2e_3 = e_2e_3I_3 = -e_1$. By multiplying these equations from left or right by I_3 we get $e_2e_3 = I_3e_1 = e_1I_3$. Such duality holds for arbitrary, not just basis, vectors and bivectors. Any vector \vec{b} is conjugate to some bivector: $\vec{b} = I_3B = BI_3$ and for any bivector B there exists \vec{b} such that $B = I_3\vec{b} = \vec{b}I_3$. Good to remember that in $\vec{b} = I_3B = BI_3$ vector and bivector are in handedness opposite to I_3 while in $B = I_3\vec{b} = \vec{b}I_3$ they are in the same handedness as I_3 . One can also think about I_3 as a right (left) single thread screw helix of the height one. In this way $-I_3$ is left (right) screw helix.

All above remains true (see Fig.1b) if we replace basis bivectors $\{e_2e_3, e_3e_1, e_1e_2\}$ by an arbitrary triple of unit bivectors $\{B_1, B_2, B_3\}$ satisfying the same multiplication rules which are valid for $\{e_2e_3, e_3e_1, e_1e_2\}$ (see Fig.2, where right screw handedness is assumed):

$$B_1B_2 = -B_3; \quad B_1B_3 = B_2; \quad B_2B_3 = -B_1 \tag{1.1}$$

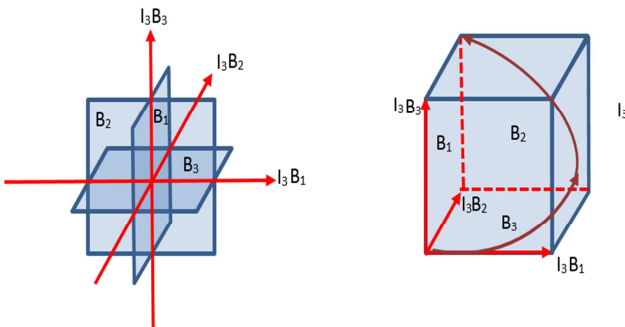


Fig. 2. Oriented basis bivectors their right screw oriented volume.

2. Parameterization of Unit Value Elements in G_3^+ and C^2 by Points of S^3

Let’s take a \mathcal{G} -qubit:

$$\alpha + I_S\beta = \alpha + \beta(b_1B_1 + b_2B_2 + b_3B_3) = \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3, \beta_i = \beta b_i, \alpha^2 + \beta^2 = 1, b_1^2 + b_2^2 + b_3^2 = 1$$

I will use notation $so(\alpha, \beta, S)$ for such elements. They can be considered as points on unit radius sphere in the sense that any point $(\alpha, \beta_1, \beta_2, \beta_3) \in S^3$ parameterizes a \mathcal{G} -qubit.

A pure qubit state in terms of conventional quantum mechanics is two dimensional unit value vector with complex components:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, z_1^2 + z_2^2 = z_1\tilde{z}_1 + z_2\tilde{z}_2 = 1, z_k = z_k^1 + iz_k^2, k = 1, 2$$

We can also think about such pure qubit states as points on S^3 because:

$$S^3 \ni \{z_1^1, z_1^2, z_2^1, z_2^2\} \leftarrow \{z = (z_1, z_2) \in C^2; |z|^2 = \tilde{z}_1z_1 + \tilde{z}_2z_2 = (z_1^1)^2 + (z_1^2)^2 + (z_2^1)^2 + (z_2^2)^2 = 1\} \tag{2.1}$$

Explicit relations, based on their parameterization by S^3 points, between elements $|\psi\rangle$ and elements $so(\alpha, \beta, S)$ can be established through the following construction².

Let basis bivector B_1 is chosen as defining the complex plane S , then we have (see multiplication rules (1.1)):

$$so(\alpha, \beta, S) = (\alpha + \beta_1B_1) + \beta_2B_1B_3 + \beta_3B_3 = (\alpha + \beta_1B_1) + (\beta_3 + \beta_2B_1)B_3$$

Hence we get the map:

$$so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta_1 \\ \beta_3 + i\beta_2 \end{pmatrix}, \text{ where } i \equiv B_1$$

In the similar way, for two other selections of complex plane we get:

$$so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_1^2 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta_2 \\ \beta_1 + i\beta_3 \end{pmatrix}, i \equiv B_2,$$

$$\text{and } so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_1^3 \\ z_2^3 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{pmatrix}, i \equiv B_3$$

So we have three different maps $so(\alpha, \beta, S) \xrightarrow{B_i} (z_1^i, z_2^i)$

² Though both $so(\alpha, \beta, S)$ and $|\psi\rangle$ can be parametrized by points of S^3 , it is not correct to say that $\{z = (z_1, z_2) \in C^2; |z|^2 = 1\}$ is sphere S^3 or \mathcal{G} -qubit is sphere S^3 .

defined by explicitly declared complex planes B_i satisfying (1.1):

$$\left. \begin{aligned} so(\alpha, \beta, S) &= \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = \\ \alpha + \beta_1 B_1 + (\beta_3 + \beta_2 B_1) B_3 &= \\ \alpha + \beta_2 B_2 + (\beta_1 + \beta_3 B_2) B_1 &= \\ \alpha + \beta_3 B_3 + (\beta_2 + \beta_1 B_3) B_2 & \end{aligned} \right\}$$

$$\Rightarrow \begin{cases} \left(\begin{matrix} \alpha + i\beta_1 \\ \beta_3 + i\beta_2 \end{matrix} \right), i \leftarrow B_1 \\ \left(\begin{matrix} \alpha + i\beta_2 \\ \beta_1 + i\beta_3 \end{matrix} \right), i \leftarrow B_2 \\ \left(\begin{matrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{matrix} \right), i \leftarrow B_3 \end{cases} \quad (2.2)$$

There exists infinite number of options to select the triple $\{B_i\}$. It means that to recover a g -qubit $so(\alpha, \beta, S)$ in 3D associated with $|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ it is necessary, firstly, to define which bivector B_i in 3D should be taken as defining ‘‘complex’’ plane and then to choose another bivector B_{i_2} , orthogonal to B_{i_1} . The third bivector B_{i_3} , orthogonal to both B_{i_1} and B_{i_2} , is then defined by the first two by orientation (handedness, right screw in the used case): $I_3 B_{i_1} I_3 B_{i_2} I_3 B_{i_3} = I_3$.

The conclusion is that to each single element (2.1) there corresponds infinite number of elements $so(\alpha, \beta, S)$ depending on choosing of a triple of orthonormal bivectors $\{B_1, B_2, B_3\}$ in 3D satisfying (1.1), and associating one of them with complex plane. This allows constructing map, fibration $G_3^+ \rightarrow C^2$ restricted to unit value elements.

3. Fiber Bundle

Take general definition of fiber bundle as a set (E, M, π, G, F) where E is bundle (or total) space; M - the base space; F - standard fiber; G - Lie group which acts effectively on F ; π - bundle projection: $\pi: E \rightarrow M$, such that each space $F_x \equiv \pi^{-1}(x)$, fiber at $x \in M$, is homeomorphic to standard fiber F .

We can think about the map $so(\alpha, \beta, S) \xrightarrow{B_i} (z_1^i, z_2^i)$ as a fiber bundle. In the current case, the fiber bundle will have $\{so(\alpha, \beta, S) = g \in G_3^+ : |g|=1\}$ as total space and $\{z = (z_1, z_2) \in C^2; |z|^2 = 1\}$ as base space. I will denote them as $G_3^+|_{S^3}$ and $C^2|_{S^3}$ respectively. The projection $\pi: G_3^+|_{S^3} \rightarrow C^2|_{S^3}$ depends on which particular B_i is taken

from an arbitrary triple $\{B_1, B_2, B_3\}$ satisfying (1.1) as associated complex plane of complex vectors of C^2 and explicitly given by (2.2), so we should write $\pi: G_3^+|_{S^3} \xrightarrow{B_i} C^2|_{S^3}$.

By some reasons that will be explained a bit later I will use complex plane associated with B_3 , so by (2.2) the projection is:

$$\pi: so(\alpha, \beta, S) = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \rightarrow \begin{pmatrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{pmatrix} \quad (3.1)$$

Then for any $z = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \in C^2|_{S^3}$ the fiber in $G_3^+|_{S^3}$ is comprised of all elements $F_z = x_1 + y_2 B_1 + x_2 B_2 + y_1 B_3$ with an arbitrary triple of orthonormal bivectors $\{B_1, B_2, B_3\}$ in 3D satisfying (1.1). That particularly means that standard fiber is equivalent to the group of rotations of the triple $\{y_2 B_1, x_2 B_2, y_1 B_3\}$ as a whole (see Fig.3):

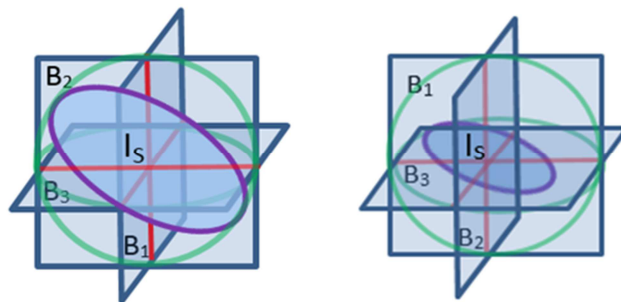


Fig. 3. Two sections of a fiber: The left one received from the right by 90 degree counterclockwise rotation around vertical axis.

All such rotations in G_3^+ are also identified by elements of $G_3^+|_{S^3}$ since for any bivector B the result of its rotation is 3 (see, for example [4], [5]):

$$so(\gamma, \delta, S) \sim Bso(\gamma, \delta, S)$$

where

$$\begin{aligned} so(\gamma, \delta, S)^\wedge &= (\gamma + \delta_1 B_1 + \delta_2 B_2 + \delta_3 B_3)^\wedge \\ &= \gamma - \delta_1 B_1 - \delta_2 B_2 - \delta_3 B_3 \end{aligned}$$

So, standard fiber is identified as $G_3^+|_{S^3}$ and composition of rotations is:

$$e^{-I_{S_2}\psi} \left(e^{-I_{S_1}\phi} B e^{I_{S_1}\phi} \right) e^{I_{S_2}\psi} = \left(e^{I_{S_1}\phi} e^{I_{S_2}\psi} \right)^\wedge B e^{I_{S_1}\phi} e^{I_{S_2}\psi}$$

All that means that the fibration $G_3^+|_{S^3} \xrightarrow{B_3} C^2|_{S^3}$ is

3 It is often convenient to write elements $so(\alpha, \beta, S)$ as exponents: $so(\alpha, \beta, S) = e^{I_S \phi}$.

principal fiber bundle with standard fiber $G_3^+|_{S^3}$ and the group acting on it is the group of (right) multiplications by elements of $G_3^+|_{S^3}$.

Now the explanation why B_3 was taken as complex plane. As was shown in [1], [6] the variant of classical Hopf fibration

$$C^2|_{S^3} \rightarrow S^2:$$

$$\text{Pr}(z_1, z_2) = \left(\tilde{z}_1 z_2 + z_1 \tilde{z}_2, i(\tilde{z}_1 z_2 - z_1 \tilde{z}_2), |z_1|^2 - |z_2|^2 \right)$$

can be received as one of the cases of generalized Hopf fibrations:

$$G_3^+|_{S^3} \xrightarrow{B} S^2:$$

$$so(\alpha, \beta, S) \xrightarrow{B} so(\alpha, \beta, S) \sim Bso(\alpha, \beta, S) \rightarrow S^2$$

The basis bivector B_1 gives:

$$\begin{aligned} so(\alpha, \beta, S) &\xrightarrow{B_1} (\alpha - \beta I_S) B_1 (\alpha + \beta I_S) = \\ (\alpha - \beta_1 B_1 - \beta_2 B_2 - \beta_3 B_3) B_1 (\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3) = \\ &\left[(\alpha^2 + \beta_1^2) - (\beta_2^2 + \beta_3^2) \right] B_1 + 2(\alpha\beta_3 + \beta_1\beta_2) B_2 + \\ &2(\beta_1\beta_3 - \alpha\beta_2) B_3 \rightarrow \\ &\left(\left[(\alpha^2 + \beta_1^2) - (\beta_2^2 + \beta_3^2) \right], 2(\alpha\beta_3 + \beta_1\beta_2), 2(\beta_1\beta_3 - \alpha\beta_2) \right) \end{aligned} \quad (3.2)$$

and for two other basis bivectors:

$$so(\alpha, \beta, S) \xrightarrow{B_2} 2(\beta_1\beta_2 - \alpha\beta_3) B_1 + \left[(\alpha^2 + \beta_2^2) - (\beta_1^2 + \beta_3^2) \right] B_2 + 2(\alpha\beta_1 + \beta_2\beta_3) B_3 \rightarrow \\ (2(\beta_1\beta_2 - \alpha\beta_3), \left[(\alpha^2 + \beta_2^2) - (\beta_1^2 + \beta_3^2) \right], 2(\alpha\beta_1 + \beta_2\beta_3)) \quad (3.3)$$

$$so(\alpha, \beta, S) \xrightarrow{B_3} 2(\alpha\beta_2 + \beta_1\beta_3) B_1 + 2(\beta_2\beta_3 - \alpha\beta_1) B_2 + \left[(\alpha^2 + \beta_3^2) - (\beta_1^2 + \beta_2^2) \right] B_3 \rightarrow \\ (2(\alpha\beta_2 + \beta_1\beta_3), 2(\beta_2\beta_3 - \alpha\beta_1), \left[(\alpha^2 + \beta_3^2) - (\beta_1^2 + \beta_2^2) \right]) \quad (3.4)$$

In the literature mostly the third or the first variants are called Hopf fibrations. For my considerations it does not matter which variant to choose. I take the case (3.4) with B_3 as complex plane:

$$\begin{aligned} so(\alpha, \beta, S) &= \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \\ &= \alpha + \beta_3 B_3 + (\beta_2 + \beta_1 B_3) B_2 \\ &\rightarrow \left(\begin{array}{c} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{array} \right), i \leftarrow B_3 \end{aligned}$$

4. Tangent Spaces

We need to temporarily get back to the case of G_2 - geometric algebra on a plane [2].

Let an orthonormal basis $\{e_1, e_2\}$ is taken. It generates G_2 basis $\{1, e_1, e_2, e_1 e_2\}$ where $e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2$ is usual geometric product of two vectors: first member is scalar

product of the vectors and second member is oriented area (bivector) swept by rotating e_1 to e_2 by the angle which is less than π .

The G_2 basis vectors satisfy particularly the properties:

$$I_2^2 \equiv (e_1 e_2)^2 = -1, \quad I_2 e_1 = -e_2 \quad (\text{clockwise rotation}),$$

$$e_1 I_2 = e_2 \quad (\text{counterclockwise rotation}), \quad I_2 e_2 = e_1, \quad e_2 I_2 = -e_1.$$

I am using notation I_2 for unit bivector $e_1 e_2$.

For further convenience, let's construct a matrix basis isomorphic to $\{1, e_1, e_2, e_1 e_2\}$. Commonly used agreement will be that scalars are identified with scalar matrices, for example:

$$\alpha \equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

For the G_2 case the second order matrices will suffice to get necessary isomorphism. Let's take

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and}$$

$\varepsilon_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as corresponding to geometric basis vectors e_1 and e_2 . They satisfy the properties mentioned above, for example

$$\begin{aligned} \varepsilon_i^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_2^2 = (\varepsilon_1 \varepsilon_2)^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ I_2 \varepsilon_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_2, \quad \text{etc.} \end{aligned}$$

The operation representing scalar product gives:

$$\varepsilon_1 \cdot \varepsilon_2 = \frac{1}{2} (\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_1) = \frac{1}{2} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.1)$$

as it should be. If we take two arbitrary vectors expanded in basis $\{\varepsilon_1, \varepsilon_2\}$:

$$\vec{a} = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_2 \\ \alpha_2 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix}$$

$$\vec{b} = \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 = \begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_1 \end{pmatrix} + \begin{pmatrix} 0 & \beta_2 \\ \beta_2 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}$$

then their product is:

$$\vec{a}\vec{b} = \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_2 & 0 \\ 0 & \alpha_1\beta_1 + \alpha_2\beta_2 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_2\beta_1 + \alpha_1\beta_2 \\ -\alpha_1\beta_2 + \alpha_2\beta_1 & 0 \end{pmatrix} = (\vec{a} \cdot \vec{b}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\alpha_1\beta_2 - \alpha_2\beta_1) I_2$$

The first member is usual scalar product and the second one is bivector of the value equal to the area of parallelogram with

4 I begin with ordinary real component matrices. That is why the order of Pauli matrices in my following definition is different from usual one. An analogue of "imaginary" unit should logically be the last one, after pure real items.

the sides \vec{a} and \vec{b} .

Important thing to keep in mind is that multiplication of any vector by unit bivector I_2 rotates the vector by $\pm \frac{\pi}{2}$ (see Fig.4):

$$\vec{a}I_2 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_2 \end{pmatrix} = -\alpha_2\epsilon_1 + \alpha_1\epsilon_2$$

(counterclockwise rotation)

$$I_2\vec{a} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & -\alpha_1 \\ -\alpha_1 & -\alpha_2 \end{pmatrix} = \alpha_2\epsilon_1 - \alpha_1\epsilon_2$$

(clockwise rotation)

This remains valid for any unit bivector of the same orientation as I_2 . We can conclude that such multiplications give basis vectors of the tangent spaces to the original vectors.

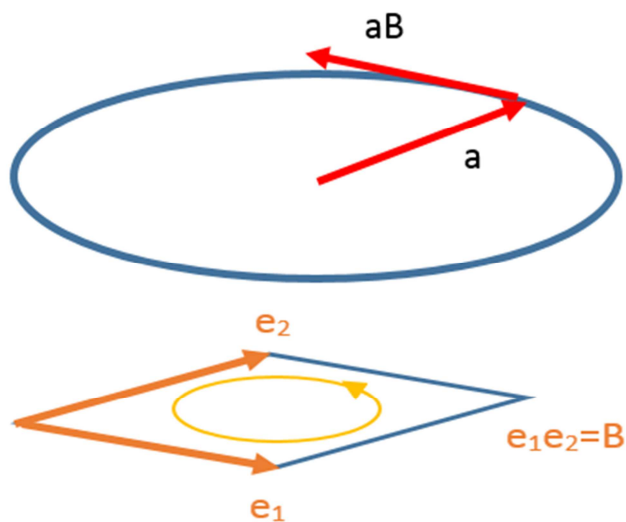


Fig. 4. Vector is rotated by 90 degrees when multiplied by basis bivector.

This is identical to considering even elements $\alpha_1 + B\alpha_2$ corresponding to vectors and their multiplication by B : $(\alpha_1 + B\alpha_2)B = -\alpha_2 + B\alpha_1$. Elements $\alpha_1 + B\alpha_2$ and $-\alpha_2 + B\alpha_1$ are orthogonal: $\langle \alpha_1 + B\alpha_2, -\alpha_2 + B\alpha_1 \rangle_0 = (\alpha_1 + B\alpha_2) \cdot (-\alpha_2 - B\alpha_1) \Big|_0 = 0$ (index 0 means scalar part).

General element of G_2 in matrix basis $\{1, \epsilon_1, \epsilon_2, \epsilon_1\epsilon_2 \equiv I_2\}$, isomorphic to geometrical basis $\{1, e_1, e_2, e_1e_2\}$, is:

$$g_2 = \alpha_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_2 + \alpha_3 \\ \alpha_2 - \alpha_3 & \alpha_0 - \alpha_1 \end{pmatrix}$$

This is arbitrary real valued matrix of the second order. Inversely, any matrix of second order can be uniquely mapped

to the element of G_2 in the basis $\{1, \epsilon_1, \epsilon_2, \epsilon_1\epsilon_2 \equiv I_2\}$:

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \Rightarrow \frac{x+t}{2} + \frac{x-t}{2}\epsilon_1 + \frac{y+z}{2}\epsilon_2 + \frac{y-z}{2}I_2$$

To upgrade to G_3 , we need basis matrices of a higher order because a second order non-zero matrix orthogonal in the sense of scalar product (4.1) to both ϵ_1 and ϵ_2 does not exist.

It is easy to verify that the three, playing the role of three-dimensional orthonormal basis, necessary matrices can be taken as 4th-order block matrices:

$$\sigma_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ - scalar matrix corresponding to scalar 1,

$I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_1\epsilon_2$. Easy to see that σ_1 and σ_2 are received from ϵ_1 and ϵ_2 by replacing scalars by corresponding scalar matrices.

By multiplications in full 4×4 form one can easily proof that block-wise multiplications are correct, the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ is orthonormal, anticommutative, and $\sigma_2\sigma_3, \sigma_3\sigma_1$ and $\sigma_1\sigma_2$ satisfy requirements (1.1).

Taking $I_3 = \sigma_1\sigma_2\sigma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$, multiplications give:

$$\sigma_2\sigma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = I_3 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I_3\sigma_1$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = I_3 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = I_3\sigma_2, \sigma_1\sigma_2 =$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = I_3 \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} = I_3\sigma_3$$

Then, since $I_3^2 = -1$, we easily get:

$$(\sigma_2\sigma_3)(\sigma_3\sigma_1) = -\sigma_1\sigma_2, (\sigma_2\sigma_3)(\sigma_1\sigma_2) = \sigma_3\sigma_1,$$

$$(\sigma_3\sigma_1)(\sigma_1\sigma_2) = -\sigma_2\sigma_3$$

that is exactly (1.1) with basis bivectors $B_1 = \sigma_2\sigma_3$, $B_2 = \sigma_3\sigma_1$ and $B_3 = \sigma_1\sigma_2$.

Following all that we can write general element g_3 of algebra G_3 expanded in formal matrix basis $\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3\} \equiv \{1, \sigma_1, \sigma_2, \sigma_3, I_3\sigma_1, I_3\sigma_2, I_3\sigma_3, I_3\}$ as:

$$g_3 = \alpha I + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3 + \beta_1 \sigma_2 \sigma_3 + \beta_2 \sigma_3 \sigma_1 + \beta_3 \sigma_1 \sigma_2 + \beta_0 \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} (\alpha + \alpha_1) + I_2(\beta_0 + \beta_1) & (\alpha_2 + \beta_3) + I_2(\beta_2 - \alpha_3) \\ (\alpha_2 - \beta_3) + I_2(\beta_2 + \alpha_3) & (\alpha - \alpha_1) + I_2(\beta_0 - \beta_1) \end{pmatrix} \quad (4.2)$$

This is arbitrary “complex” valued matrix where “imaginary unit” is matrix $I_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, corresponding to bivector expanded in the first two basis vectors σ_1, σ_2 , that geometrically is bivector B_3 . Inversely, any “complex” valued matrix can be written as g_3 element in matrix basis $\{1, \sigma_1, \sigma_2, \sigma_3, I_3 \sigma_1, I_3 \sigma_2, I_3 \sigma_3, I_3\}$ isomorphic to geometrical basis $\{1, e_1, e_2, e_3, e_2 e_3, e_3 e_1, e_1 e_2, e_1 e_2 e_3\}$:

$$\begin{pmatrix} s + I_2 t & u + I_2 v \\ x + I_2 y & z + I_2 w \end{pmatrix} = \frac{s+z}{2} + \frac{s-z}{2} \sigma_1 + \frac{u+x}{2} \sigma_2 + \frac{v-y}{2} \sigma_3 + \frac{t-w}{2} \sigma_2 \sigma_3 + \frac{y+v}{2} \sigma_3 \sigma_1 + \frac{u-x}{2} \sigma_1 \sigma_2 + \frac{t+w}{2} I_3$$

If g_3 belongs to even subalgebra G_3^+ then, by identifying $\sigma_2 \sigma_3 \Leftrightarrow B_1, \sigma_3 \sigma_1 \Leftrightarrow B_2, \sigma_1 \sigma_2 \Leftrightarrow B_3$, we have from (4.2) the correspondence:

$$g_3^+ = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \Leftrightarrow \begin{pmatrix} \alpha + I_2 \beta_1 & \beta_3 + I_2 \beta_2 \\ -\beta_3 + I_2 \beta_2 & \alpha - I_2 \beta_1 \end{pmatrix} \quad (4.3)$$

In the same way as in the G_2 case, multiplications of $so(\alpha, \beta_1, \beta_2, \beta_3) = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ by basis bivectors B_i give basis bivectors of tangent spaces to original bivectors:

$$T_1 = so(\alpha, \beta_1, \beta_2, \beta_3) B_1 = (\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3) B_1 = -\beta_1 + \alpha B_1 - \beta_3 B_2 + \beta_2 B_3$$

$$T_2 = so(\alpha, \beta_1, \beta_2, \beta_3) B_2 = (\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3) B_2 = -\beta_2 + \beta_3 B_1 + \alpha B_2 - \beta_1 B_3$$

$$T_3 = so(\alpha, \beta_1, \beta_2, \beta_3) B_3 = (\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3) B_3 = -\beta_3 - \beta_2 B_1 + \beta_1 B_2 + \alpha B_3$$

These $G_3^+|_{S^3}$ elements are orthogonal to $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ and to each other, and are the tangent space basis elements at points $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$.

Projections of T_i onto $C^2|_{S^3}$ are:

$$\pi(T_1) = \begin{pmatrix} -\beta_1 + i\beta_2 \\ -\beta_3 + i\alpha \end{pmatrix}, \quad \pi(T_2) = \begin{pmatrix} -\beta_2 - i\beta_1 \\ \alpha + i\beta_3 \end{pmatrix},$$

$$\pi(T_3) = \begin{pmatrix} -\beta_3 + i\alpha \\ \beta_1 - i\beta_2 \end{pmatrix}$$

These elements of $C^2|_{S^3}$ are orthogonal, in the sense of Euclidean scalar product in $C^2: \langle z, w \rangle = \text{Re}(\tilde{z}_1 w_1 + \tilde{z}_2 w_2)$, to each other and to the projection $\pi(so(\alpha, \beta_1, \beta_2, \beta_3)) = \begin{pmatrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{pmatrix}$ of the original state in $G_3^+|_{S^3}$. They are the tangent space basis elements in $C^2|_{S^3}$ at points $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{pmatrix}$.

5. Hamiltonians in $C^2|_{S^3}$ and Their Lifts in G_3

Take self-adjoint (Hamiltonian) linear operator in C^2 in its most general form:

$$H = aI + b\sigma_1 + c\sigma_2 + d\sigma_3 = \begin{pmatrix} a+b & c - I_2 d \\ c + I_2 d & a-b \end{pmatrix} \quad \text{Its}$$

corresponding G_3 element, see (4.3), is $a + I_3(bB_1 + cB_2 + dB_3)$ and does not belong to $G_3^+|_{S^3}$, though the result of its action on an element from $C^2|_{S^3}$ (qubit), has π^{-1} preimage in $G_3^+|_{S^3}$. Let’s prove that. Take a qubit and a Hamiltonian in $C^2|_{S^3}$:

$$z = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} = \begin{pmatrix} r_1 e^{i\psi_1} \\ r_2 e^{i\psi_2} \end{pmatrix}, \quad r_k = \sqrt{x_k^2 + y_k^2},$$

$$\cos \psi_k = \frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \quad \sin \psi_k = \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, \quad k = 1, 2$$

$$H = \begin{pmatrix} a+b & \text{Re} e^{i\psi} \\ \text{Re}^{-i\psi} & a-b \end{pmatrix}, \quad R = \sqrt{c^2 + d^2}, \quad \cos \psi = \frac{c}{\sqrt{c^2 + d^2}},$$

$$\sin \psi = \frac{d}{\sqrt{c^2 + d^2}}$$

Then:

$$Hz = \begin{pmatrix} a+b & \text{Re} e^{i\psi} \\ \text{Re}^{-i\psi} & a-b \end{pmatrix} \begin{pmatrix} r_1 e^{i\psi_1} \\ r_2 e^{i\psi_2} \end{pmatrix} = \quad (5.1)$$

$$\begin{pmatrix} (a+b)r_1 \cos \psi_1 + Rr_2 \cos(\psi + \psi_2) + i((a+b)r_1 \sin \psi_1 + Rr_2 \sin(\psi + \psi_2)) \\ (a-b)r_2 \cos \psi_2 + Rr_1 \cos(\psi - \psi_1) + i((a-b)r_2 \sin \psi_2 - Rr_1 \sin(\psi - \psi_1)) \end{pmatrix}$$

For each single qubit $z = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \in C^2|_{S^3}$ its fiber (full preimage) in $G_3^+|_{S^3}$ is $F_z = x_1 + y_2 B_1 + x_2 B_2 + y_1 B_3$ with an

arbitrary triple of orthonormal bivectors $\{B_1, B_2, B_3\}$ in 3D satisfying (1.1). In exponential form:

$$\begin{aligned}
 F_z &= x_1 + y_2 B_1 + x_2 B_2 + y_1 B_3 \\
 &= x_1 + \sqrt{1-x_1^2} \left(\frac{y_2}{\sqrt{1-x_1^2}} B_1 + \frac{x_2}{\sqrt{1-x_1^2}} B_2 + \frac{y_1}{\sqrt{1-x_1^2}} B_3 \right) \\
 &= \cos \phi + \sin \phi (b_1 B_1 + b_2 B_2 + b_3 B_3) = e^{I_S \phi}, \quad (5.2)
 \end{aligned}$$

where

$$\left. \begin{aligned}
 \cos \phi &= x_1, \sin \phi = \sqrt{1-x_1^2} \\
 b_1 &= \frac{y_2}{\sqrt{1-x_1^2}}, b_2 = \frac{x_2}{\sqrt{1-x_1^2}}, b_3 = \frac{y_1}{\sqrt{1-x_1^2}} \\
 I_S &= b_1 B_1 + b_2 B_2 + b_3 B_3
 \end{aligned} \right\} \quad (5.3)$$

Using (5.2) we get from (5.3) the fiber at $H z$:

$$F_{Hz} = e^{I_{S(H)} \phi(H)}, \quad (5.4)$$

where

$$\left. \begin{aligned}
 \cos \phi(H) &= (a+b)r_1 \cos \psi_1 + Rr_2 \cos(\psi + \psi_2); \\
 I_{S(H)} &= b_1(H)B_1 + b_2(H)B_2 + b_3(H)B_3; \\
 b_1(H) &= \frac{(a-b)r_2 \sin \psi_2 - Rr_1 \sin(\psi - \psi_1)}{\sqrt{1 - \cos^2 \phi(H)}}, \\
 b_2(H) &= \frac{(a-b)r_2 \cos \psi_2 + Rr_1 \cos(\psi - \psi_1)}{\sqrt{1 - \cos^2 \phi(H)}}, \\
 b_3(H) &= \frac{(a+b)r_1 \sin \psi_1 + Rr_2 \sin(\psi + \psi_2)}{\sqrt{1 - \cos^2 \phi(H)}}
 \end{aligned} \right\} \quad (5.5)$$

We can also explicitly write the element $g_3(H) \in G_3^+ |_{S^3}$ which, acting (from the right) on $\pi^{-1}(z)$ gives $\pi^{-1}(Hz)$:

$$\pi^{-1}(z)g_3(H) = \pi^{-1}(Hz)$$

From (5.1) and (5.4):

$$e^{I_S \phi} g_3(H) = e^{I_{S(H)} \phi(H)} \Rightarrow g_3(H) = e^{-I_S \phi} e^{I_{S(H)} \phi(H)}$$

where I_S , ϕ , $I_{S(H)}$ and $\phi(H)$ are defined by (5.3) and (5.5).

6. Clifford Translations

Suppose, action on z in $C^2 |_{S^3}$ is multiplication by an

exponent: $Cl_\psi(z) = e^{i\psi} z$, often called Clifford translation⁵, that's:

$$Cl_\psi(z) = e^{i\psi} \begin{pmatrix} r_1 e^{i\psi_1} \\ r_2 e^{i\psi_2} \end{pmatrix} = \begin{pmatrix} r_1 \cos(\psi + \psi_1) + ir_1 \sin(\psi + \psi_1) \\ r_2 \cos(\psi + \psi_2) + ir_2 \sin(\psi + \psi_2) \end{pmatrix}$$

Then the fiber of $Cl_\psi(z)$ in $G_3^+ |_{S^3}$ is:

$$\begin{aligned}
 F_{Cl_\psi(z)} &= r_1 \cos(\psi + \psi_1) + r_2 \sin(\psi + \psi_2) B_1 + \\
 & r_2 \cos(\psi + \psi_2) B_2 + r_1 \sin(\psi + \psi_1) B_3 = \\
 & r_1 \cos(\psi + \psi_1) + r_1 \sin(\psi + \psi_1) B_3 \quad (6.1) \\
 & + [r_2 \cos(\psi + \psi_2) + r_2 \sin(\psi + \psi_2) B_3] B_2 = \\
 & r_1 e^{B_3 \psi} e^{B_3 \psi_1} + r_2 e^{B_3 \psi} e^{B_3 \psi_2} B_2 = e^{B_3 \psi} F_z
 \end{aligned}$$

In other words, Clifford translations in $C^2 |_{S^3}$ are equivalent to multiplications of fibers in $G_3^+ |_{S^3}$ by standard fiber elements with bivector part B_3 (which is associated with formal imaginary unit i) and the same phase ψ .

In commonly used quantum mechanics all states from the set $Cl_\psi(z)$ are considered as identical to the state z , they have the same values of observables. That is not commonly true for g -qubits. Only in the case when all the observables common plane, say I_S , (unspecified in the C^2 model⁶) is the same as the plane associated with complex plane in $Cl_\psi(z)$, the g -qubit $e^{I_S \psi} F_z$ leaves any such observable $e^{I_S \phi}$ unchanged:

$$\tilde{F}_z e^{-I_S \psi} e^{I_S \phi} e^{I_S \psi} F_z = \tilde{F}_z e^{I_S \phi} F_z$$

Suppose angle ψ in $Cl_\psi(z)$ is varying. Then the tangent to Clifford orbit is

$$\frac{\partial}{\partial \psi} Cl_\psi(z) = \frac{\partial}{\partial \psi} e^{i\psi} z = i e^{i\psi} z = i Cl_\psi(z).$$

It is orthogonal to $Cl_\psi(z)$ in the sense of usual scalar product in C^2 : $\langle u, v \rangle = \text{Re}(\tilde{u}_1 v_1 + \tilde{u}_2 v_2)$, and remains on $C^2 |_{S^3}$: $|i Cl_\psi(z)| = 1$ (translational velocity of Clifford translation is one).

For any $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in C^2 |_{S^3}$ define $w = \begin{pmatrix} \tilde{z}_2 \\ -\tilde{z}_1 \end{pmatrix} \in C^2 |_{S^3}$. The elements iz , w and iw are orthogonal to z and pairwise orthogonal. That means they span tangent space and the plane spanned by $\{w, iw\}$ is orthogonal to the Clifford orbit plane spanned by $\{z, iz\}$.

⁵ It is called "translation" because does not change distance between elements.

⁶ In the G_3^+ model observables are also elements of G_3^+ , see [1], [6]

The vector of translational velocity rotates while moving in the orbit plane. Both w and iw also rotates with the same unit value angular velocity since

$$\frac{\partial}{\partial \psi} Cl_\psi(w) = \frac{\partial}{\partial \psi} e^{i\psi} w = e^{i\psi} iw = Cl_\psi(iw)$$

and

$$\frac{\partial}{\partial \psi} Cl_\psi(iw) = \frac{\partial}{\partial \psi} e^{i\psi} iw = ie^{i\psi} iw = -Cl_\psi(w).$$

To make the planes of rotations clearly identified, that is difficult to do with formal imaginary unit, let's lift Clifford translation to $G_3^+ |_{S^3}$ using (6.1). Translational velocity, similar to the $C^2 |_{S^3}$ case, is $\frac{\partial}{\partial \psi} F_{Cl_\psi(z)} = \frac{\partial}{\partial \psi} (e^{B_3 \psi} F_z) = B_3 F_{Cl_\psi(z)}$ and is orthogonal to $F_{Cl_\psi(z)}$:

$$\left\langle F_{Cl_\psi(z)}, B_3 F_{Cl_\psi(z)} \right\rangle_0 = \left(F_{Cl_\psi(z)} \tilde{F}_{Cl_\psi(z)} \tilde{B}_3 \right)_0^\sim = (B_3)_0 = 0$$

Two other components of the tangent space, orthogonal to $F_{Cl_\psi(z)}$ and $B_3 F_{Cl_\psi(z)}$ at any point of the orbit, are $B_1 F_{Cl_\psi(z)}$ and $B_2 F_{Cl_\psi(z)}$ 7. Their velocities while moving along Clifford orbit are:

$$\frac{\partial}{\partial \psi} (B_1 F_{Cl_\psi(z)}) = \frac{\partial}{\partial \psi} (B_1 e^{B_3 \psi} F_z) = B_1 B_3 F_{Cl_\psi(z)} = B_2 F_{Cl_\psi(z)}$$

(derivative of $B_1 F_{Cl_\psi(z)}$ is orthogonal to $B_1 F_{Cl_\psi(z)}$ and looking in the direction $B_2 F_{Cl_\psi(z)}$)

$$\frac{\partial}{\partial \psi} (B_2 F_{Cl_\psi(z)}) = \frac{\partial}{\partial \psi} (B_2 e^{B_3 \psi} F_z) = B_2 B_3 F_{Cl_\psi(z)} = -B_1 F_{Cl_\psi(z)}$$

(derivative of $B_2 F_{Cl_\psi(z)}$ is orthogonal to $B_2 F_{Cl_\psi(z)}$ and looking in the direction $-B_1 F_{Cl_\psi(z)}$)

These two equations explicitly show that the two tangents, orthogonal to Clifford translation velocity, rotate in moving plane $\{B_1 F_{Cl_\psi(z)}, B_2 F_{Cl_\psi(z)}\}$ with the same unit value rotational velocity. Interesting to notice that the triple of the translational velocity and two rotational velocities has orientation opposite to the triple of tangents: if the tangents $\{B_1 F_{Cl_\psi(z)}, B_2 F_{Cl_\psi(z)}, B_3 F_{Cl_\psi(z)}\}$ have right screw orientation, the speed triple $\{-B_1 F_{Cl_\psi(z)}, B_2 F_{Cl_\psi(z)}, B_3 F_{Cl_\psi(z)}\}$ is left screw.

If a fiber, g -qubit, makes full circle in Clifford translation:

$$F_z \rightarrow F_{Cl_\psi(z)} = e^{B_3 \psi} F_z, \quad 0 \leq \psi \leq 2\pi, \quad \text{both } B_1 F_{Cl_\psi(z)} \text{ and } B_2 F_{Cl_\psi(z)}$$

also make full rotation in their own plane by 2π . This is special case of the g -qubit Berry phase incrementing in a closed curve quantum state path.

7. Conclusions

Evolution of a quantum state described in terms of G_3^+ gives more detailed information about two state system compared to the C^2 Hilbert space model. It confirms the idea that distinctions between "quantum" and "classical" states become less deep if a more appropriate mathematical formalism is used. This paradigm spreads from trivial phenomena like tossed coin experiment [7] to recent results on entanglement and Bell theorem [8] where the former was demonstrated as not exclusively quantum property.

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7 Clearly, the three $B_i F_{Cl_\psi(z)}$ are identical to earlier considered tangents T_i